The physics of music

- Synthesize realistic notes by modeling the mechanical and acoustic behavior of a musical instrument
- Sound produced by *waves* traveling through some medium
  - Common math for different physical phenomena: gas, solids, EM
- Waves transfer *energy* without permanent displacement of matter

- e.g. guitar string, cymbal
Some scary math: The wave equation

- Lossless string in a 1-D medium with displacement $y(x, t)$:

$$c^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$$

$y(0, t) = m(t)$

$y^+(x, t)$

$y(L, t) = 0$

$c = \sqrt{\frac{K}{\varepsilon}}$ (wave speed), $K = \text{string tension}$, $\varepsilon = \text{density}$
Solving the wave equation

- d’Alembert’s solution (1747):
  \[ y(x, t) = y^+(x - ct) + y^-(x + ct) \]

- Sum of left-moving \((y^+)\) and right-moving \((y^-)\) traveling waves
- Shape doesn’t change (set by initial conditions)
Digital waveguides

- Represent each traveling wave using a delay line
  
  \[ y^+(n) \rightarrow z^{-m} \rightarrow y^+(n-m) \]

  \[ y^-(n) \leftarrow z^{-m} \leftarrow y^-(n+m) \]

  - String length determines length of delay line \( m \)
  - Wave impedance \( R \)

- Compute solution to wave equation by sampling delay line and summing contribution of each traveling wave
  
  \[ y^+(n) \rightarrow z^{-m} \rightarrow y^+(n-m) \]

  \[ y^-(n) \leftarrow z^{-m} \leftarrow y^-(n+m) \]

  \[ y(x_m,t_n) \]

  \[ y_m \]
Physical outputs

- Can work with other physical variables (acceleration, velocity)
- Derived from displacement:

\[ v = \frac{\partial y}{\partial t} = \text{velocity} \quad a = \frac{\partial^2 y}{\partial t^2} = \text{acceleration} \]

- Implement using digital filters

\[ y(n) \rightarrow z^{-1} \rightarrow \hat{v}(n) \]

\[ \hat{v}(n) \rightarrow \hat{y}(n) \rightarrow g \]
Waves in musical instruments aren’t the Energizer bunny . . .
Solution to wave equation must match constraints
  leads to reflections at rigid terminations

• Boundary conditions include fixed points e.g. held ends of string

• Superposition of traveling waves must match constraints hence reflections

• Any impedance change results in some reflection

• Energy loss...

\[ y(x,t) = y^+ + y^- \]

\[ y^+(n-N/2) \]

\[ y^- (n+N/2) \]

\[ y^+(n) \]

\[ y^+(n-N/2) \]

\[ y(nT,mX) \]

\[ -1 \]

\[ -1 \]

Easy to incorporate into digital waveguide
Alternative interpretation: Mass-spring (lumped) model

Real strings have losses (e.g. friction within springs) . . .
(More sophisticated: 2-D mass-spring)

A 2-D square surface

A drum membrane
Lossy 1-D wave

- Simple model: constant loss at each “spring”:

\[ y(x, t) = g^x y^+(x - ct) + g^{-x} y^-(x + ct) \]

- Consolidate delays and losses where possible
- More realistic: frequency-dependent losses
  - Replace \( g \) with filter
Putting it all together: Damped plucked strings

- We almost have a guitar string:

- But, real strings have losses
  - exponentially decaying traveling waves

Because there is no input/output coupling, can consolidate all delays and loses at a single point in the loop:

- \( y^+(n) \)

\[ y^+(n-N/2)g^{N/2} \]

\[ y^+(n+N/2)g^{-N/2} \]
Digital waveguide review

- Digital Waveguides
  - Direct physical model + simplifications

\[
\text{Delay } z^{-L} \\
\text{Initialize with random values} \\
L = \frac{SR}{f_0} \\
h_{LP} \\

\text{String Waveguide} \\
\text{Nut} \\
-1 \\
\text{Bridge dispersion + radiation load} \\

\text{Delay Lines} \\
y^+(x,t) \\
y^-(x,t) \\
y^+(0,t) = -y^-(0,t) \\
y^-(L,t) = h_{reflec}(t) * y^+(L,t) \\

\text{Initialize with pluck shape} \\

\text{Karplus-Strong}
The Karplus-Strong algorithm (1978)

- Initialize the waveguide with random noise
- Noise “wave” will propagate through the loop
t  - decaying as it passes through the filter
- Pitch is proportional to length of delay line: \( f = \frac{f_s}{N} \)
- Does this look familiar?
  - it's just an IIR comb filter . . .
    - with an LPF in the loop instead of a fixed gain
  - pass a short noise burst in instead of long term noise
Karplus-Strong examples

Averaging acts as a low-pass filter, limiting the speed of change of the signal, hence limiting the presence of high frequencies. Because we are feeding back the averaged values, our waveform evolves. These accumulative low-pass filtering will keep stabilising the process until we reach equilibrium.

4. Filtering

Amplitude modulation alone is not enough; real instruments have time-varying spectra, e.g., plucked strings. Generally just LPF (+ resonance) will ensure that high frequencies die away after initial transient. Resonance can give some BPF effect.

Strings: 🎸 🎸 🎸 Drums: 🤼‍♀️ 🤼‍♀️
Extended Karplus-Strong (Jaffe and Smith, 1983)

\[ N = \text{pitch period (2× string length) in samples} \]

\[ H_p(z) = \frac{1-p}{1-pz^{-1}} = \text{pick-direction lowpass filter} \]

\[ H_\beta(z) = 1 - z^{-\beta N} = \text{pick-position comb filter, } \beta \in (0,1) \]

\[ H_d(z) = \text{string-damping filter (one/two poles/zeros typical)} \]

\[ H_s(z) = \text{string-stiffness allpass filter (several poles and zeros)} \]

\[ H_\rho(z) = \frac{\rho(N) - z^{-1}}{1 - \rho(N) z^{-1}} = \text{first-order string-tuning allpass filter} \]

\[ H_L(z) = \frac{1 - R_L}{1 - R_L z^{-1}} = \text{dynamic-level lowpass filter} \]
Bowed strings (1986)

Bowed strings have **more complex excitation**

![Diagram of bowed string instrument with labels for nut, string, bow, string, and bridge]
More strings: Clavichord (Valimaki et al, 2004)

State-of-the-art models

Valimaki et al, 2003; used for clavichord synthesis
Woodwinds (Smith, 1986)

- **Wave equation in air**
  - Pressure waves traveling in tube
  - Resonance of tube depends on length
  - Coupled energy input

- **Example of wind instruments** include the clarinet, trumpet, flute, and organ pipe.

- **Digital waveguide Woodwind instrument**

### Equations

\[ U_0 e^{j\omega t} \]

\[ kx = \pi \quad x = \frac{\lambda}{2} \]

\[ \text{pressure} = 0 \text{ (node)} \]
\[ \text{vol. veloc.} = \text{max} \quad \text{antinode} \]

\[ U \Rightarrow e^{j\omega t} \]

\[ f(x) = \frac{\partial}{\partial x} (x) \]

\[ \text{Mouth pressure} \rightarrow \text{Nonlinear Scattering Junction} \rightarrow \text{Output Signal} \]

\[ z^{-N} \]

\[ R(z) \]

\[ \text{Rate of air flow in steady state} \]

\[ \text{Mouth pressure} - \text{mouthpiece pressure} \]

\[ \text{Reed open: flow proportional to pressure difference} \]

\[ \text{higher pressure forces the reed shut} \]
Physical Modeling: The Verdict

- Realistic synthesis of acoustic instruments
- Parameters based on the physical attributes of real instruments
  - less guesswork involved
- But expensive to implement
- Need different model for each instrument
Reading

- Much more at [https://ccrma.stanford.edu/~jos/wg.html](https://ccrma.stanford.edu/~jos/wg.html)