Abstract—Achievable rate regions of multiuser fading interference channels depend on their fading realizations. Motivated by this premise we consider the problem of adapting the users’ rates to fading variations. Channel-dependent rate adjustments are accomplished after each transition of the fading channel from one state to another. Such rate adjustments (increments or decrements) are constrained to meet some notion of fairness among the users and are designed to ensure that all users remain decodable. Here, we employ the notions of symmetric fair and max-min fair rate adaptations and offer algorithms for computing such fair rate adaptations. Besides fairness, the two other major features of these algorithms are that they are amenable to distributed implementation with limited information exchange among the users, and their complexities scale polynomially in the number of users.

I. INTRODUCTION

We consider a $K$-user fading interference channel and address the problem of adapting the users’ rates to fading variations such that the rate adjustment after each channel variation satisfies a fairness rule. We focus on interference channels where each user employs one Gaussian codebook as opposed to the two-codebooks-per-user strategy of [1], [2]. Deploying single codebooks, results in smaller achievable rate regions but has certain practical merits [3], [4].

We propose procedures for rate adaptations based on the notions of symmetric and max-min fairness. Besides achieving the optimal fairness, the two major features of these procedures are that they are amenable to distributed implementation and have polynomial complexity. In this context, we note that efficient rate allocation in an interference channel with single codebooks has been investigated in [3]–[6]. In particular, [3] considers a $K$-user interference channel where each user employs a decoder based on successive interference cancelation and obtains a decentralized max-min fair rate allocation algorithm. [4] considers a $K$-user interference channel and solves the problem of maximizing the desired user’s rate at a particular receiver given the transmission rates of the other users. In this paper, we assume that each receiver uses the unconstrained group decoder (UGD) and is allowed to jointly decode any subset of users containing its designated user, after suppressing the remaining users. UGD can be considered as an example of a successive group decoder (SGD) without complexity constraints [7]. Efficient and fair rate allocation algorithms for a multiuser interference channel are given in [5] for the case where each receiver employs an SGD and in [6] for the case where each receiver employs the UGD. The algorithms in [5], [6] were designed to yield rate increments for all users given their minimum acceptable rates, where the minimum acceptable rates were guaranteed to be simultaneously achievable. Thus, these algorithms are not applicable, without justification, if we wish to alter a rate vector in a fair manner but where the given rate vector is not achievable.

II. DESCRIPTION

A. System Model

Consider a $K$-user Gaussian interference channel where each single-antenna transmitter intends to communicate with its designated multi-antenna receiver. Due to the broadcast nature of the wireless channel each receiver is exposed to the communication between all other transmitter-receiver pairs. Denote the channel vector from the $l^{th}$ transmitter to the $k^{th}$ receiver by $h_{lk}^t$ and for the $k^{th}$ receiver define $H_k^t = [h_{1k}^t, \ldots, h_{Kk}^t]$. Also, define $x_k^x = [x_1, \ldots, x_K]^T$, where $x_k$ represents the input from the $k^{th}$ transmitter that employs a codebooks of rate $R_k$. The $k^{th}$ receiver ultimately desires to decode the codeword transmitted by the $k^{th}$ transmitter and the discrete-time model of its received signal is given by

$$y_k = H_k^tx + z_k \quad \text{for} \quad k \in \{1, \ldots, K\},$$

where $z_k$ is the additive noise vector whose elements are zero-mean unit variance complex Gaussian random variables.

B. Problem Statement

We consider the problem of rate adaptation in a $K$-user interference channel such that some notion of fairness is guaranteed. Assume that at some time slot $T_i$, for $i \in \mathbb{N}$, the $K$-user interference channel is in some state, which we denote by $S_i$. This state is influenced among other factors by the transmit powers available to the transmitters as well as the fading realizations of the wireless channels at time slot $T_i$. In order to account for the variations in the state, we let $h_{1i}(i)$ $\cdots$ the product of the channel vector $h_{ki}$ at time slot $T_i$ and the square-root of the average transmit power used by the $k^{th}$ transmitter in that slot. We assume that the fading realizations as well as the transmit powers do not vary within a slot but can change arbitrarily between slots. Now suppose that at time slot $T_i$, the users are operating at some decodable rate $R^t = [R_{1i}^t, \ldots, R_{Ki}^t]$, i.e., $R^t \geq 0$ and for all $k$ the $k^{th}$ transmitter-receiver link can sustain the rate $R_{ki}^t$. The channel remains in the same state for the current slot but due to some variation in the available transmit powers or fading realizations, at time slot $T_{i+1}$ changes to the state $S_{i+1}$. Due to such change the rate vector $R^t$ may remain decodable if $\forall k$, $R_{ki}^t$ is decodable by the $k^{th}$ receiver and not be decodable if $\exists k$ such that $R_{ki}^t$ is not decodable by the $k^{th}$ receiver. We seek to update $R^t$ and obtain a new decodable rate vector $R^{t+1}$ such that the following conditions are satisfied.

1) Some notion of fairness is maintained, i.e., no user unduly sacrifices its rate in favor of other users.
2) Updates are accomplished in a distributed way such that each user updates its rate merely based on its local knowledge of the network and some limited information exchange with other users.

We consider two fairness measures.

1) Symmetric Fairness: This measure suggests adapting the rates of all users based on some pre-determined priority. More specifically, we are interested in finding the largest possible scalar $x \in \mathbb{R}$ such that after updating the rates of the secondary users as $R^{t+1} = R^t + x \cdot t$, based on some user priorities accounted via the vector $t \overset{\Delta}{=} [t_1, \ldots, t_K] > 0$, the updated rate vector $R^{t+1}$ is decodable at time
slot $T_{i+1}$. We call $x$ the rate adaptation factor. The vector $t$ can model different notions of fairness. For instance, setting $t_1 = \cdots = t_K = 1$ provides all users with identical rate change, or setting $t_k = R^i_k$ leads to proportional fairness that scales all the rates identically. The optimal rate adaptation factor under symmetric fairness is given by

$$x^*_i = \left\{ \max_{\text{s.t.}} x \mid R^{i+1} = R^i + x \cdot t \text{ is decodable at } T_{i+1} \right\} \quad (2)$$

Clearly, if $R^i$ is decodable at $T_{i+1}$ then $x^*_i$ is expected to be non-negative and results in rate increment and if $R^i$ is not decodable at $T_{i+1}$ then $x^*_i$ is expected to be negative and results in rate decrement.

2) Max-Min Fairness: By denoting the rate variation of the $k^{th}$ user by $r_k$, our objective is to maximize $\min_k \frac{R_k}{t_k}$ such that after updating $R^{i+1} = R^i + r$, for some given vector $t \succ 0$, $R^{i+1}$ is decodable. By defining $r \triangleq [r_1, \ldots, r_M]$ the problem can be formalized as

$$y^*_i = \left\{ \max_{\text{s.t.}} \min_k \frac{r_k}{t_k} \mid R^{i+1} = R^i + r \text{ is decodable at } T_{i+1} \right\} \quad (3)$$

Further, among multiple $R^{i+1}$ that yield the same $y^*_i$, we are interested in the one that is also pareto-optimal. Similar to symmetric fairness, if $R^i$ is decodable at $T_{i+1}$ then $y^*_i$ is non-negative and otherwise it is negative. The objective is to obtain $x^*_i$ and $y^*_i$ for given $R^i$ and $t$ in a distributed fashion with manageable complexity.

Notice that the optimization problem in (2) need not always be feasible in that there may not exist a scalar $x$ such that $R^i + x \cdot t \succeq 0$ and at time slot $T_{i+1}$, the $k^{th}$ transmitter-receiver link can sustain the rate $R_k + x t_k$ for all $k$. The algorithms given in the sequel also provide a feasibility check and their results are optimal whenever (2) is feasible. We remark that the fairness measures may themselves be inappropriate when (2) is not feasible. Henceforth, unless otherwise mentioned we will make the reasonable assumption that the problem in (2) is feasible. We also note that to prove the results in the sequel, we leverage the techniques developed and used in [6] but we remove the restriction that the vector $R^i$ be itself decodable at time slot $T_{i+1}$.

III. UNCONSTRAINED GROUP DECODING

We assume that all the receivers employ the unconstrained group decoder (UGD) where each receiver is allowed to decode any arbitrary subset of transmitters (that contains its designated one) after suppressing the remaining ones as Gaussian interferers. Single-user decoding, where each receiver decodes only its designated transmitter, and (joint) maximum likelihood decoding, where each receiver decodes all transmitters, are the two extremes of unconstrained group decoding.

For any two disjoint subset of users $U, V$, let $C_k(T_i, U, V)$ denote the achievable rate region supported by the $k^{th}$ receiver at time slot $T_i$ when jointly decodes the users in $U$ via maximum likelihood decoding after suppressing those in $V$ as Gaussian interferers. $C_k(T_i, U, V)$ can be characterized as

$$C_k(T_i, U, V) = \left\{ r \in \mathbb{R}^{|U|}_{+} \mid \sum_{j \in Q} r_j \leq R_k(T_i, Q, V), \forall Q \subseteq U \right\}, \quad (4)$$

where upon defining $H^k_Q(i) \triangleq [h^k_q(i)]_{q \in Q}$ for any subset of users $Q \subseteq K \triangleq \{1, \ldots, K\}$, $R_k(T_i, U, V)$ is given by

$$R_k(T_i, Q, V) = \log \left| I + (H^k_Q(i))^{H} (I + H^k_V(i) (H^k_V(i))^{H})^{-1} H^k_Q(i) \right|. \quad (5)$$

Also let $\{G_k, G^c_k\}$ denote a partition of $K$ such that $k \in G_k$ and the users in $G_k$ are jointly decoded after treating those in $G^c_k$ as Gaussian interferers. Therefore, a rate vector $R$ is decodable if for all $k$, the $k^{th}$ user is decodable at the $k^{th}$ receiver, i.e.,

$$\exists \: G_1, \ldots, G_K \text{ such that } \forall k, k \in G_k \text{ and } R_{G_k} \in C_k(T_i, G_k, G^c_k), \quad (6)$$

where for any subset $Q \subseteq K$ we have defined $R_Q \triangleq \{R_j\}_{j \in Q}$.

IV. SYMMETRIC FAIR RATE ADAPTATION

In this section we offer a procedure that obtains $x^*_i$ defined in (2) in a distributed way and with polynomial complexity.

A. Rate Adaptation

Given the rate vector $R^i$ decodable at time slot $T_i$, for the $k^{th}$ receiver and any two disjoint subsets $U, V$ of $K$ we define a rate adaptation factor for slot $T_{i+1}$ as

$$\delta_k(T_{i+1}, U, V, R^i, t) \triangleq \left\{ \max_{\text{s.t.}} x \mid R^i + x \cdot t \in C_k(T_{i+1}, U, V) \right\}, \quad (7)$$

where we had defined $R_Q \triangleq \{R_j\}_{j \in Q}$. Note that when (7) is feasible, $\delta_k(T_{i+1}, U, V, R^i, t)$ identifies the maximum rate increment (or the minimum rate decrement) vector along the given vector $t$ such that the rate vector $R^i + x \cdot t$ lies within the rate region $C_k(T_{i+1}, U, V)$ defined in (4). Given this definition, from the viewpoint of the $k^{th}$ receiver, the maximum rate adaptation factor such that the $k^{th}$ user remains decodable can be found by maximizing $\delta_k(T_{i+1}, U, V, R^i, t)$ over all possible choices of $U = G \subseteq K$ such that $k \in G$ and $V = K \setminus G$. The maximum rate adaptation factor for the $k^{th}$ receiver is given by

$$\delta^*_k(T_{i+1}) \triangleq \max_{G \subseteq K, k \in G} \delta_k(T_{i+1}, G, K \setminus G, R^i, t). \quad (8)$$

Note that since $\delta^*_k(T_{i+1}) \geq x^*_i$, due to our assumption on the state of the channel at $T_{i+1}$, we must have

$$R^i + \delta^*_k(T_{i+1}) \cdot t \succeq 0. \quad (9)$$

We now offer a two-step procedure for obtaining the optimal rate adaptation factor $x^*_i$. In the first step we demonstrate how efficiently (with polynomial complexity) we can solve $\delta^*_k(T_{i+1})$ and in the second step (provided in the next section) we leverage the solution of $\delta^*_k(T_{i+1})$ for $k \in K$ in order to obtain $x^*_i$.

Solving $\delta^*_k(T_{i+1})$ involves two levels of complexity. One pertains to solving $\delta_k(T_{i+1}, U, V, R^i, t)$ given in (7) for the choices of $U = G$ such that $k \in G$ and $V = K \setminus G$, and the other relates to obtaining the optimal choice of $G$ for maximizing $\delta_k(T_{i+1}, G, K \setminus G, R^i, t)$. Based on the techniques developed in [3] we state the following lemma, which helps in solving the optimization problem in (7).

Lemma 1: Consider the region

$$\mathcal{H}(f, U) \triangleq \left\{ R \in \mathbb{R}^{|U|} \mid \sum_{j \in Q} R_j \leq f(Q), \forall Q \subseteq U \right\},$$

where $f : Q \subseteq U \rightarrow \mathbb{R}_+$ is a rank function. If

$$\delta = \left\{ \max_{\text{s.t.}} x \mid R^i + x \cdot t \in \mathcal{H}(f, U) \right\},$$

then

$$\delta = \frac{\min_{Q \neq \emptyset, Q \subseteq U} f(Q) - \sum_{j \in Q} R^i_j}{\sum_{j \in Q} t_j}. \quad (10)$$

1We will set $\delta_k(T_{i+1}, U, V, R^i, t) = -\infty$ whenever (7) is infeasible.
Note that the region $C_k(T_{i+1}, U, V)$ can be shown to be a polymatroid [8] with the rank function $f(Q) = R_k(T_{i+1}, Q, V)$ given in (5). Then, $C_k(T_{i+1}, U, V)$ is in fact the set of vectors in $H(f, U)$ with non-negative elements. Hence, by applying Lemma 1, we have
\[
\delta_k(T_{i+1}, U, V, R^*, t) \leq \min_{Q \neq \emptyset, Q \subseteq U} \frac{\Delta_k(T_{i+1}, Q, V, R^*)}{\sum_{j \in Q} t_j}, \quad (10)
\]
where
\[
\Delta_k(T_{i+1}, Q, V, R^*) = R_k(T_{i+1}, Q, V) - \sum_{j \in Q} R_j, \quad (11)
\]
with equality in (10) whenever (7) is feasible. Using (10) with (8) and (9) we can show that
\[
\delta_k^*(T_{i+1}) = \max_{g \subseteq K, k \in U} \min_{Q \neq \emptyset, \emptyset \subseteq Q \subseteq U} \frac{\Delta_k(T_{i+1}, g, Q, R^*)}{\sum_{j \in Q} t_j}. \quad (12)
\]
Solving the RHS in (10) boils down to solving at-most $|U|$ sub-modular minimization problems each of which can be solved with a complexity polynomial in $|U|$ [4], [9]. Next, given (12) a direct (naive) way to compute $\delta_k^*(T_{i+1})$ is to search all valid partitions $\{g, K\setminus g\}$ such that $k \in \bar{g}$. We instead propose an efficient iterative procedure that finds $\delta_k^*(T_{i+1})$ for each user $k \in K$, with a complexity that is polynomial in $K$. The core idea of this procedure is that each receiver begins by including all other users for decoding jointly with its desired user. Then it successively identifies and eliminates a group of users that can be safely treated as Gaussian interferers and their elimination provides a higher rate adaptation factor. The procedure continues until the desired user also lies among the users to be eliminated. At this point the procedure declares the users eliminated in the previous iterations as those to be treated as noise and based on that computes the optimal rate adaptation factor. This steps of this procedure, which have polynomial complexity in $K$, and find $\delta_k^*(T_{i+1})$ and its respective optimal user partitioning $\{g_k^*, K\setminus g_k^*\}$ at each $k$ without imposing any information exchange among the users (distributed processing), are formalized in Algorithm 1. The optimality of this algorithm is demonstrated in the subsequent theorem.

Algorithm 1

1: Initialize $g = K$, $V = \emptyset$, $V_0 = \emptyset$, $R^* \subseteq R$ and $\ell = 1$
2: repeat
3: Find $\delta_\ell = \min_{Q \neq \emptyset, \emptyset \subseteq Q \subseteq U} \frac{\Delta_k(T_{i+1}, Q, V, R^*)}{\sum_{j \in Q} t_j}$
4: Find $V_\ell = \arg \min_{Q \neq \emptyset, \emptyset \subseteq Q \subseteq U} \Delta_k(T_{i+1}, Q, V, R^*)$
5: if $k \in V_\ell$
6: $\delta_k(T_{i+1}) = \delta_\ell$
7: else
8: $g \leftarrow g_\cap V_\ell$ and $V \leftarrow V \cup V_\ell$ and $\ell \leftarrow \ell + 1$
9: end if
10: until $k \in V_\ell$
11: Output $\delta_k(T_{i+1})$ and $g_k = g$

Before we proceed to the optimal properties we note that $\Delta(\cdot)$ has the following properties. For any disjoint sets $U, V, W$,
\[
\Delta_k(T_{i+1}, U \cup V, W, R^*) = \Delta_k(T_{i+1}, U, V, W, R^*) + \Delta_k(T_{i+1}, V, W, R^*), \quad (13)
\]
and for any $V \subseteq W$
\[
\Delta_k(T_{i+1}, U, V, R^*) \geq \Delta_k(T_{i+1}, U, W, R^*). \quad (14)
\]
For proving the optimality of Algorithm 1 we first provide the following lemma.

Lemma 2: By denoting the number of iterations of Algorithm 1 by $m$ we have $\delta_1 \leq \delta_2 \leq \cdots \leq \delta_m = \delta_k(T_{i+1}) \leq \delta_k^*(T_{i+1})$.

Proof: For any $\delta_\ell$ for $\ell = 1, \ldots, m - 1$ we have
\[
\delta_\ell = \Delta_k(T_{i+1}, V_\ell, U_{\ell+1}^{m-1}V_{\ell}, R^*)
\]
\[
= \min_{\forall \emptyset \neq \emptyset \subseteq Q, \emptyset \subseteq \emptyset \subseteq U} \frac{\Delta_k(T_{i+1}, Q, V_{\ell}, R^*)}{\sum_{j \in V_{\ell}} t_j}
\]
\[
\leq \Delta_k(T_{i+1}, V_\ell \cup V_{\ell+1}, U_{\ell+1}^{m-1}V_{\ell}, R^*)
\]
\[
= \Delta_k(T_{i+1}, V_\ell, V_{\ell+1}, U_{\ell+1}^{m-1}V_{\ell}, R^*) + \Delta_k(T_{i+1}, V_{\ell+1}, U_{\ell+1}^{m-1}V_{\ell}, R^*)
\]
\[
\leq \sum_{j \in V_{\ell+1}} t_j + \sum_{j \in V_{\ell}} t_j
\]
where (16) holds by recalling (13). The inequality between (15) and (17) with some simple manipulation provides that $\delta_\ell \leq \delta_\ell$.

Finally, the inequality $\delta_m = \delta_k(T_{i+1}) \leq \delta_k^*(T_{i+1})$ holds due to (12).

By deploying the result of Lemma 2, the following theorem establishes the optimality of Algorithm 1.

Theorem 1: Algorithm 1 identifies a partitioning that maximizes $\delta_k(T_{i+1}, g, K\setminus g, R^*, t)$ over all $g \subseteq K$ such that $k \in g$, i.e.,
\[
\delta_k(T_{i+1}) = \delta_k^*(T_{i+1}) = \max_{g \subseteq K, k \in g} \delta(H^*, g, K\setminus g, R^*),
\]
where $\delta_k^*(T_{i+1})$ is the rate adaptation factor returned by Algorithm 1.

Proof: Note that for the output of the algorithm we have $g_k = K\setminus U_{\ell+1}^{m-1}V_k$, where $m$ is the number of iterations. Suppose $\{g_k^*, K\setminus g_k^*\}$ represents an optimal partitioning with $k \in g_k^*$ such that $R_k^* + \delta_k^*(T_{i+1}) t_k^* \in C_k(T_{i+1}, g_k^*, K\setminus g_k^*)$. Then if we show that $\delta_k^*(T_{i+1}) \leq \delta_k^*(T_{i+1})$, we can conclude that $R^* + \delta_k^*(T_{i+1}) t \geq 0$ so that $R_k^* + \delta_k^*(T_{i+1}) t_k^* \in C_k(T_{i+1}, g_k, K\setminus g_k^*)$, where $g_k$ is the output of Algorithm 1, hence $\{g_k, K\setminus g_k\}$ is also an optimal partition and invoking Lemma 2, $\delta_k^*(T_{i+1}) = \delta_k^*(T_{i+1})$. We establish the proof by contradiction. Suppose that we have $\delta_k^*(T_{i+1}) > \delta_k^*(T_{i+1})$. Then, as we shall show later, we must have $g_k^* \subseteq g_k$, and therefore we get
\[
V_m \subseteq g_k \subseteq Q \quad \text{and} \quad K \setminus g_k = \{K \setminus g_k^* \} \cup Q.
\]
Consequently we get
\[
\delta_k(T_{i+1}) = \delta_m = \Delta_k(T_{i+1}, V_m, U_{\ell+1}^{m-1}V_k, R^*)
\]
\[
= \min_{\emptyset \neq \emptyset \subseteq Q, \emptyset \subseteq U} \frac{\Delta_k(T_{i+1}, P, K \setminus P, R^*)}{\sum_{j \in Q} t_j}
\]
\[
= \min_{\emptyset \neq \emptyset \subseteq Q, \emptyset \subseteq U} \frac{\Delta_k(T_{i+1}, P, K \setminus P, R^*)}{\sum_{j \in Q} t_j}
\]
\[
\geq \min_{\emptyset \neq \emptyset \subseteq Q, \emptyset \subseteq U} \frac{\Delta_k(T_{i+1}, P, K \setminus P, R^*)}{\sum_{j \in Q} t_j}
\]
\[
= \delta_k^*(T_{i+1}) = \delta_k^*(T_{i+1}),
\]
where (18) holds since $V_m \subseteq g_k \subseteq Q$ and therefore minimizing over the $g_k$ and $g_k \subseteq Q$ essentially yield the same result. Hence, we have shown that $\delta_k^*(T_{i+1}) \leq \delta_k^*(T_{i+1})$, which contradicts the assumption that $\delta_k^*(T_{i+1}) > \delta_k^*(T_{i+1})$ and therefore by contradiction we must have $\delta_k^*(T_{i+1}) \leq \delta_k^*(T_{i+1})$. Finally we need to only show that if $\delta_k^*(T_{i+1}) > \delta_k^*(T_{i+1})$, then $g^*_k \subseteq g_k$ and $V_m \subseteq g_k$.
1) $G^*_k \subseteq G_k$.

Note that $G^*_k = K \setminus \bigcup_{j=1}^{m-1} V_j$. In order to show that $G^*_k \subseteq G_k$, we equivalently show that $G^*_k \cap V_j = \emptyset$ for $j = 1, \ldots, m - 1$. By contradiction, let us assume that $G^*_k$ has non-empty intersection with some of the sets $\{V_j\}_{j=1}^{m-1}$ and denote $j$ as the smallest value such that $G^*_k \cap V_j \neq \emptyset$. By the above expansion of $V_j$, we have $G^*_k \cap V_j = \emptyset$.

Using the definition of $\delta_j$ and $V_j$ (Algorithm 1) we get

$$
\delta_j \sum_{n = V_j} t_n = \Delta_k(T_{i+1}, V_j, \bigcup_{j=1}^{m-1} V_j, R^i)
$$

$$
= \Delta_k(T_{i+1}, \{G^*_k \cap V_j\} \cup \{\{K \setminus G^*_k\} \cap V_j\}, \bigcup_{j=1}^{m-1} V_j, R^i)
$$

$$
= \Delta_k(T_{i+1}, \{G^*_k \cap V_j\} \cup \{\{K \setminus G^*_k\} \cap V_j\}, \bigcup_{j=1}^{m-1} V_j, R^i)
$$

$$
+ \Delta_k(T_{i+1}, G^*_k \cap V_j, \{K \setminus G^*_k\} \cap V_j \cup \bigcup_{j=1}^{m-1} V_j, R^i)
$$

$$
\geq \Delta_k(T_{i+1}, \{K \setminus G^*_k\} \cap V_j \cup \bigcup_{j=1}^{m-1} V_j, R^i)
$$

$$
+ \Delta_k(T_{i+1}, G^*_k \cap V_j, \{K \setminus G^*_k\} \cap V_j \cup \bigcup_{j=1}^{m-1} V_j, R^i)
$$

$$(\forall j \subseteq G^*_k) .$$

(19)

Now note that for any $Q \subseteq V_j$, we can readily show that

$$
\delta_j = \Delta_k(T_{i+1}, V_j, \bigcup_{j=1}^{m-1} V_j, R^i) \leq \sum_{n \in V_j} t_n
$$

or equivalently

$$
\Delta_k(T_{i+1}, Q, \bigcup_{j=1}^{m-1} V_j, R^i) \geq \delta_j \sum_{n \in Q} t_n .$$

(20)

By following the same line of argument we can also show that

$$
\Delta_k(T_{i+1}, G^*_k \cap V_j, \{K \setminus G^*_k\} \cap V_j, R^i) \geq \delta_k(T_{i+1}) \sum_{n \in G^*_k \cap V_j} t_n .$$

(21)

Combining (19)-(21) provides that

$$
\delta_j \leq \Delta_k(T_{i+1}, V_j, \bigcup_{j=1}^{m-1} V_j, R^i) \leq \Delta_k(T_{i+1}, Q, \bigcup_{j=1}^{m-1} V_j, R^i) .$$

which is a contradiction. Therefore, for all $j \in \{1, \ldots, m-1\}$, $G^*_k \cap V_j = \emptyset$ and as a result $G^*_k \subseteq \{K \setminus \bigcup_{j=1}^{m-1} V_j\} = G_k$.

2) $\forall m \subseteq G^*_k$,

If $\forall m \subseteq G^*_k$, then by noting that $V_m \subseteq G_k$ we deduce $V_m \cap \{G^*_k \cap V_j\} = \emptyset$. By expanding $V_m = \{V_m \cap G^*_k\} \cup \{V_m \cap \{K \setminus G^*_k\}\}$ and following the same line of argument as in case 1 we have

$$
\delta_k \sum_{j \in V_m} t_j = \Delta_k(T_{i+1}, V_m, \bigcup_{j=1}^{m-1} V_j, R^i)
$$

$$
= \Delta_k(T_{i+1}, V_m \cap \{G^*_k \cap V_j\} \cup \{K \setminus G^*_k\}, \bigcup_{j=1}^{m-1} V_j, R^i)
$$

$$
+ \Delta_k(T_{i+1}, V_m \cap \{G^*_k \cap V_j\} \cup \{K \setminus G^*_k\}, \bigcup_{j=1}^{m-1} V_j, R^i)
$$

$$
\geq \Delta_k(T_{i+1}, V_m \cap \{G^*_k \cap V_j\} \cup \{K \setminus G^*_k\}, \bigcup_{j=1}^{m-1} V_j, R^i)
$$

$$
+ \delta_k(T_{i+1}) \sum_{j \in V_m \cap \{G^*_k \cap V_j\}} t_j + \delta_m \sum_{j \in V_m \cap \{K \setminus G^*_k\}} t_j
$$

$$
\geq \delta_k(T_{i+1}) \sum_{j \in V_m \cap \{G^*_k \cap V_j\}} t_j + \delta_k(T_{i+1}) \sum_{j \in V_m \cap \{K \setminus G^*_k\}} t_j
$$

which is a contradiction and therefore $V_m \subseteq G^*_k$.

B. Distributed Processing

Algorithm 1 determines the optimal rate adaptation factor and rate update policy corresponding to each user. However, distinct users not necessarily suggest identical rate updates. Therefore, it becomes imperative to reach a consensus among all users about how to update the rates. In this section, we offer a distributed algorithm that is constructed based on Algorithm 1 and yields the global rate adaptation policy and achieves symmetric fairness among the users. Its optimality is proved in Theorem 2.

**Algorithm 2 - Symmetric Fair Rate Adaptation**

1: Input $R^i$

2: for $k = 1, \ldots, K$ do

3: Use Algorithm 1 to determine $\delta_k^*(T_{i+1})$ and $G_k^*$

4: end for

5: Update $R^{i+1} = R^i + \min_{1 \leq k \leq K} \{\delta_k^*(T_{i+1})\} \cdot t$

6: Output $R^{i+1}$ and $\{G_k\}_{k=1}^K$

**Theorem 2:** The rate vector yielded by Algorithm 2 satisfies $R^{i+1} \geq R$. where $R$ is any decodable rate-vector such that $R = R^i + x \cdot t$ for some $x \in \mathbb{R}$, i.e., $x^* = \min_k \delta_k^*(T_{i+1})$.

Proof: If there exists a decodable rate-vector $\hat{R}$ as defined above such that $R^{i+1} < \hat{R}$, then we have that $x > \min_k \delta_k^*(T_{i+1})$. By defining $k^* = \arg \min_{1 \leq k \leq K} \{\delta_k^*(T_{i+1})\}$ we conclude $x > \delta_k^*(T_{i+1})$.

Since $R$ is decodable, then $R^{i+1} = R^i + x \cdot t$ is decodable by the $k^*$th user and $x > \delta_k^*(T_{i+1})$ which contradicts the optimality of $\delta_k^*(T_{i+1})$ as the optimal rate adaptation factor (Theorem 1).

It can be shown that the problem in (2) is feasible if and only if the rate vector yielded by Algorithm 2 satisfies $R^{i+1} \geq 0$.

V. MAX-MIN FAIR RATE ADAPTATION

A. Rate Adaptation

In this section, we aim to solve (3) and identify $y^*_i$. By denoting the rate change of the $k$th user by $r_k$, our objective is to maximize $\min_{1 \leq k \leq K} r_k$ such that the updated rate vector $R^{i+1} = R^i + r$ is decodable. Therefore, for the $k$th receiver and for any two disjoint subsets $\mathcal{U}, \mathcal{V}$ of $\mathcal{K}$, we define

$$
\theta_k(T_{i+1}, \mathcal{U}, \mathcal{V}, R^i, t) \triangleq \max \left\{ \min_{1 \leq k \leq K} \frac{r_k}{t} \right\}
$$

s.t. $R_{U} + t \cdot r_U \in \mathcal{C}_k(T_{i+1}, \mathcal{U}, \mathcal{V})$ .

(22)

which picks a rate-vector within the rate region $\mathcal{C}_k(T_{i+1}, \mathcal{U}, \mathcal{V})$ and achieves weighted max-min fairness for the users in $\mathcal{U}$. Therefore, from the viewpoint of the $k$th user, the maximum rate adaptation factor for sustaining max-min fairness such that remains decodable is given by

$$
\theta_k(T_{i+1}) \triangleq \max_{G_k \subseteq K, \subseteq \mathcal{G}} \min_{1 \leq k \leq K} \theta_k(T_{i+1}, G^*_k, \mathcal{G}, R^i, t) .
$$

(23)

The following lemma reveals the underlying connection between the solution of the symmetric fair rate adaptation problem in (7) and the max-min fair rate adaptation problem in (22).

**Lemma 3:** For the $k$th user and any two disjoint subsets $\mathcal{U}, \mathcal{V}$ of $\mathcal{K}$, where $\mathcal{U} \in \mathcal{K}$ we have

$$
\theta_k(T_{i+1}, \mathcal{U}, \mathcal{V}, R^i, t) = \delta_k(T_{i+1}, \mathcal{U}, \mathcal{V}, R^i, t)
$$

whenever (7) is feasible. Further,

$$
\theta_k(T_{i+1}, \mathcal{U}, \mathcal{V}, R^i, t) \leq \min \frac{\Delta_k(T_{i+1}, \mathcal{Q}, \mathcal{V}, R^i)}{\sum_{j \in \mathcal{Q}} t_j} .
$$

(24)
Based on Lemma 3, the definition of $\theta_k^e(T_{i+1})$ in (23) combined with (9) and (12), we have

$$\theta_k^e(T_{i+1}) = \max_{g \in K, \ell \in G} \min_{Q \subseteq K, \ell \in G} \frac{\Delta_k(T_{i+1}, Q, G, R^\ell)}{\sum_{j \in Q} t_j}.$$  \hspace{1cm} (24)

Algorithm 3 given below is a computationally efficient procedure (polynomial in $K$) for finding the set of rate increments $\{r_k^1, \ldots, r_k^K\}$ for each given receiver $k$, such that the $k^{th}$ receiver can successfully decode its designated user and max-min fairness is sustained.

**Algorithm 3**

1. Initialize $Q = K$ and $G = \emptyset$ and $G_k = \emptyset$ and $\ell = 1$
2. repeat
3. Find $\delta_t = \min_{V \neq \emptyset, V \subseteq Q} \frac{\Delta_k(T_{i+1} \cup V, G, R^\ell)}{\sum_{j \in V} t_j}$
4. Find $V_t = \arg \min_{V \neq \emptyset, V \subseteq Q} \frac{\Delta_k(T_{i+1} \cup V, G, R^\ell)}{\sum_{j \in V} t_j}$
5. if $k \in V_t$ or $k \in G$
6. $r_t^k = \delta_t t$, for all $j \in V_t$
7. $Q = Q \cup V_t$ and $G = G \cup V_t$
8. $G_k = G_k \cup V_t$ and $\ell = \ell + 1$
9. else
10. $r_t^k = +\infty$ for all $j \in V_t$, $Q = Q \cup V_t$
11. $G = G \cup V_t$, $\ell = \ell + 1$
12. end if
13. until $Q = \emptyset$
14. Output $\{r_k^1, \ldots, r_k^K\}$ and $G_k$

**Theorem 3:** The $k^{th}$ user is decodable under the rate vector $R^\ell + \{r_k^1, \ldots, r_k^K\}$, where $\{r_k^i\}$ is yielded by Algorithm 3. Furthermore, $R^\ell + \{r_k^1, \ldots, r_k^K\} \succeq R^e(T_{i+1}) \cdot t \geq 0$ and

$$\theta_k^e(T_{i+1}) = \min_{\ell \subseteq K} \frac{r_t^k}{t_k} \geq \min_{\ell \subseteq K} \frac{r_t^k}{t_k},$$

where $\{r_k^1, \ldots, r_k^K\}$ is any other arbitrary rate update vector for which the $k^{th}$ user is decodable under the rates $R^\ell + \{r_k^1, \ldots, r_k^K\}$.

In the following section we show how the rate changes suggested by various users should be processed.

**B. Distributed Processing**

Recall that using Algorithm 3 each specific user $k$ identifies rate changes for all users with the constraint that the $k^{th}$ user remains decodable at its designated receiver. As a result, in the $K$-user channel, each user receives $K$ rate change suggestions, one computed by itself and $K - 1$ ones by others. Now each user picks the smallest rate change suggested for it. The steps for such rate adaptation are formalized in Algorithm 4. The optimal properties of this algorithm are enumerated in Theorem 4.

**Algorithm 4 - Max-Min Fair Rate Adaptation**

1. Input $R^\ell$ and Initialize $q = 0$ and $R^{(0)} = R^\ell$
2. repeat
3. for $k = 1, \ldots, K$ do
4. Run Algorithm 3
5. end for
6. Update $R_k^{(q+1)} = R_k^{(q)} + \min_{1 \leq \ell \leq K} \{r_k^\ell\}$, $\forall k$
7. Update $q = q + 1$
8. until $R_k^{(q)}$ converges
9. Output $R_k^{(q)} = R_k^{(1)}$ and $\{G_k\}_{k=1}^K$

**Theorem 4:** The distributed (iterative) max-min fair rate adaptation algorithm (Algorithm 4) has the following properties

1. For all $q \geq 1$, $R_k^{(q)}$ is decodable and max-min optimal, i.e., for any other arbitrary decodable rate vector $\hat{R}$ we must have $\min_{k \in K} \frac{R_k^{(q)} - R_k^{(1)}}{t_k} \geq \min_{k \in K} \frac{\hat{R}_k - R_k^{(1)}}{t_k}$, $\forall q \geq 1$.
2. The algorithm is monotonic in the sense that $R_k^{(q+1)} \succeq R_k^{(q)}$, $\forall q \geq 1$ and is convergent.\footnote{Note that since $R^{(0)} = R^\ell$ need not be decodable, $R_k^{(1)} \succeq R_k^{(0)}$ is not guaranteed.}
3. The rate vector yielded by Algorithm 4 upon convergence is decodable, max-min fair and also pareto-optimal.

Note that the pareto-optimality property ensures that in the solution provided by Algorithm 4, any increase in the rate of one user must be penalized by a decrease in the rate of some other user(s). Finally, it can be shown that the problem in (2) is feasible if and only if the rate vectors in Algorithm 4 satisfy $R_k^{(q+1)} \succeq 0$, $\forall q \geq 1$.

**VI. CONCLUSIONS**

We consider multiuser fading interference channels. The objective is to propose distributed and computationally efficient procedures for adaptively adjusting the users’ rates with channel variations. Rate adjustments are constrained to meet fairness requirements. In particular, based on the channel state, the users are required to increase or decrease their rates such that all users remain decodable under the new channel state and furthermore all rate increments or decrements satisfy symmetric or max-min fairness. The proposed algorithms are amenable to distributed processing with limited information exchange and have complexities that scale polynomially in the number of users.

**REFERENCES**