# Feature Selection for SVMs

by J. Weston, S. Mukherjee, O. Chapelle, M. Pontil, T. Poggio, V. Vapnik

Sambarta Bhattacharjee
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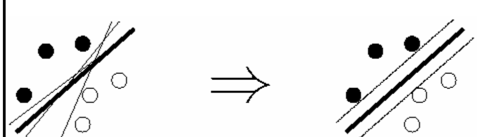
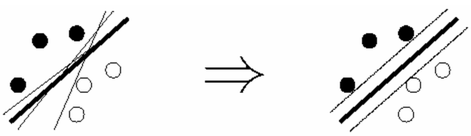
# Review of Support Vector Machines

A support vector machine classifies data as
+1 or -1



- A decision boundary with maximum margin looks like it should generalize well

Support Vector Machines



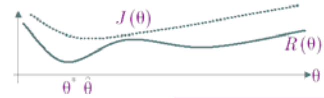
- A decision boundary with maximum margin looks like it should generalize well

Support Vector Machines

- Minimize True Risk ?

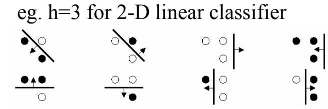
$$R(\theta) = E_P\{L(x,y,\theta)\} = \int_{X\times Y} P(x,y) L(x,y,\theta)\,dx\,dy \in [0,1]$$

- Miminize Guaranteed Risk instead




$$R(\theta) \le J(\theta) = R_{emp}(\theta) + \sqrt{\frac{h\left(\log\left(\frac{2N}{h}\right)+1\right) - \log\left(\frac{\eta}{4}\right)}{N}}$$

- VC dimension h = # of training points that can be shattered
  - eg. h=3 for 2-D linear classifier



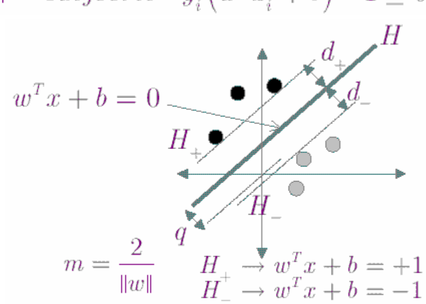
- To minimize J, minimize h
- To minimize h, maximize margin M


- Structural Risk Minimization: minimize $R_{emp}$ while maximizing margin

Support Vector Machines

# Support Vector Machines

- Maximize margin subject to classifying all points correctly
- $\min \frac{1}{2}\|w\|^2 \quad subject\ to \quad y_i\left(w^T x_i + b\right) - 1 \ge 0$

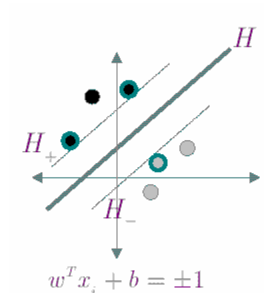



The support vector machine

- To classify: $f(x) = sign\left(x^T w + b\right) = sign\left(\sum_i \alpha_i y_i x^T x_i + b\right)$

# Support Vector Machines

- Support Vectors: ○




# Support Vector Machines

- Dual Problem

$$\max \sum_i \alpha_i - \tfrac{1}{2}\sum_{i,j} \alpha_i \alpha_j y_i y_j x_i^T x_j \quad \textit{subject to} \ \sum_i \alpha_i y_i = 0 \ \ \& \ \ \alpha_i \geq 0$$

# Support Vector Machines

- Dual Problem

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$$\Downarrow$$

- Nonseparable?

$$0 \leq \alpha_i \leq C$$

# Support Vector Machines

- Dual Problem

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$$\Downarrow$$

- Nonseparable?

$$0 \leq \alpha_i \leq C$$

$$\Downarrow$$

- Nonlinear? $k(x,y) = \phi(x)^T \phi(y)$

Cover's theorem on the separability of patterns: A pattern classification problem cast in a high-dimensional space is more likely to be linearly separable

# SVM Matlab Implementation

$$\max \sum_i \alpha_i - \boxed{\tfrac{1}{2}\sum_{i,j} \alpha_i \alpha_j y_i y_j x_i^T x_j} \quad subject\ to \quad \sum_i \alpha_i y_i = 0 \quad \& \quad \alpha_i \geq 0$$

$$k(x,y) = \phi(x)^T \phi(y) \qquad 0 \leq \alpha_i \leq C$$

Another parameter in the qp program sets this constraint to an equality

```
%To train..
for i=1:N
    for j=1:N
        H(i,j)=Y(i)*Y(j)*svm_kernel(ker,X(i,:),X(j,:));
    end
end
alpha=qp(H,f,A,b,vlb,vub);
%X=QP(H,f,A,b) solves the quadratic programming problem:
%              min 0.5*x'Hx + f'x   subject to:  Ax <= b
%               x
%X=QP(H,f,A,b,VLB,VUB) defines a set of lower and upper bounds on the
%design variables, X, so the soln is always in the range VLB <= X <= VUB.
```

# SVM Matlab Implementation

$$f(x) = sign\left(x^T w + b\right) = sign\left(\sum_i \alpha_i y_i x^T x_i + b\right)$$

The bias term is found from the KKT conditions

```
%To classify..
for i=1:M
    for j=1:N
        H(i,j)=Ytrn(j)*svm_kernel(ker,Xtst(i,:),Xtrn(j,:));
    end
end
Ytst=sign(H*alpha+b0);
```

- Karush-Kuhn-Tucker Conditions (KKT): solve value of b on margin (for nonzero alphas) have: $w^T x_i + b = y_i$ using known w, compute b for each support vector $\tilde{b}_i = y_i - w^T x_i \quad \forall i : \alpha_i > 0$ then... $b = average\left(\tilde{b}_i\right)$

# Support Vector Machines

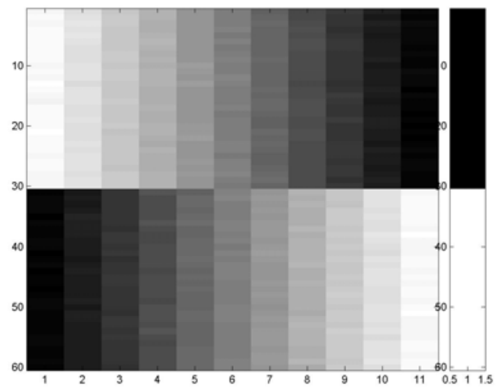
- Summary
  - Use Matlab's qp( ) to perform optimization on training points and get parameters of hyperplane
  - Use hyperplane to classify test points

# Feature Selection for SVMs

# Here's some data

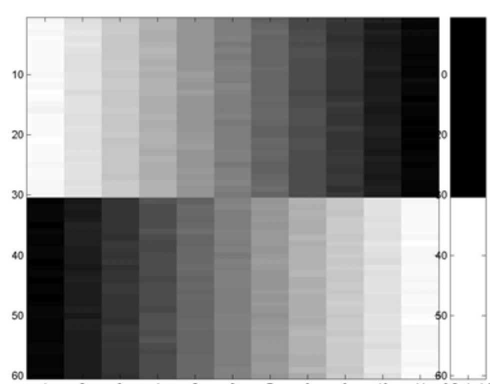
Row 20 is a 11-D data point

60 data points




The data is classified as +1 (black) or -1 (white)

Col 3 is the 3rd dimension




Dimension 6 is pretty useless in classification

We want to find the relative discriminative ability of each dimension, and throw away the least discriminative dimensions

## Dimensionality Reduction

- Improve generalization error
- Need less training data (avoid curse of dimensionality)
- Speed, computational cost
- (qualitative) Find out which features matter
- For SVMs, irrelevant features hurt performance

# Formal problem

$$\tau(\sigma, \alpha) = \int \underbrace{V}_{\text{loss functional}}(y, f((\overset{\text{input}}{x} \cdot \underbrace{\sigma}_{\text{weights}}), \alpha))dP(x, y)$$

- Weight each feature by 0 or 1

  $\sigma = (1, 0, 1, 1, 0),\ x = (x_1, x_2, x_3, x_4, x_5),\ x \cdot \sigma = (x_1, 0, x_3, x_4, 0)$

- Which set of weights minimizes (average expected) loss?
  - Specifically, if we want to keep m features out of n, which set of weights minimizes loss subject to the constraint that weight vector sums to m?
- We don't know P(x,y)

# Formal solution
(approximations)

$$(x \cdot \sigma)$$

- Weight each feature by 0 or 1

$\cong$

$(x \cdot \sigma)$

- Weight each feature by 0 or 1
- Weight each feature by a real valued vector
- First approach suggests combinatorial search over all weights (intractable for large dimensionality)
- Second approach brings you closer to a gradient descent solution

- There's a weight vector that minimizes (average expected) loss $\tau(\sigma, \alpha)$
- There's a weight vector that minimizes expected leave-one-out error probability for weighted inputs $EP_{err}$

$\cong$

- There's a weight vector that minimizes (average expected) loss $\tau(\sigma, \alpha)$
- There's a weight vector that minimizes expected leave-one-out error probability for weighted inputs $EP_{err}$
- Let's pretend these are the same ("wrapper method")

A wrapper method is *a search through the space of feature subsets using the estimated accuracy from an induction algorithm trained on preprocessed data as a measure of goodness of a particular subset* [1].

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- Theorem

$$EP_{err} \leq \frac{1}{l} E[R^2 W^2(\alpha^0)]$$

- Data in sphere of size R, separable with margin M ($1/M^2 = W^2$)

≅

- Theorem

$$EP_{err} \leq \frac{1}{l} E[R^2 W^2(\alpha^0)]$$

- Data in sphere of size R, separable with margin M ($1/M^2=W^2$)
- To minimize error probability, let's minimize $R^2W^2$ instead


- Someone gives us a contour map, telling us which direction to walk in weight vector space to get highest increase in $R^2W^2$
- We take a small step in the opposite direction
- Check map again
- Repeat above steps (until we stop moving)

This is gradient descent


$$\frac{\partial R^2 W^2(\sigma)}{\partial \sigma_k} = R^2(\sigma)\frac{\partial W^2(\alpha^0,\sigma)}{\partial \sigma_k} + W^2(\alpha^0,\sigma)\boxed{\frac{\partial R^2(\sigma)}{\partial \sigma_k}} \quad (7)$$

another optimization problem

SVM training

$$\boxed{\frac{\partial R^2(\sigma)}{\partial \sigma_k}} = \sum_i \beta_i^0 \frac{\partial K_\sigma(x_i,x_i)}{\partial \sigma_k} - \sum_{i,j} \beta_i^0 \beta_j^0 \frac{\partial K_\sigma(x_i,x_j)}{\partial \sigma_k} \quad (8)$$

$$\frac{\partial W^2(\alpha^0,\sigma)}{\partial \sigma_k} = -\sum_{i,j} \alpha_i^0 \alpha_j^0 y_i y_j \frac{\partial K_\sigma(x_i,x_j)}{\partial \sigma_k}. \quad (9)$$

## This is the contour map

# Feature Selection for SVMs

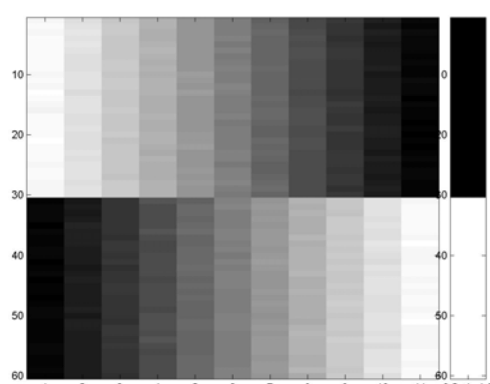
$$\frac{\partial K_\sigma(x_i,x_j)}{\partial \sigma_k}$$

- Choose kernel, find gradient, proceed with above algorithm to find weights
- Throw away lowest weighted dimension(s) after gradient descent finds minimum, repeat until you have specified number of dimensions left
  - E.g. You have 123 dimensions (41 average X Y Z coordinates of person’s joints) for walking/running classification. You want to reduce to 6 (maybe these will be the X Y Z coordinates of both ankles)
  - Throw away worst 2 dimensions after each run of algorithm until you have desired number left

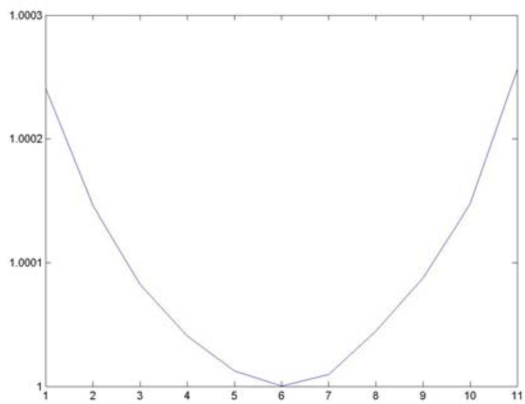
# Feature Selection for SVMs

- Throw away worst q dimensions after each run of algorithm until you have desired number left
- As we increase q, fewer calls to qp algorithm and faster performance

# For this data

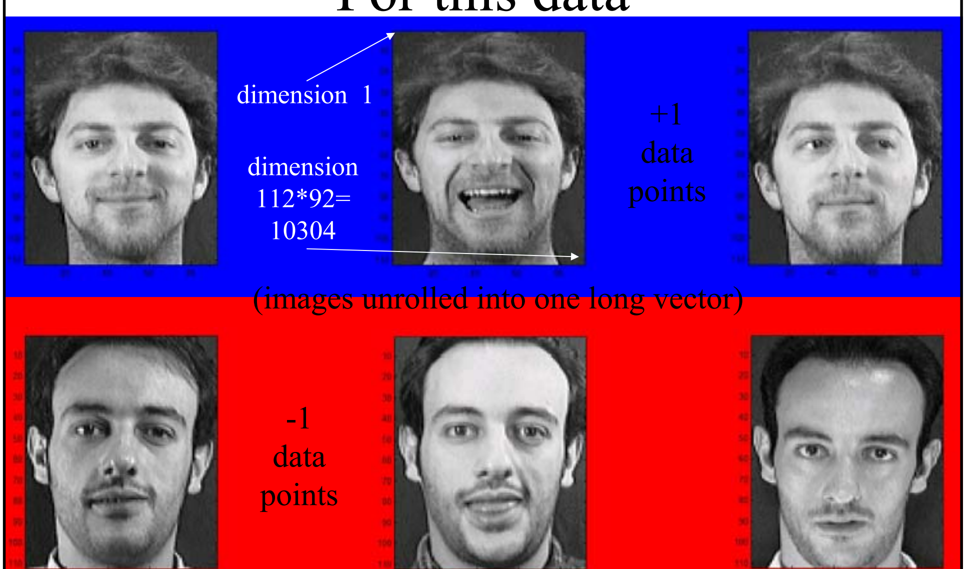


# We get this weighting

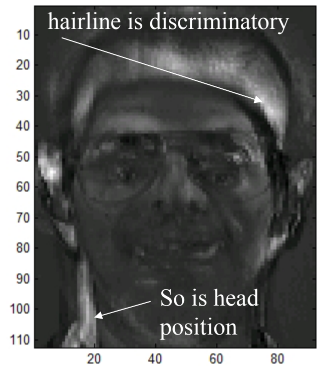



Dimension 6 is the first to go

# For this data

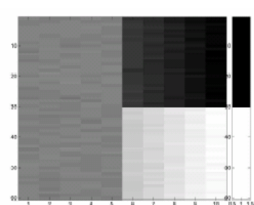


# We get this weighting




# And…

- Automatic dimensionality reduction? (user doesn't have to specify number of dimensions)



# References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- [2] O. Chapelle, V. Vapnik, Choosing Multiple Parameters for Support Vector Machines. Machine Learning, 2001
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- [4] V. Vapnik, Statistical Learning Theory, John Wiley, 1998
- [5] O. Chapelle, V. Vapnik, O. Bousquet, S. Mukherjee, Choosing Kernel Parameters for Support Vector Machines. Machine Learning, 2000