Lecture 4

Hidden Markov Models

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Administrivia

- Feedback (2+ votes):
  - Pace good; examples + graphs good.
  - Less math (3).
  - More on how to implement formulas in programs (2).
- Muddiest topics: EM (9); multivariate GMM (1); algebra (1).
- Many people said "maybe" to non-reading project (∼10).

Administrivia

- Hardcopies of slides not most recent version.
- Do you have your username and password?
- Xiao-Ming's office hours: 2-4pm Monday, 7LE3 Schapiro (CEPSR).
- Late policy: three free late days; -0.5 per day after that.
- Lab 1 is due Wednesday at 6pm!
  - Use Courseworks for questions!
- Lab 2 posted on web site by Wednesday.
- We found pictures.

Recap: Probabilistic Modeling for ASR

- Old paradigm: DTW.

$$w^* = \arg\min_{w \in \text{vocab}} \text{distance}(A'_{\text{test}}, A'_w)$$

- New paradigm: Probabilities.

$$w^* = \arg\max_{w \in \text{vocab}} P(A'_{\text{test}}|w)$$

  - $P(A'|w)$ is (relative) frequency with which w ...
  - Is realized as feature vector A'.
- The more "accurate" $P(A'|w)$ is ...
  - The more accurate classification is.

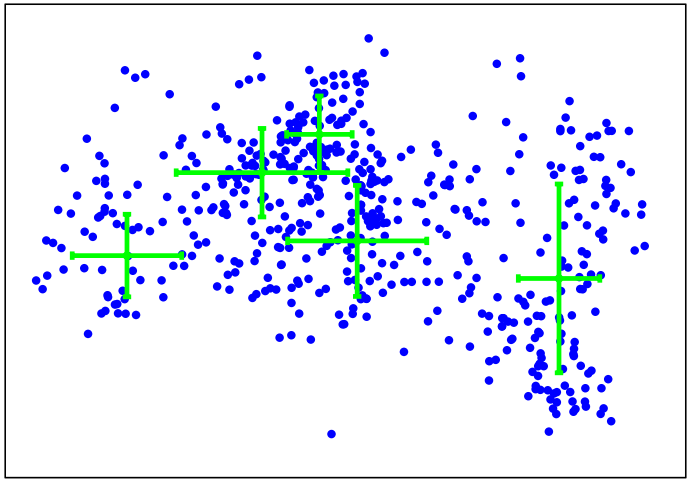
Recap: Gaussian Mixture Models

- Probability distribution over . . .
  - Individual (*e.g.*, 40d) feature vectors.

$$P(\mathbf{x}) = \sum_j p_j \frac{1}{(2\pi)^{d/2}|\Sigma_j|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_j)^T \boldsymbol{\Sigma}_j^{-1}(\mathbf{x}-\boldsymbol{\mu}_j)}$$

- Can model arbitrary (non-Gaussian) data pretty well.
- Can use EM algorithm to do ML estimation.
  - Finds local optimum in likelihood, iteratively.

Example: Modeling Acoustic Data With GMM



What We Have And What We Want

- What we have: $P(\mathbf{x})$.
  - GMM is distribution over indiv feature vectors $\mathbf{x}$.
- What we want: $P(A' = \mathbf{x}_1, \ldots, \mathbf{x}_T)$.
  - Distribution over *sequences* of feature vectors.
  - Build separate model $P(A'|w)$ for each word w.
  - There you go.

$$w^* = \arg\max_{w \in \text{vocab}} P(A'_{\text{test}}|w)$$

Sequence Modeling

- Today's lecture: using Hidden Markov Models . . .
  - To model sequences of feature vectors.
  - *i.e.*, how feature vectors evolve over time.
  - Probabilistic counterpart to DTW.
- How things fit together.
  - GMM's: for each sound, what are likely feature vectors?
  - *e.g.*, why the sound "*b*" is different from "*d*".
  - HMM's: what "sounds" are likely to follow each other?
  - *e.g.*, why *bad* is different from *dab*.

Simplification: Discrete Sequences

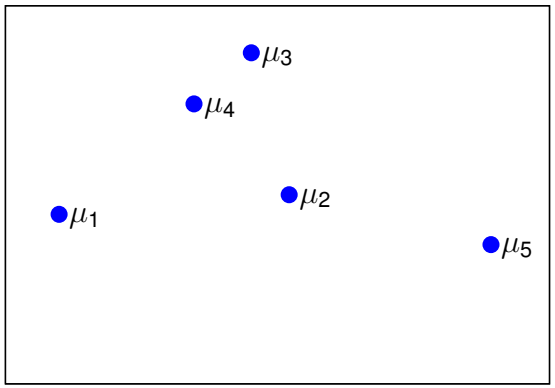
- Goal: continuous data.
  - *e.g.*, $P(\mathbf{x}_1, \ldots, \mathbf{x}_T)$ for $\mathbf{x} \in \mathcal{R}^{40}$.
- Most of today: discrete data.
  - $P(x_1, \ldots, x_T)$ for $x \in$ finite alphabet.
- *Discrete* HMM's *vs. continuous* HMM's.

Vector Quantization

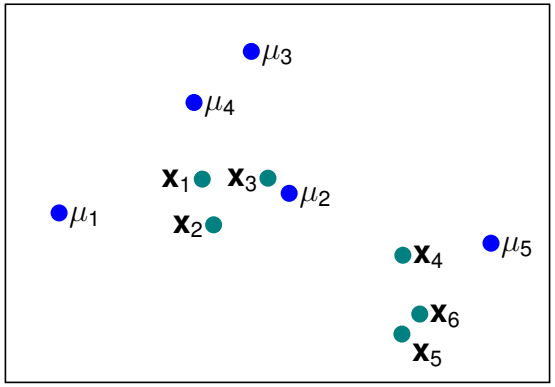
- Before continuous HMM's and GMM's ($\sim$1990) ...
  - People used discrete HMM's and VQ (1980's).
- Convert multidimensional feature vector ...
  - To discrete symbol $\{1, \ldots, V\}$ using *codebook*.
- Each symbol has representative feature vector μ_j.
- Convert each feature vector ...
  - To symbol j with nearest μ_j.

The Basic Idea

- How to pick the μ_j?



The Basic Idea



$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5, \mathbf{x}_6 \ldots \Rightarrow 4, 2, 2, 5, 5, 5, \ldots$

Recap

- Need probabilistic sequence modeling for ASR.
- Let's start with discrete sequences.
  - Simpler than continuous.
  - What was used first in ASR.
- Let's go!

Part I

Nonhidden Sequence Models

Case Study: Coin Flipping

- Let's flip (unfair) coin 10 times: $x_1, \ldots x_{10} \in \{\mathrm{T}, \mathrm{H}\}$, *e.g.*,

  T, T, H, H, H, H, T, H, H, H

- Design $P(x_1, \ldots x_T)$ matching actual frequencies ...
  - Of sequences $(x_1, \ldots x_T)$.
- What should form of distribution be?
- How to estimate its parameters?

Where Are We?

Independence

- Coin flips are *independent*!
  - Outcome of previous flips doesn't influence ...
  - Outcome of future flips (given parameters).

$$P(x_1, \ldots x_{10}) = \prod_{i=1}^{10} P(x_i)$$

- System has no *memory* or *state*.
- Example of dependence: draws from deck of cards.
  - *e.g.*, if last card was A♠, next card isn't.
  - State: all cards seen.

Modeling a Single Coin Flip $P(x_i)$

- *Multinomial* distribution.
- One parameter for each outcome: $p_H, p_T \geq 0$ ...
  - Modeling frequency of that outcome, *i.e.*, $P(x) = p_x$.
- Parameters must sum to 1: $p_H + p_T = 1$.
- Where have we seen this before?

Computing the Likelihood of Data

- Some parameters: $p_H = 0.6$, $p_T = 0.4$.
- Some data:

T, T, H, H, H, H, T, H, H, H

- The likelihood:

$$\begin{aligned} P(x_1, \ldots x_{10}) &= \prod_{i=1}^{10} P(x_i) = \prod_{i=1}^{10} p_{x_i} \\ &= p_T \times p_T \times p_H \times p_H \times p_H \times \cdots \\ &= 0.6^7 \times 0.4^3 = 0.00179 \end{aligned}$$

Computing the Likelihood of Data

- More generally:

$$\begin{aligned} P(x_1, \ldots x_N) &= \prod_{i=1}^{N} p_{x_i} \\ &= \prod_{x} p_x^{c(x)} \\ \log P(x_1, \ldots x_N) &= \sum_{x} c(x) \log p(x) \end{aligned}$$

  where $c(x)$ is *count* of outcome x.
- Likelihood only depends on counts of outcomes ...
  - Not on order of outcomes.

Estimating Parameters

- Choose parameters that maximize likelihood of data . . .
  - Because ML estimation is awesome!
- If H heads and T tails in $N = H + T$ flips, log likelihood is:

$$L(x_1^N) = \log(p_H)^H (p_T)^T = H \log p_H + T \log(1 - p_H)$$

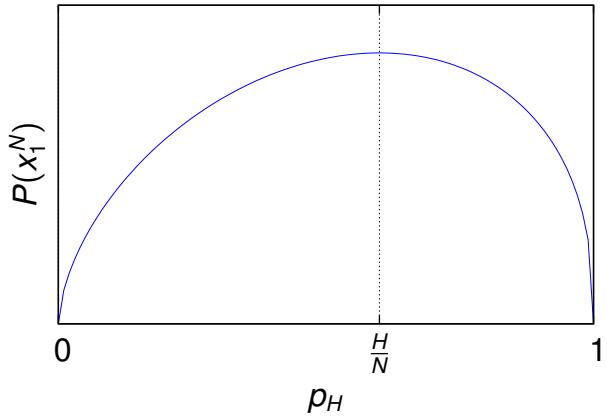
- Taking derivative w.r.t. p_H and setting to 0.

$$\frac{H}{p_H} - \frac{T}{1 - p_H} = 0 \qquad p_H = \frac{H}{H + T} = \frac{H}{N}$$

$$H - H \times p_H = T \times p_H \qquad p_T = 1 - p_H = \frac{T}{N}$$

Maximum Likelihood Estimation

- MLE of multinomial parameters is intuitive estimate!
  - Just relative frequencies: $p_H = \frac{H}{N}$, $p_T = \frac{T}{N}$.
  - Count and normalize, baby!



Example: Maximum Likelihood Estimation

- Training data: 50 samples.

  T, T, H, H, H, H, T, H, H, H, T, H, H, T, H, H, T, T, T, T, H,
  T, T, H, H, H, H, H, T, T, H, T, H, T, H, H, T, H, T, H, H, H,
  T, H, H, T, H, H, H, T

- Counts: 30 heads, 20 tails.

$$p_H^{\text{MLE}} = \frac{30}{50} = 0.6 \qquad p_T^{\text{MLE}} = \frac{20}{50} = 0.4$$

- Sample from MLE distribution:

  H, H, T, T, H, H, H, T, T, T, H, H, H, H, T, H, T, H, T, H, T,
  T, H, T, H, H, T, H, T, T, H, T, H, T, H, H, T, H, H, H, H, T,
  H, T, H, T, T, H, H, H

Recap: Multinomials, No State

- Log likelihood just depends on counts.

$$L(x_1^N) = \sum_x c(x) \log p_x$$

- MLE: count and normalize.

$$p_x^{\text{MLE}} = \frac{c(x)}{N}$$

- Easy peasy.

Where Are We?

Case Study: Austin Weather

- From National Climate Data Center.
  - R = rainy = precipitation $>$ 0.00 in.
  - W = windy = not rainy; avg. wind $\geq$ 10 mph.
  - C = calm = not rainy and not windy.
- Some data:

  W, W, C, C, W, W, C, R, C, R, W, C, C, C, R, R, R, R, C, C, R, R, R, R, R, R, R, R, C, C, C, C, C, R, R, R, R, R, R, R, C, C, C, W, C, C, C, C, C, C, R, C, C, C, C

- Does system have state/memory?
  - Does yesterday's outcome influence today's?

State and the Markov Property

- How *much* state to remember?
  - How much past information to encode in state?
- Independent events/no memory: remember nothing.

$$P(x_1, \ldots, x_N) \stackrel{?}{=} \prod_{i=1}^{N} P(x_i)$$

- General case: remember everything (always holds).

$$P(x_1, \ldots, x_N) = \prod_{i=1}^{N} P(x_i | x_1, \ldots, x_{i-1})$$

- Something in between?

The Markov Property, Order n

- Holds if:

$$\begin{aligned} P(x_1, \ldots, x_N) &= \prod_{i=1}^{N} P(x_i | x_1, \ldots, x_{i-1}) \\ &= \prod_{i=1}^{N} P(x_i | x_{i-n}, x_{i-n+1}, \cdots, x_{i-1}) \end{aligned}$$

- *e.g.*, if know weather for past n days ...
  - Knowing more doesn't help predict future weather.
- *i.e.*, if data satisfies this property ...
  - No loss from just remembering past n items!

A Non-Hidden Markov Model, Order 1

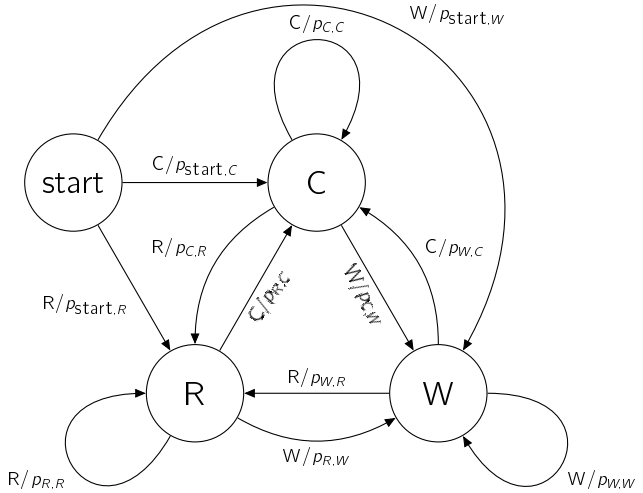
- Let's assume: knowing yesterday's weather is enough.

$$P(x_1, \ldots, x_N) = \prod_{i=1}^{N} P(x_i | x_{i-1})$$

- Before (no state): single multinomial $P(x_i)$.
- After (with state): separate multinomial $P(x_i|x_{i-1})$ ...
  - For each $x_{i-1} \in \{\text{rainy}, \text{windy}, \text{calm}\}$.
  - Model $P(x_i|x_{i-1})$ with parameter p_{x_{i-1},x_i}.
- What about $P(x_1|x_0)$?
  - Assume $x_0 =$ start, a special value.
  - One more multinomial: $P(x_i|\text{start})$.
- Constraint: $\sum_{x_i} p_{x_{i-1},x_i} = 1$ for all x_{i-1}.

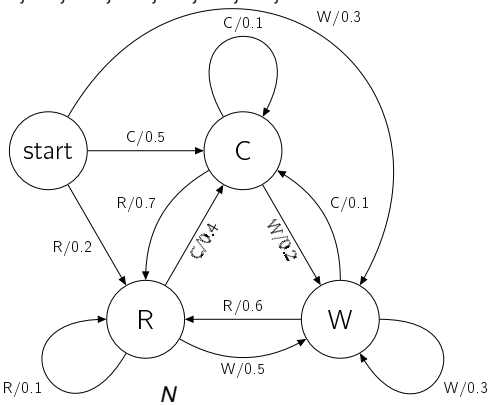
A Picture

- After observe x, go to state labeled x.
- Is state non-hidden?



Computing the Likelihood of Data

- Some data: **x** = W, W, C, C, W, W, C, R, C, R.



- The likelihood:

$$P(x_1, \ldots, x_{10}) = \prod_{i=1}^{N} P(x_i|x_{i-1}) = \prod_{i=1}^{N} p_{x_{i-1},x_i}$$
$$= p_{\text{start,W}} \times p_{\text{W,W}} \times p_{\text{W,C}} \times \ldots$$
$$= 0.3 \times 0.3 \times 0.1 \times 0.1 \times \ldots = 1.06 \times 10^{-6}$$

Computing the Likelihood of Data

- More generally:

$$P(x_1, \ldots, x_N) = \prod_{i=1}^{N} P(x_i|x_{i-1}) = \prod_{i=1}^{N} p_{x_{i-1},x_i}$$
$$= \prod_{x_{i-1},x_i} p_{x_{i-1},x_i}^{c(x_{i-1},x_i)}$$
$$\log P(x_1, \ldots x_N) = \sum_{x_{i-1},x_i} c(x_{i-1}, x_i) \log p_{x_{i-1},x_i}$$

  - $x_0 =$ *start*.
  - $c(x_{i-1}, x_i)$ is count of x_i following x_{i-1}.
- Likelihood only depends on counts of pairs (bigrams).

Maximum Likelihood Estimation

- Choose p_{x_{i-1},x_i} to optimize log likelihood:

$$\begin{aligned} L(x_1^N) &= \sum_{x_{i-1},x_i} c(x_{i-1},x_i) \log p_{x_{i-1},x_i} \\ &= \sum_{x_i} c(\text{start},x_i) \log p_{\text{start},x_i} + \sum_{x_i} c(\text{R},x_i) \log p_{\text{R},x_i} + \\ &\quad \sum_{x_i} c(\text{W},x_i) \log p_{\text{W},x_i} + \sum_{x_i} c(\text{C},x_i) \log p_{\text{C},x_i} \end{aligned}$$

- Each sum is log likelihood of multinomial.
  - Each multinomial has nonoverlapping parameter set.
- Can optimize each sum independently!

$$p^{\text{MLE}}_{x_{i-1},x_i} = \frac{c(x_{i-1},x_i)}{\sum_x c(x_{i-1},x)}$$

Example: Maximum Likelihood Estimation

- Some raw data:

  W, W, C, C, W, W, C, R, C, R, W, C, C, C, R, R, R, R, C, C, R, R, R, R, R, R, R, R, C, C, C, C, C, R, R, R, R, R, R, C, C, C, W, C, C, C, C, C, C, R, C, C, C, C

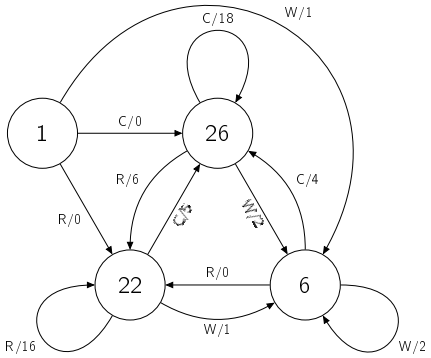
- Counts and ML estimates:

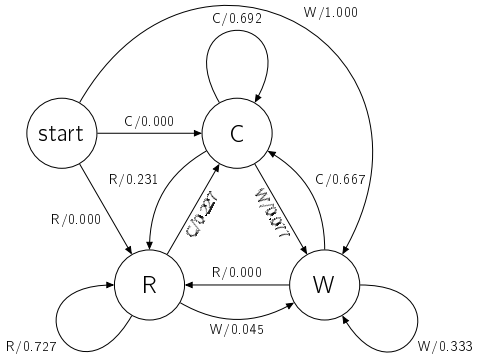
$c(\cdot,\cdot)$	R	W	C	sum
start	0	1	0	1
R	16	1	5	22
W	0	2	4	6
C	6	2	18	26

p^{MLE}	R	W	C
start	0.000	1.000	0.000
R	0.727	0.045	0.227
W	0.000	0.333	0.667
C	0.231	0.077	0.692

$$p^{\text{MLE}}_{x_{i-1},x_i} = \frac{c(x_{i-1},x_i)}{\sum_x c(x_{i-1},x)} \qquad p^{\text{MLE}}_{R,C} = \frac{5}{16+1+5} = 0.227$$

Example: Maximum Likelihood Estimation





Example: Maximum Likelihood Estimation

- Some raw data:

  W, W, C, C, W, W, C, R, C, R, W, C, C, C, R, R, R, R, C, C, R, R, R, R, R, R, R, R, C, C, C, C, C, R, R, R, R, R, R, C, C, C, W, C, C, C, C, C, C, R, C, C, C, C

- Data sampled from MLE Markov model, order 1:

  W, W, C, R, R, R, R, R, R, R, R, R, R, R, R, R, R, R, R, C, C, C, C, C, C, W, W, C, C, C, R, R, R, C, C, W, C, C, C, C, C, R, R, R, R, C, R, R, C, R, R, R, R, R

- Data sampled from MLE Markov model, order 0:

  C, R, C, R, R, R, R, C, R, R, C, C, R, C, C, R, R, R, R, C, C, C, R, C, R, W, R, C, C, C, W, C, R, C, C, W, C, C, C, C, R, R, C, C, C, R, C, R, R, C, R, C, R, W, R

Recap: Non-Hidden Markov Models

- Use *states* to encode limited amount of information . . .
  - About the past.
- Current state is known given observations.
- Log likelihood just depends on pair counts.

$$L(x_1^N) = \sum_{x_{i-1}, x_i} c(x_{i-1}, x_i) \log p_{x_{i-1}, x_i}$$

- MLE: count and normalize.

$$p^{\text{MLE}}_{x_{i-1}, x_i} = \frac{c(x_{i-1}, x_i)}{\sum_x c(x_{i-1}, x)}$$

- Easy beezy.

Part II

Discrete Hidden Markov Models

Case Study: Austin Weather 2.0

- Ignore rain; one sample every two weeks:

  C, W, C, C, C, C, W, C, C, C, C, C, C, W, W, C, W, C, W,
  W, C, W, C, W, C, C, C, C, C, C, C, C, C, C, C, C, C, C,
  W, C, C, C, W, W, C, C, W, W, C, W, C, W, C, C, C, C, C,
  C, C, C, C, C, C, C, C, W, C, W, C, C, W, W, C, W, W, W,
  C, W, C, C, C, C, C, C, C, C, C, C, W, C, W, W, W, C, C,
  C, C, C, W, C, C, W, C, C, C, C, C, C, C, C, C, C, C, W

- Does system have state/memory?

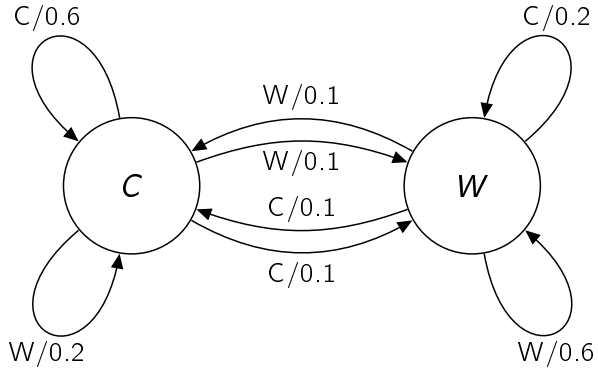
Another View

```
C W C C C C W C C C C C C W W C W C W W C W C W C C
C C C C C C C C C C C C W C C C W W C C W W C W C W
C C C C C C C C C C C C C W C W C C W W C W W W C W
C C C C C C C C C C W C W W W C C C C W C C W C C
C C C C C C C C C W C C W W C W C C C W C W C W C C
C C C C C C W C C C C C W C C C W C W C W C C W C W
C C C C C C C C C C C C C W C C C W W C C C W C W C
```

- Does system have memory?
- How many states?

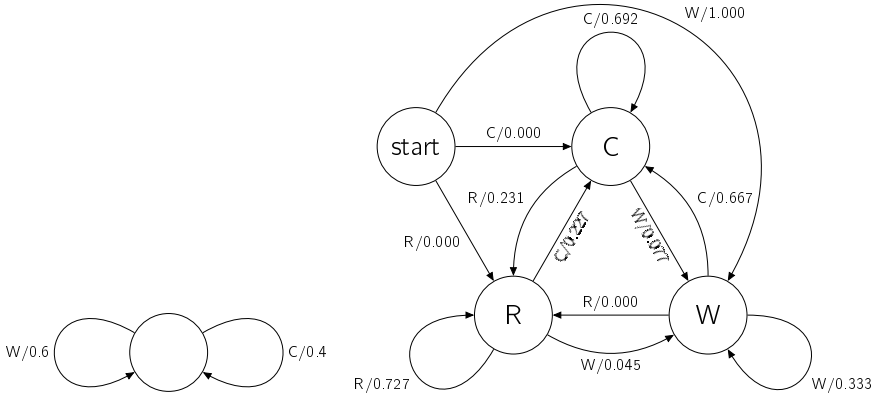
A Hidden Markov Model

- For simplicity, no separate *start* state.
  - Always start in calm state *c*.



- Why is state "hidden"?
  - What are conditions for state to be non-hidden?

Contrast: Non-Hidden Markov Models

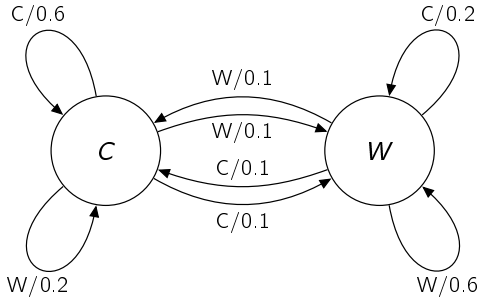


Why Hidden State?

- No "simple" way to determine state given observed.
  - If see "*W*", doesn't mean windy season started.
- Speech recognition: one HMM per word.
  - Each state represents different sound in word.
  - How to tell from observed when state switches?
- Hidden models can model same stuff as non-hidden . . .
  - Using much fewer states.
- Pop quiz: name a hidden model with no memory.

The Problem With Hidden State

- For observed $\mathbf{x} = x_1, \ldots, x_N$, what is hidden state $\mathbf{h}$?
  - Corresponding state sequence $\mathbf{h} = h_1, \ldots, h_{N+1}$.
- In non-hidden model, how many $\mathbf{h}$ possible given $\mathbf{x}$?
- In hidden model, what $\mathbf{h}$ are possible given $\mathbf{x}$?



- This makes everything difficult.

Three Key Tasks for HMM's

1. Find single best path in HMM given observed **x**.
   - *e.g.*, when did windy season begin?
   - *e.g.*, when did each sound in word begin?
2. Find total likelihood $P(\mathbf{x})$ of observed.
   - *e.g.*, to pick which word assigns highest likelihood.
3. Find ML estimates for parameters of HMM.
   - *i.e.*, estimate arc probabilities to match training data.

Where Are We?

What We Want to Compute

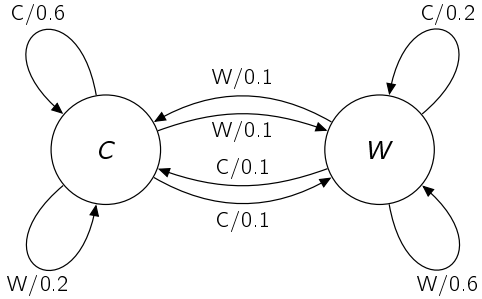
- Given observed, *e.g.*, **x** = C, W, C, C, W, ...
  - Find state sequence $\mathbf{h}^*$ with highest likelihood.

$$\mathbf{h}^* = \arg\max_{\mathbf{h}} P(\mathbf{h}, \mathbf{x})$$

- Why is this easy for non-hidden model?
- Given state sequence **h**, how to compute $P(\mathbf{h}, \mathbf{x})$?
  - Same as for non-hidden model.
  - Multiply all arc probabilities along path.

Likelihood of Single State Sequence

- Some data: **x** = W, C, C.
- A state sequence: $\mathbf{h} = c, c, c, w$.



- Likelihood of path:

$$P(\mathbf{h}, \mathbf{x}) = 0.2 \times 0.6 \times 0.1 = 0.012$$

What We Want to Compute

- Given observed, *e.g.*, **x** = C, W, C, C, W, ...
  - Find state sequence $\mathbf{h}^*$ with highest likelihood.

$$\mathbf{h}^* = \arg\max_{\mathbf{h}} P(\mathbf{h}, \mathbf{x})$$

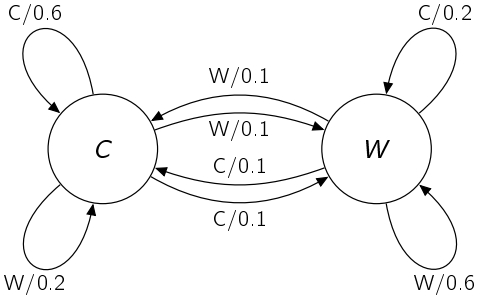
- Let's start with simpler problem:
  - Find likelihood of best state sequence $P_{\text{best}}(\mathbf{x})$.
  - Worry about identity of best sequence later.

$$P_{\text{best}}(\mathbf{x}) = \max_{\mathbf{h}} P(\mathbf{h}, \mathbf{x})$$

What's the Problem?

$$P_{\text{best}}(\mathbf{x}) = \max_{\mathbf{h}} P(\mathbf{h}, \mathbf{x})$$

- For observation sequence of length N ...
  - How many different possible state sequences **h**?



- How in blazes can we do max ...
  - Over exponential number of state sequences?

Dynamic Programming

- Let S_0 be start state; *e.g.*, the calm season *c*.
- Let $\mathcal{P}(S, t)$ be set of paths of length t ...
  - Starting at start state S_0 and ending at S ...
  - Consistent with observed $x_1, \ldots, x_t$.
- Any path $p \in \mathcal{P}(S, t)$ must be composed of ...
  - Path of length $t-1$ to predecessor state $S' \rightarrow S$ ...
  - Followed by arc from S' to S labeled with x_t.
  - This decomposition is unique.

$$\mathcal{P}(S, t) = \bigcup_{S' \xrightarrow{x_t} S} \mathcal{P}(S', t-1) \cdot (S' \xrightarrow{x_t} S)$$

Dynamic Programming

$$\mathcal{P}(S, t) = \bigcup_{S' \xrightarrow{x_t} S} \mathcal{P}(S', t-1) \cdot (S' \xrightarrow{x_t} S)$$

- Let $\hat{\alpha}(S, t)$ = likelihood of best path of length t ...
  - Starting at start state S_0 and ending at S.
  - $P(p)$ = prob of path p = product of arc probs.

$$\begin{aligned}
\hat{\alpha}(S, t) &= \max_{p \in \mathcal{P}(S,t)} P(p) \\
&= \max_{p' \in \mathcal{P}(S',t-1), S' \xrightarrow{x_t} S} P(p' \cdot (S' \xrightarrow{x_t} S)) \\
&= \max_{S' \xrightarrow{x_t} S} P(S' \xrightarrow{x_t} S) \max_{p' \in \mathcal{P}(S',t-1)} P(p') \\
&= \max_{S' \xrightarrow{x_t} S} P(S' \xrightarrow{x_t} S) \times \hat{\alpha}(S', t-1)
\end{aligned}$$

What Were We Computing Again?

- Assume observed **x** of length T.
  - Want likelihood of best path of length T ...
  - Starting at start state S_0 and ending anywhere.

$$P_{\text{best}}(\mathbf{x}) = \max_{\mathbf{h}} P(\mathbf{h}, \mathbf{x}) = \max_{S} \hat{\alpha}(S, T)$$

- If can compute $\hat{\alpha}(S, T)$, we are done.
- If know $\hat{\alpha}(S, t-1)$ for all S, easy to compute $\hat{\alpha}(S, t)$:

$$\hat{\alpha}(S, t) = \max_{S' \xrightarrow{x_t} S} P(S' \xrightarrow{x_t} S) \times \hat{\alpha}(S', t-1)$$

- This looks promising ...

The Viterbi Algorithm

- $\hat{\alpha}(S, 0) = 1$ for $S = S_0$, 0 otherwise.
- For $t = 1, \ldots, T$:
  - For each state S:

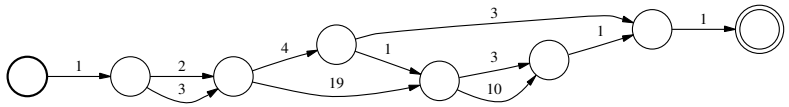
$$\hat{\alpha}(S, t) = \max_{S' \xrightarrow{x_t} S} P(S' \xrightarrow{x_t} S) \times \hat{\alpha}(S', t-1)$$

- The end.

$$P_{\text{best}}(\mathbf{x}) = \max_{\mathbf{h}} P(\mathbf{h}, \mathbf{x}) = \max_{S} \hat{\alpha}(S, T)$$

Viterbi and Shortest Path

- Equivalent to shortest path problem.



- One "state" for each state/time pair (S, t).
- Iterate through "states" in topological order:
  - All arcs go forward in time.
  - If order "states" by time, valid ordering.

$$d(S) = \min_{S' \to S} \{d(S') + \text{distance}(S', S)\}$$

$$\hat{\alpha}(S, t) = \max_{S' \xrightarrow{x_t} S} P(S' \xrightarrow{x_t} S) \times \hat{\alpha}(S', t-1)$$

Identifying the Best Path

- Wait! We can calc likelihood of best path:

$$P_{\text{best}}(\mathbf{x}) = \max_{\mathbf{h}} P(\mathbf{h}, \mathbf{x})$$

- What we really wanted: *identity* of best path.
  - *i.e.*, the best state sequence **h**.
- Basic idea: for each S, t ...
  - Record identity $S_{\text{prev}}(S, t)$ of previous state S' ...
  - In best path of length t ending at state S.
- Find best final state.
  - Backtrace best previous states until reach start state.

The Viterbi Algorithm With Backtrace

- $\hat{\alpha}(S, 0) = 1$ for $S = S_0$, 0 otherwise.
- For $t = 1, \ldots, T$:
  - For each state S:

$$\hat{\alpha}(S, t) = \max_{S' \xrightarrow{x_t} S} P(S' \xrightarrow{x_t} S) \times \hat{\alpha}(S', t-1)$$

$$S_{\text{prev}}(S, t) = \arg\max_{S' \xrightarrow{x_t} S} P(S' \xrightarrow{x_t} S) \times \hat{\alpha}(S', t-1)$$

- The end.

$$P_{\text{best}}(\mathbf{x}) = \max_S \hat{\alpha}(S, T)$$

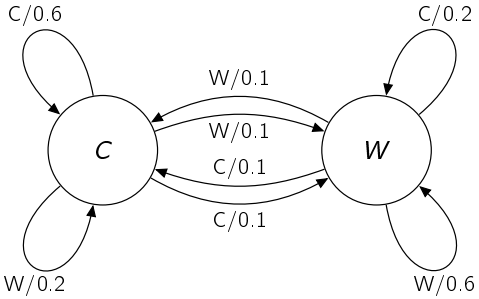
$$S_{\text{final}}(\mathbf{x}) = \arg\max_S \hat{\alpha}(S, T)$$

The Backtrace

- $S_{\text{cur}} \leftarrow S_{\text{final}}(\mathbf{x})$
- for t in $T, \ldots, 1$:
  - $S_{\text{cur}} \leftarrow S_{\text{prev}}(S_{\text{cur}}, t)$
- The best state sequence is ...
  - List of states traversed in reverse order.

Example

- Some data: C, C, W, W.



$\hat{\alpha}$	0	1	2	3	4
c	1.000	0.600	0.360	0.072	0.014
w	0.000	0.100	0.060	0.036	**0.022**

$$\begin{aligned}\hat{\alpha}(c, 2) &= \max\{P(c \xrightarrow{C} c) \times \hat{\alpha}(c, 1), P(w \xrightarrow{C} c) \times \hat{\alpha}(w, 1)\} \\ &= \max\{0.6 \times 0.6, 0.1 \times 0.1\} = 0.36\end{aligned}$$

Example: The Backtrace

S_{prev}	0	1	2	3	4
c		**c**	**c**	c	c
w		c	c	**c**	**w**

$$\mathbf{h}^* = \arg\max_{\mathbf{h}} P(\mathbf{h}, \mathbf{x}) = (c, c, c, w, w)$$

- The data: C, C, W, W.
- Calm season switching to windy season.

Recap: The Viterbi Algorithm

- Given observed **x**, ...
  - Exponential number of hidden sequences **h**.
- Can find likelihood and identity of best path ...
  - Efficiently using dynamic programming.
- What is time complexity?

Where Are We?

What We Want to Compute

- Given observed, *e.g.*, **x** = C, W, C, C, W, ...
  - Find total likelihood $P(\mathbf{x})$.
- Need to sum likelihood over all hidden sequences:

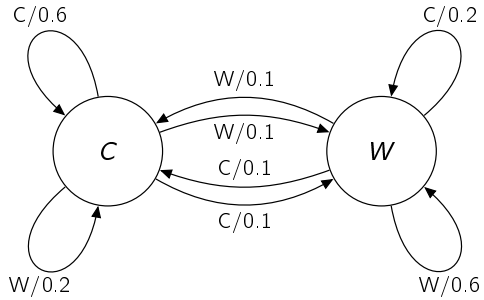
$$P(\mathbf{x}) = \sum_{\mathbf{h}} P(\mathbf{h}, \mathbf{x})$$

- Given state sequence **h**, how to compute $P(\mathbf{h}, \mathbf{x})$?
  - Multiply all arc probabilities along path.
- Why is this sum easy for non-hidden model?

What's the Problem?

$$P(\mathbf{x}) = \sum_{\mathbf{h}} P(\mathbf{h}, \mathbf{x})$$

- For observation sequence of length N ...
  - How many different possible state sequences **h**?



- How in blazes can we do sum ...
  - Over exponential number of state sequences?

Dynamic Programming

- Let $\mathcal{P}(S,t)$ be set of paths of length t ...
  - Starting at start state S_0 and ending at S ...
  - Consistent with observed $x_1, \ldots, x_t$.
- Any path $p \in \mathcal{P}(S,t)$ must be composed of ...
  - Path of length $t-1$ to predecessor state $S' \rightarrow S$ ...
  - Followed by arc from S' to S labeled with x_t.

$$\mathcal{P}(S,t) = \bigcup_{S' \stackrel{x_t}{\rightarrow} S} \mathcal{P}(S', t-1) \cdot (S' \stackrel{x_t}{\rightarrow} S)$$

Dynamic Programming

$$\mathcal{P}(S,t) = \bigcup_{S' \stackrel{x_t}{\rightarrow} S} \mathcal{P}(S', t-1) \cdot (S' \stackrel{x_t}{\rightarrow} S)$$

- Let $\alpha(S,t)$ = sum of likelihoods of paths of length t ...
  - Starting at start state S_0 and ending at S.

$$\begin{aligned}
\alpha(S,t) &= \sum_{p \in \mathcal{P}(S,t)} P(p) \\
&= \sum_{p' \in \mathcal{P}(S',t-1), S' \stackrel{x_t}{\rightarrow} S} P(p' \cdot (S' \stackrel{x_t}{\rightarrow} S)) \\
&= \sum_{S' \stackrel{x_t}{\rightarrow} S} P(S' \stackrel{x_t}{\rightarrow} S) \sum_{p' \in \mathcal{P}(S',t-1)} P(p') \\
&= \sum_{S' \stackrel{x_t}{\rightarrow} S} P(S' \stackrel{x_t}{\rightarrow} S) \times \alpha(S', t-1)
\end{aligned}$$

What Were We Computing Again?

- Assume observed $\mathbf{x}$ of length T.
  - Want sum of likelihoods of paths of length T ...
  - Starting at start state S_0 and ending anywhere.

$$P(\mathbf{x}) = \sum_{\mathbf{h}} P(\mathbf{h}, \mathbf{x}) = \sum_{S} \alpha(S,T)$$

- If can compute $\alpha(S,T)$, we are done.
- If know $\alpha(S,t-1)$ for all S, easy to compute $\alpha(S,t)$:

$$\alpha(S,t) = \sum_{S' \stackrel{x_t}{\rightarrow} S} P(S' \stackrel{x_t}{\rightarrow} S) \times \alpha(S', t-1)$$

- This looks promising ...

The Forward Algorithm

- $\alpha(S,0) = 1$ for $S = S_0$, 0 otherwise.
- For $t = 1, \ldots, T$:
  - For each state S:

$$\alpha(S,t) = \sum_{S' \stackrel{x_t}{\rightarrow} S} P(S' \stackrel{x_t}{\rightarrow} S) \times \alpha(S', t-1)$$

- The end.

$$P(\mathbf{x}) = \sum_{\mathbf{h}} P(\mathbf{h}, \mathbf{x}) = \sum_{S} \alpha(S,T)$$

Viterbi *vs.* Forward

- The goal:

$$P_{\text{best}}(\mathbf{x}) = \max_{\mathbf{h}} P(\mathbf{h}, \mathbf{x}) = \max_{S} \hat{\alpha}(S, T)$$

$$P(\mathbf{x}) = \sum_{\mathbf{h}} P(\mathbf{h}, \mathbf{x}) = \sum_{S} \alpha(S, T)$$

- The invariant.

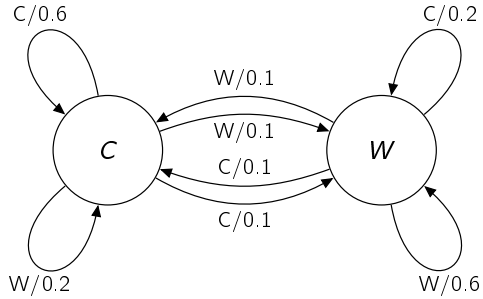
$$\hat{\alpha}(S, t) = \max_{S' \xrightarrow{x_t} S} P(S' \xrightarrow{x_t} S) \times \hat{\alpha}(S', t-1)$$

$$\alpha(S, t) = \sum_{S' \xrightarrow{x_t} S} P(S' \xrightarrow{x_t} S) \times \alpha(S', t-1)$$

- Just replace all max's with sums (any *semiring* will do).

Example

- Some data: C, C, W, W.



α	0	1	2	3	4
c	1.000	0.600	0.370	0.082	**0.025**
w	0.000	0.100	0.080	0.085	**0.059**

$$\alpha(c, 2) = P(c \xrightarrow{C} c) \times \alpha(c, 1) + P(w \xrightarrow{C} c) \times \alpha(w, 1)$$
$$= 0.6 \times 0.6 + 0.1 \times 0.1 = 0.37$$

Recap: The Forward Algorithm

- Can find total likelihood $P(\mathbf{x})$ of observed ...
  - Using very similar algorithm to Viterbi algorithm.
- Just replace max's with sums.
- Same time complexity.

Where Are We?

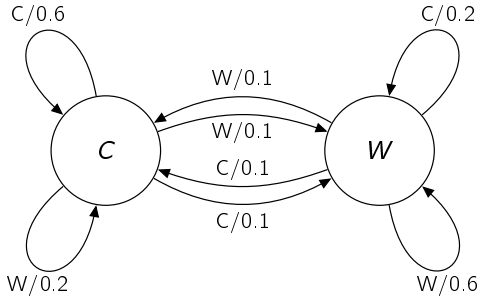
Training the Parameters of an HMM

- Given training data $\mathbf{x}$ ...
- Estimate parameters of model ...
  - To maximize likelihood of training data.

$$P(\mathbf{x}) = \sum_{\mathbf{h}} P(\mathbf{h}, \mathbf{x})$$

What Are The Parameters?

- One parameter for each arc:



- Identify arc by source S, destination S', and label x: $p_{S \xrightarrow{x} S'}$.
- Probs of arcs leaving same state must sum to 1:

$$\sum_{x,S'} p_{S \xrightarrow{x} S'} = 1 \quad \text{for all } S$$

What Did We Do For Non-Hidden Again?

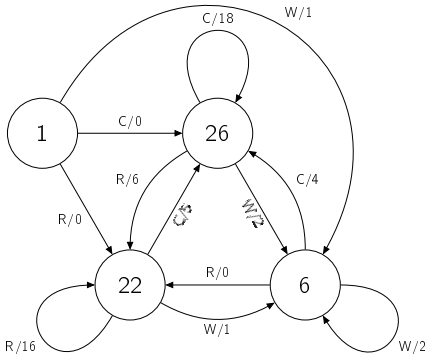
- Likelihood of single path: product of arc probabilities.
- Log likelihood can be written as:

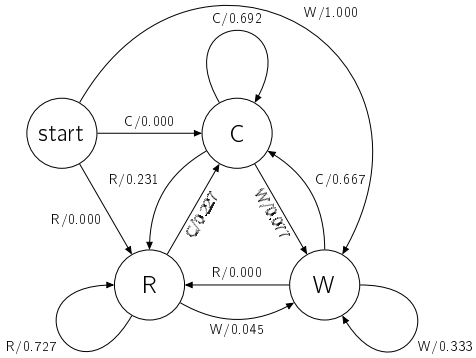
$$L(x_1^N) = \sum_{S \xrightarrow{x} S'} c(S \xrightarrow{x} S') \log p_{S \xrightarrow{x} S'}$$

  - Just depends on counts $c(S \xrightarrow{x} S')$ of each arc.
- Each source state corresponds to multinomial ...
  - With nonoverlapping parameters.
- ML estimation for multinomials: count and normalize!

$$p^{\text{MLE}}_{S \xrightarrow{x} S'} = \frac{c(S \xrightarrow{x} S')}{\sum_{x,S'} c(S \xrightarrow{x} S')}$$

Example: Non-Hidden Estimation





How Do We Train Hidden Models?

- Hmmm, I know this one . . .

Review: The EM Algorithm

- General way to train parameters in hidden models . . .
  - To optimize likelihood.
- Guaranteed to improve likelihood in each iteration.
  - Only finds local optimum.
  - Seeding matters.

The EM Algorithm

- Initialize parameter values somehow.
- For each iteration . . .
- Expectation step: compute posterior (count) of each $\mathbf{h}$.

$$\tilde{P}(\mathbf{h}|\mathbf{x}) = \frac{P(\mathbf{h}, \mathbf{x})}{\sum_{\mathbf{h}} P(\mathbf{h}, \mathbf{x})}$$

- Maximization step: update parameters.
  - Instead of data $\mathbf{x}$ with unknown $\mathbf{h}$, pretend . . .
  - *Non-hidden* data where . . .
  - (Fractional) count of each $(\mathbf{h}, \mathbf{x})$ is $\tilde{P}(\mathbf{h}|\mathbf{x})$.

Applying EM to HMM's: The E Step

- Compute posterior (count) of each $\mathbf{h}$.

$$\tilde{P}(\mathbf{h}|\mathbf{x}) = \frac{P(\mathbf{h}, \mathbf{x})}{\sum_{\mathbf{h}} P(\mathbf{h}, \mathbf{x})}$$

- How to compute prob of single path $P(\mathbf{h}, \mathbf{x})$?
  - Multiply arc probabilities along path.
- How to compute denominator?
  - This is just total likelihood of observed $P(\mathbf{x})$.

$$P(\mathbf{x}) = \sum_{\mathbf{h}} P(\mathbf{h}, \mathbf{x})$$

  - This looks vaguely familiar.

Applying EM to HMM's: The M Step

- Non-hidden case: single path $\mathbf{h}$ with count 1.
  - Total count of arc is count of arc in $\mathbf{h}$:
$$c(S \xrightarrow{x} S') = c_{\mathbf{h}}(S \xrightarrow{x} S')$$
  - Normalize.
$$p^{\text{MLE}}_{S \xrightarrow{x} S'} = \frac{c(S \xrightarrow{x} S')}{\sum_{x,S'} c(S \xrightarrow{x} S')}$$
- Hidden case: every path $\mathbf{h}$ has count $\tilde{P}(\mathbf{h}|\mathbf{x})$.
  - Total count of arc is weighted sum ...
  - Of count of arc in each $\mathbf{h}$.
$$c(S \xrightarrow{x} S') = \sum_{\mathbf{h}} \tilde{P}(\mathbf{h}|\mathbf{x}) c_{\mathbf{h}}(S \xrightarrow{x} S')$$
  - Normalize as before.

What's the Problem?

- Need to sum over exponential number of h:
$$c(S \xrightarrow{x} S') = \sum_{\mathbf{h}} \tilde{P}(\mathbf{h}|\mathbf{x}) c_{\mathbf{h}}(S \xrightarrow{x} S')$$
- If only we had an algorithm for doing this type of thing.

The Game Plan

- Decompose sum by time (*i.e.*, position in $\mathbf{x}$).
  - Find count of each arc at each "time" t.
$$c(S \xrightarrow{x} S') = \sum_{t=1}^{T} c(S \xrightarrow{x} S', t) = \sum_{t=1}^{T} \sum_{\mathbf{h} \in \mathcal{P}(S \xrightarrow{x} S', t)} \tilde{P}(\mathbf{h}|\mathbf{x})$$
- $\mathcal{P}(S \xrightarrow{x} S', t)$ are paths where arc at time t is $S \xrightarrow{x} S'$.
- $\mathcal{P}(S \xrightarrow{x} S', t)$ is empty if $x \neq x_t$.
- Otherwise, use dynamic programming to compute
$$c(S \xrightarrow{x_t} S', t) \equiv \sum_{\mathbf{h} \in \mathcal{P}(S \xrightarrow{x_t} S', t)} \tilde{P}(\mathbf{h}|\mathbf{x})$$

Let's Rearrange Some

- Recall we can compute $P(\mathbf{x})$ using Forward algorithm:
$$\tilde{P}(\mathbf{h}|\mathbf{x}) = \frac{P(\mathbf{h}, \mathbf{x})}{P(\mathbf{x})}$$
- Some paraphrasing:
$$\begin{aligned} c(S \xrightarrow{x_t} S', t) &= \sum_{\mathbf{h} \in \mathcal{P}(S \xrightarrow{x_t} S', t)} \tilde{P}(\mathbf{h}|\mathbf{x}) \\ &= \frac{1}{P(\mathbf{x})} \sum_{\mathbf{h} \in \mathcal{P}(S \xrightarrow{x_t} S', t)} P(\mathbf{h}, \mathbf{x}) \\ &= \frac{1}{P(\mathbf{x})} \sum_{p \in \mathcal{P}(S \xrightarrow{x_t} S', t)} P(p) \end{aligned}$$

What We Need

- Goal: sum over all paths $p \in \mathcal{P}(S \xrightarrow{x_t} S', t)$.
  - Arc at time t is $S \xrightarrow{x_t} S'$.
- Let $\mathcal{P}_i(S, t)$ be set of (initial) paths of length t ...
  - Starting at start state S_0 and ending at S ...
  - Consistent with observed $x_1, \ldots, x_t$.
- Let $\mathcal{P}_f(S, t)$ be set of (final) paths of length $T - t$ ...
  - Starting at state S and ending at any state ...
  - Consistent with observed $x_{t+1}, \ldots, x_T$.
- Then:

$$\mathcal{P}(S \xrightarrow{x_t} S', t) = \mathcal{P}_i(S, t-1) \cdot (S \xrightarrow{x_t} S') \cdot \mathcal{P}_f(S', t)$$

Translating Path Sets to Probabilities

$$\mathcal{P}(S \xrightarrow{x_t} S', t) = \mathcal{P}_i(S, t-1) \cdot (S \xrightarrow{x_t} S') \cdot \mathcal{P}_f(S', t)$$

- Let $\alpha(S, t)$ = sum of likelihoods of paths of length t ...
  - Starting at start state S_0 and ending at S.
- Let $\beta(S, t)$ = sum of likelihoods of paths of length $T - t$ ...
  - Starting at state S and ending at any state.

$$\begin{aligned} c(S \xrightarrow{x_t} S', t) &= \frac{1}{P(\mathbf{x})} \sum_{p \in \mathcal{P}(S \xrightarrow{x_t} S', t)} P(p) \\ &= \frac{1}{P(\mathbf{x})} \sum_{p_i \in \mathcal{P}_i(S, t-1), p_f \in \mathcal{P}_f(S', t)} P(p_i \cdot (S \xrightarrow{x_t} S') \cdot p_f) \\ &= \frac{1}{P(\mathbf{x})} \times p_{S \xrightarrow{x_t} S'} \sum_{p_i \in \mathcal{P}_i(S, t-1)} P(p_i) \sum_{p_f \in \mathcal{P}_f(S', t)} P(p_f) \\ &= \frac{1}{P(\mathbf{x})} \times p_{S \xrightarrow{x_t} S'} \times \alpha(S, t-1) \times \beta(S', t) \end{aligned}$$

Mini-Recap

- To do ML estimation in M step ...
  - Need count of each arc: $c(S \xrightarrow{x} S')$.
- Decompose count of arc by time:

$$c(S \xrightarrow{x} S') = \sum_{t=1}^{T} c(S \xrightarrow{x} S', t)$$

- Can compute count at time efficiently ...
  - If have *forward* probabilities $\alpha(S, t)$ ...
  - And *backward* probabilities $\beta(S, T)$.

$$c(S \xrightarrow{x_t} S', t) = \frac{1}{P(\mathbf{x})} \times p_{S \xrightarrow{x_t} S'} \times \alpha(S, t-1) \times \beta(S', t)$$

The Forward-Backward Algorithm (1 iter)

- Apply Forward algorithm to compute $\alpha(S, t)$, $P(\mathbf{x})$.
- Apply Backward algorithm to compute $\beta(S, t)$.
- For each arc $S \xrightarrow{x_t} S'$ and time t ...
  - Compute posterior count of arc at time t if $x = x_t$.

$$c(S \xrightarrow{x_t} S', t) = \frac{1}{P(\mathbf{x})} \times p_{S \xrightarrow{x_t} S'} \times \alpha(S, t-1) \times \beta(S', t)$$

- Sum to get total counts for each arc.

$$c(S \xrightarrow{x} S') = \sum_{t=1}^{T} c(S \xrightarrow{x} S', t)$$

- For each arc, find ML estimate of parameter:

$$p^{\text{MLE}}_{S \xrightarrow{x} S'} = \frac{c(S \xrightarrow{x} S')}{\sum_{x, S'} c(S \xrightarrow{x} S')}$$

The Forward Algorithm

- $\alpha(S, 0) = 1$ for $S = S_0$, 0 otherwise.
- For $t = 1, \ldots, T$:
  - For each state S:

$$\alpha(S, t) = \sum_{S' \xrightarrow{x_t} S} p_{S' \xrightarrow{x_t} S} \times \alpha(S', t-1)$$

- The end.

$$P(\mathbf{x}) = \sum_{\mathbf{h}} P(\mathbf{h}, \mathbf{x}) = \sum_{S} \alpha(S, T)$$

The Backward Algorithm

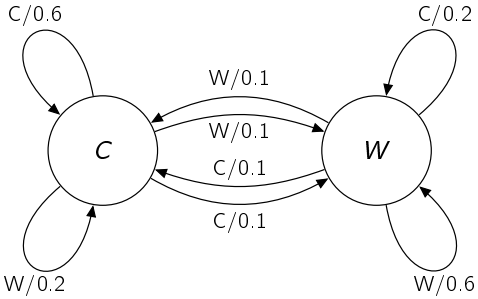
- $\beta(S, T) = 1$ for all S.
- For $t = T-1, \ldots, 0$:
  - For each state S:

$$\beta(S, t) = \sum_{S \xrightarrow{x_{t+1}} S'} p_{S \xrightarrow{x_{t+1}} S'} \times \beta(S', t+1)$$

- Pop quiz: how to compute $P(\mathbf{x})$ from β's?

Example: The Forward Pass

- Some data: C, C, W, W.

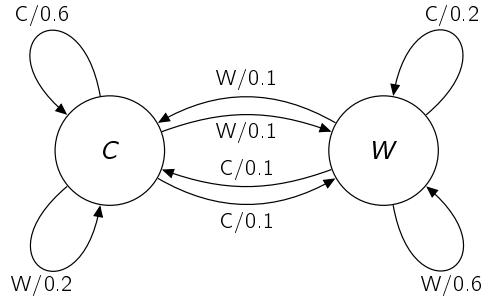


α	0	1	2	3	4
c	1.000	0.600	0.370	0.082	**0.025**
w	0.000	0.100	0.080	0.085	**0.059**

$$\alpha(c, 2) = p_{c \xrightarrow{c} c} \times \alpha(c, 1) + p_{w \xrightarrow{c} c} \times \alpha(w, 1)$$
$$= 0.6 \times 0.6 + 0.1 \times 0.1 = 0.37$$

The Backward Pass

- The data: C, C, W, W.



β	0	1	2	3	4
c	**0.084**	0.123	0.130	0.300	1.000
w	0.033	0.103	0.450	0.700	1.000

$$\beta(c, 2) = p_{c \xrightarrow{w} c} \times \beta(c, 3) + p_{c \xrightarrow{w} w} \times \beta(w, 3)$$
$$= 0.2 \times 0.3 + 0.1 \times 0.7 = 0.13$$

Computing Arc Posteriors

α, β	0	1	2	3	4
c	1.000	0.600	0.370	0.082	0.025
w	0.000	0.100	0.080	0.085	0.059
c	**0.084**	0.123	0.130	0.300	1.000
w	0.033	0.103	0.450	0.700	1.000

$c(S \xrightarrow{x} S', t)$	$p_{S \xrightarrow{x} S'}$	1	2	3	4
$c \xrightarrow{C} c$	0.6	0.878	0.556	0.000	0.000
$c \xrightarrow{W} c$	0.2	0.000	0.000	0.264	0.195
$c \xrightarrow{C} w$	0.1	0.122	0.321	0.000	0.000
$c \xrightarrow{W} w$	0.1	0.000	0.000	0.308	0.098
$w \xrightarrow{C} w$	0.2	0.000	0.107	0.000	0.000
$w \xrightarrow{W} w$	0.6	0.000	0.000	0.400	0.606
$w \xrightarrow{C} c$	0.1	0.000	0.015	0.000	0.000
$w \xrightarrow{W} c$	0.1	0.000	0.000	0.029	0.101

Computing Arc Posteriors

α, β	0	1	2	3	4
c	1.000	0.600	0.370	0.082	0.025
w	0.000	0.100	0.080	0.085	0.059
c	**0.084**	0.123	0.130	0.300	1.000
w	0.033	0.103	0.450	0.700	1.000

$c(S \xrightarrow{x} S', t)$	$p_{S \xrightarrow{x} S'}$	1	2	3	4
$c \xrightarrow{C} c$	0.6	0.878	0.556	0.000	0.000
$c \xrightarrow{W} c$	0.2	0.000	0.000	0.264	0.195
...	...	...	...	...	...
...	...	...	...	...	...

$$c(c \xrightarrow{C} c, 2) = \frac{1}{P(\mathbf{x})} \times p_{c \xrightarrow{C} c} \times \alpha(c, 1) \times \beta(c, 2)$$
$$= \frac{1}{0.084} \times 0.6 \times 0.600 \times 0.130 = 0.0556$$

Summing Arc Counts and Reestimation

	1	2	3	4	$c(S \xrightarrow{x} S')$	$p^{\text{MLE}}_{S \xrightarrow{x} S'}$
$c \xrightarrow{C} c$	0.878	0.556	0.000	0.000	1.434	0.523
$c \xrightarrow{W} c$	0.000	0.000	0.264	0.195	0.459	0.167
$c \xrightarrow{C} w$	0.122	0.321	0.000	0.000	0.444	0.162
$c \xrightarrow{W} w$	0.000	0.000	0.308	0.098	0.405	0.148
$w \xrightarrow{C} w$	0.000	0.107	0.000	0.000	0.107	0.085
$w \xrightarrow{W} w$	0.000	0.000	0.400	0.606	1.006	0.800
$w \xrightarrow{C} c$	0.000	0.015	0.000	0.000	0.015	0.012
$w \xrightarrow{W} c$	0.000	0.000	0.029	0.101	0.130	0.103

$$\sum_{x,S'} c(c \xrightarrow{x} S') = 2.742 \qquad \sum_{x,S'} c(w \xrightarrow{x} S') = 1.258$$

Summing Arc Counts and Reestimation

	1	2	3	4	$c(S \xrightarrow{x} S')$	$p^{\text{MLE}}_{S \xrightarrow{x} S'}$
$c \xrightarrow{C} c$	0.878	0.556	0.000	0.000	1.434	0.523
$c \xrightarrow{W} c$	0.000	0.000	0.264	0.195	0.459	0.167
...	...	...	...	...	...	...
...	...	...	...	...	...	...

$$\sum_{x,S'} c(c \xrightarrow{x} S') = 2.742 \qquad \sum_{x,S'} c(w \xrightarrow{x} S') = 1.258$$

$$c(c \xrightarrow{C} c) = \sum_{t=1}^{T} c(c \xrightarrow{C} c, t)$$
$$= 0.878 + 0.556 + 0.000 + 0.000 = 1.434$$

$$p^{\text{MLE}}_{c \xrightarrow{C} c} = \frac{c(c \xrightarrow{C} c)}{\sum_{x,S'} c(c \xrightarrow{x} S')} = \frac{1.434}{2.742} = 0.523$$

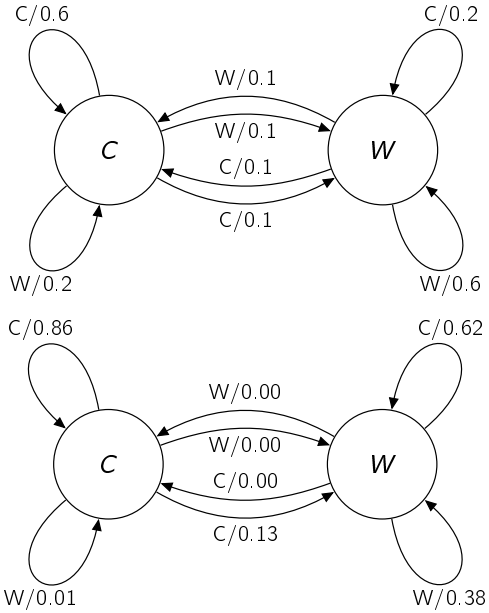
Slide for Quiet Contemplation

Another Example

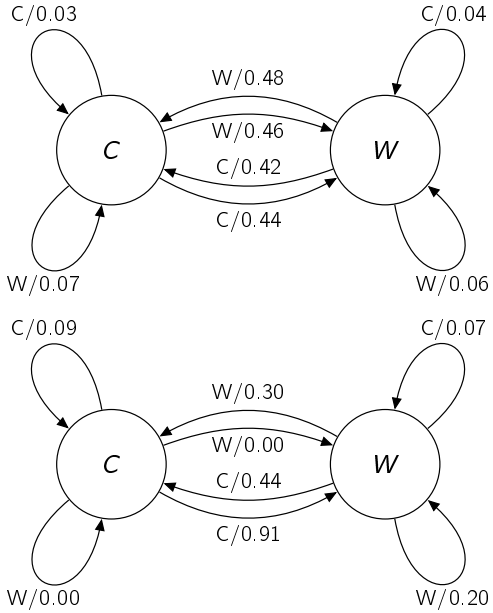
- Same initial HMM.
- Training data: instead of one sequence, many.
  - Each sequence is 26 samples $\Leftrightarrow$ 1 year.

```
C W C C C C W C C C C C C W W C W C W W C W C W C C
C C C C C C C C C C C C W C C C W W C C W W C W C W
C C C C C C C C C C C C C W C W C C W W C W W W C W
C C C C C C C C C C W C W W W C C C C W C C W C C
C C C C C C C C C W C C W W C W C C C W C W C W C C
C C C C C C W C C C C C W C C C W C W C W C C W C W
C C C C C C C C C C C C C W C C C W W C C C W C W C
```


Before and After



Another Starting Point



Recap: The Forward-Backward Algorithm

- Also called *Baum-Welch* algorithm.
- Instance of EM algorithm.
  - Uses dynamic programming to efficiently sum over ...
  - Exponential number of hidden state sequences.
  - Don't explicitly compute posterior of every **h**.
  - Compute posteriors of counts needed in M step.
- What is time complexity?
- Finds local optimum for parameters in likelihood.
  - Ending point depends on starting point.

Where Are We?

HMM's and ASR

- Old paradigm: DTW.

$$w^* = \arg\min_{w \in \text{vocab}} \text{distance}(A'_{\text{test}}, A'_w)$$

- New paradigm: Probabilities.

$$w^* = \arg\max_{w \in \text{vocab}} P(A'_{\text{test}} | w)$$

- Vector quantization: $A'_{\text{test}} \Rightarrow \mathbf{x}_{\text{test}}$.
  - Convert from sequence of 40d feature vectors ...
  - To sequence of values from discrete alphabet.

The Basic Idea

- For each word w, build HMM modeling $P(\mathbf{x}|w) = P_w(\mathbf{x})$.
- Training phase.
  - For each w, pick HMM topology, initial parameters.
  - Take all instances of w in training data.
  - Run Forward-Backward on data to update parameters.
- Testing: the Forward algorithm.

$$w^* = \arg\max_{w \in \text{vocab}} P_w(\mathbf{x}_{\text{test}})$$

- Alignment: the Viterbi algorithm.
  - When each sound begins and ends.

Recap: Discrete HMM's

- HMM's are powerful tool for making probabilistic models . . .
  - Of discrete sequences.
- Three key algorithms for HMM's:
  - The Viterbi algorithm.
  - The Forward algorithm.
  - The Forward-Backward algorithm.
- Each algorithm has important role in ASR.
- Can do ASR within probabilistic paradigm . . .
  - Using just discrete HMM's and vector quantization.

Part III

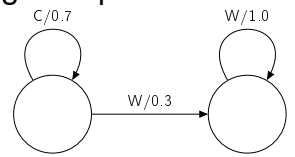
Continuous Hidden Markov Models

Going from Discrete to Continuous Outputs

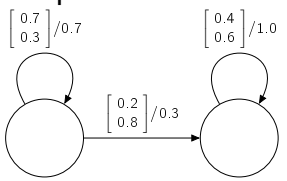
- What we have: a way to assign likelihoods . . .
  - To discrete sequences, *e.g.*, C, W, R, C, . . .
- What we want: a way to assign likelihoods . . .
  - To sequences of 40d (or so) feature vectors.

Variants of Discrete HMM's

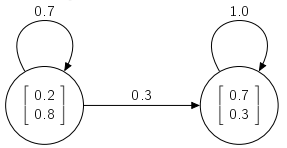
- Our convention: single output on each arc.



- Another convention: output *distribution* on each arc.



- (Another convention: output *distribution* on each state.)



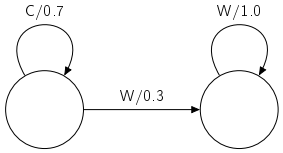
Moving to Continuous Outputs

- Idea: replace discrete output distribution . . .
  - With continuous output distribution.
- What's our favorite continuous distribution?
  - Gaussian mixture models.

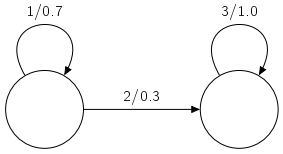
Where Are We?

Moving to Continuous Outputs

- Discrete HMM's.
  - Finite vocabulary of outputs.
  - Each arc labeled with single output x.



- Continuous HMM's.
  - Finite number of GMM's: $g = 1, \ldots, G$.
  - Each arc labeled with single GMM identity g.



What Are The Parameters?

- Assume single start state as before.
- Old: one parameter for each arc: $p_{S \xrightarrow{g} S'}$.
  - Identify arc by source S, destination S', and GMM g.
  - Probs of arcs leaving same state must sum to 1:

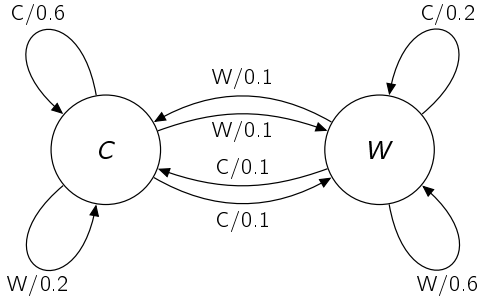
$$\sum_{g,S'} p_{S \xrightarrow{g} S'} = 1 \quad \text{for all } S$$

- New: GMM parameters for $g = 1, \ldots, G$:
  - $p_{g,j}$, $\boldsymbol{\mu}_{g,j}$, $\boldsymbol{\Sigma}_{g,j}$.

$$P_g(x) = \sum_j p_{g,j} \frac{1}{(2\pi)^{d/2} |\Sigma_{g,j}|^{1/2}} \, e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_{g,j})^T \boldsymbol{\Sigma}_{g,j}^{-1} (\mathbf{x}-\boldsymbol{\mu}_{g,j})}$$

Computing the Likelihood of a Path

- Multiply arc and output probabilities along path.
- Discrete HMM:
  - Arc probabilities: $p_{S \xrightarrow{x} S'}$.
  - Output probability 1 if output of arc matches ...
  - And 0 otherwise (*i.e.*, path is disallowed).
- *e.g.*, consider **x** = C, C, W, W.



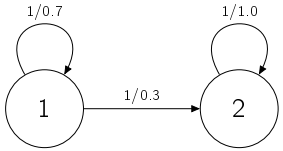
Computing the Likelihood of a Path

- Multiply arc and output probabilities along path.
- Continuous HMM:
  - Arc probabilities: $p_{S \xrightarrow{g} S'}$.
  - Every arc matches any output.
  - Output probability is GMM probability.

$$P_g(x) = \sum_j p_{g,j} \frac{1}{(2\pi)^{d/2} |\Sigma_{g,j}|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_{g,j})^T \boldsymbol{\Sigma}_{g,j}^{-1} (\mathbf{x}-\boldsymbol{\mu}_{g,j})}$$

Example: Computing Path Likelihood

- Single 1d GMM w/ single component: $\mu_{1,1} = 0$, $\sigma_{1,1}^2 = 1$.



- Observed: $\mathbf{x} = 0.3, -0.1$; state sequence: $\mathbf{h} = 1, 1, 2$.

$$P(\mathbf{x}) = p_{1 \xrightarrow{1} 1} \times \frac{1}{\sqrt{2\pi}\sigma_{1,1}} e^{-\frac{(0.3-\mu_{1,1})^2}{2\sigma_{1,1}^2}} \times$$

$$p_{1 \xrightarrow{1} 2} \times \frac{1}{\sqrt{2\pi}\sigma_{1,1}} e^{-\frac{(-0.1-\mu_{1,1})^2}{2\sigma_{1,1}^2}}$$

$$= 0.7 \times 0.381 \times 0.3 \times 0.397 = 0.0318$$

The Three Key Algorithms

- The main change:
  - Whenever see arc probability $p_{S \xrightarrow{x} S'}$ ...
  - Replace with arc probability times output probability:

$$p_{S \xrightarrow{g} S'} \times P_g(x)$$

- The other change: Forward-Backward.
  - Need to also reestimate GMM parameters.

Example: The Forward Algorithm

- $\alpha(S, 0) = 1$ for $S = S_0$, 0 otherwise.
- For $t = 1, \ldots, T$:
  - For each state S:

$$\alpha(S, t) = \sum_{S' \xrightarrow{g} S} p_{S' \xrightarrow{g} S} \times P_g(x_t) \times \alpha(S', t-1)$$

- The end.

$$P(\mathbf{x}) = \sum_{\mathbf{h}} P(\mathbf{h}, \mathbf{x}) = \sum_{S} \alpha(S, T)$$

The Forward-Backward Algorithm

- Compute posterior count of each arc at time t as before.

$$c(S \xrightarrow{g} S', t) = \frac{1}{P(\mathbf{x})} \times p_{S \xrightarrow{g} S'} \times P_g(x_t) \times \alpha(S, t-1) \times \beta(S', t)$$

- Use to get total counts of each arc as before ...

$$c(S \xrightarrow{x} S') = \sum_{t=1}^{T} c(S \xrightarrow{x} S', t) \qquad p^{\text{MLE}}_{S \xrightarrow{x} S'} = \frac{c(S \xrightarrow{x} S')}{\sum_{x, S'} c(S \xrightarrow{x} S')}$$

- But also use to estimate GMM parameters.
  - Send $c(S \xrightarrow{g} S', t)$ counts for point x_t ...
  - To estimate GMM g.

An Example, Please?

- No more examples for you!

Where Are We?

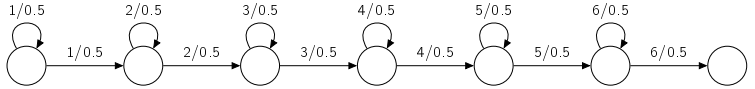
An HMM/GMM Recognizer

- For each word w, build HMM modeling $P(\mathbf{x}|w) = P_w(\mathbf{x})$.
- Training phase.
  - For each w, pick HMM topology, initial parameters, ...
  - Number of components in each GMM.
  - Take all instances of w in training data.
  - Run Forward-Backward on data to update parameters.
- Testing: the Forward algorithm.

$$w^* = \arg\max_{w \in \text{vocab}} P_w(\mathbf{x}_{\text{test}})$$

What HMM Topology, Initial Parameters?

- A standard topology (three states per phoneme):



- How many Gaussians per mixture?
- Set all means to 0; variances to 1 (*flat start*).
- That's everything!

HMM/GMM *vs.* DTW

- Old paradigm: DTW.

$$w^* = \arg\min_{w \in \text{vocab}} \text{distance}(A'_{\text{test}}, A'_w)$$

- New paradigm: Probabilities.

$$w^* = \arg\max_{w \in \text{vocab}} P(A'_{\text{test}}|w)$$

- In fact, can design HMM such that

$$\text{distance}(A'_{\text{test}}, A'_w) \approx -\log P(A'_{\text{test}}|w)$$

  - See Holmes, Sec. 9.13, p. 155.

The More Things Change . . .

DTW	HMM
template	HMM
frame in template	state in HMM
DTW alignment	HMM path
local path cost	transition (log)prob
frame distance	output (log)prob
DTW search	Viterbi algorithm

What Have We Gained?

- Principles!
  - Probability theory; maximum likelihood estimation.
  - Can choose path scores and parameter values . . .
  - In non-arbitrary manner.
  - Less ways to screw up!
- Scalability.
  - Can extend HMM/GMM framework to . . .
  - Lots of data; continuous speech; large vocab; etc.
- Generalization.
  - HMM can assign high prob to sample . . .
  - Even if sample not close to any one training example.

The Markov Assumption

- Everything need to know about past . . .
  - Is encoded in identity of state.
  - *i.e.*, conditional independence of future and past.
- What information do we encode in state?
  - What information don't we encode in state?
  - Issue: the more states, the more parameters.
  - *e.g.*, the weather.
- Solutions.
  - More states.
  - Condition on more stuff, *e.g.*, graphical models.

Recap: HMM's

- Together with GMM's, good way to model likelihood . . .
  - Of sequences of 40d acoustic feature vectors.
- Use *state* to capture information about past.
  - Lets you model how data evolves over time.
- Not nearly as *ad hoc* as dynamic time warping.
- Need three basic algorithms for ASR.
  - Viterbi, Forward, Forward-Backward.
  - All three are efficient: dynamic programming.
- Know enough to build basic GMM/HMM recognizer.

Part IV

Epilogue

What's Next

- Lab 2: Build simple HMM/GMM system.
  - Training and decoding.
- Lecture 5: Language modeling.
  - Moving from isolated to continuous word ASR.
- Lecture 6: Pronunciation modeling.
  - Moving from small to large vocabulary ASR.

Course Feedback

1. Was this lecture mostly clear or unclear? What was the muddiest topic?
2. Other feedback (pace, content, atmosphere)?