Fragmented Random Structures

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Introduction and Motivation

Modern computers with hard-disk storage and networks with dynamic spectrum access illustrate systems having structures that allow fragmented allocations. The structure is modeled as a sequence of $M > 1$ slots for which items in a FIFO queue make requests. Fragmentation in the form of alternating gaps and allocated slots builds up randomly as items come and go. The improvements in utilization created by fragmentation are acquired at a processing cost, so how fragmentation evolves is an important performance issue.

Hard-disk drives

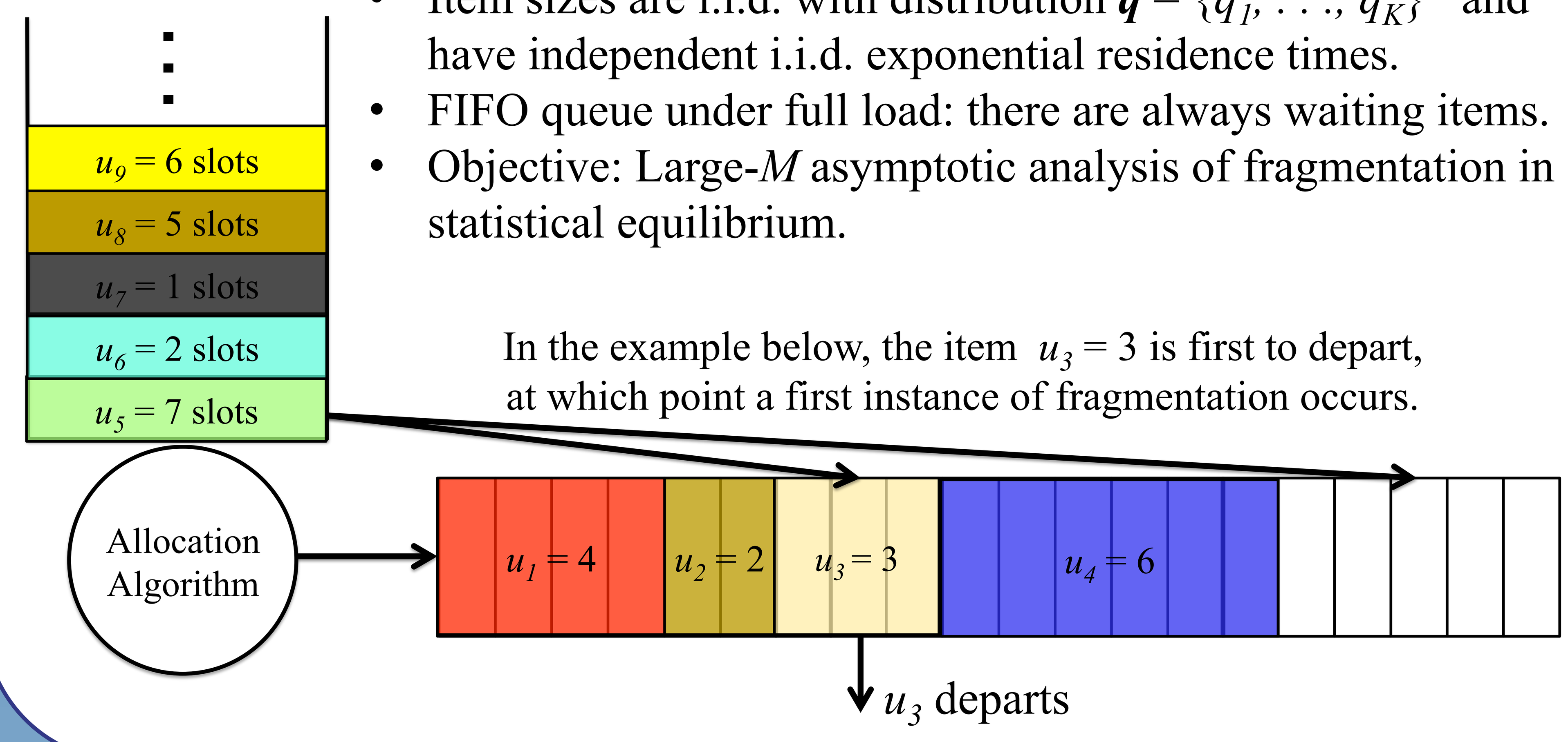
Solid-state disk drives

OFDMA and Cognitive Radio

The Model

- Item sizes are i.i.d. with distribution $\boldsymbol{q} = \{q_1, \ldots, q_K\}$ and have independent i.i.d. exponential residence times.
- FIFO queue under full load: there are always waiting items.
- Objective: Large-M asymptotic analysis of fragmentation in statistical equilibrium.

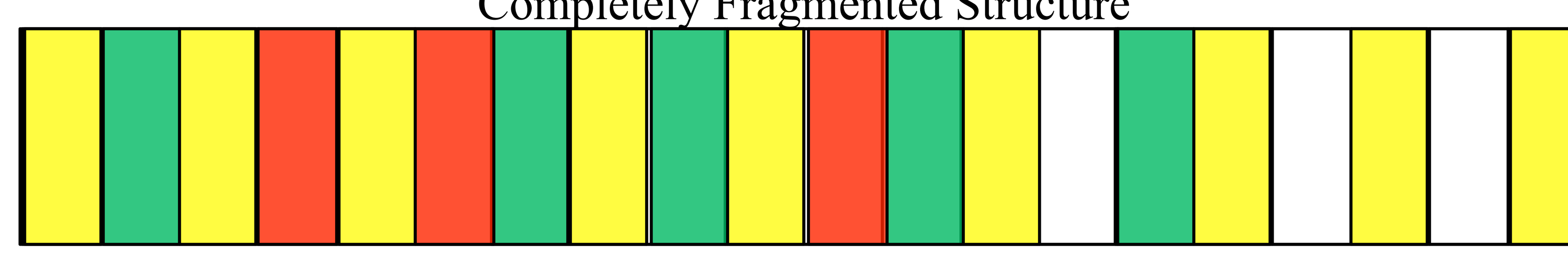
In the example below, the item $u_3 = 3$ is first to depart, at which point a first instance of fragmentation occurs.



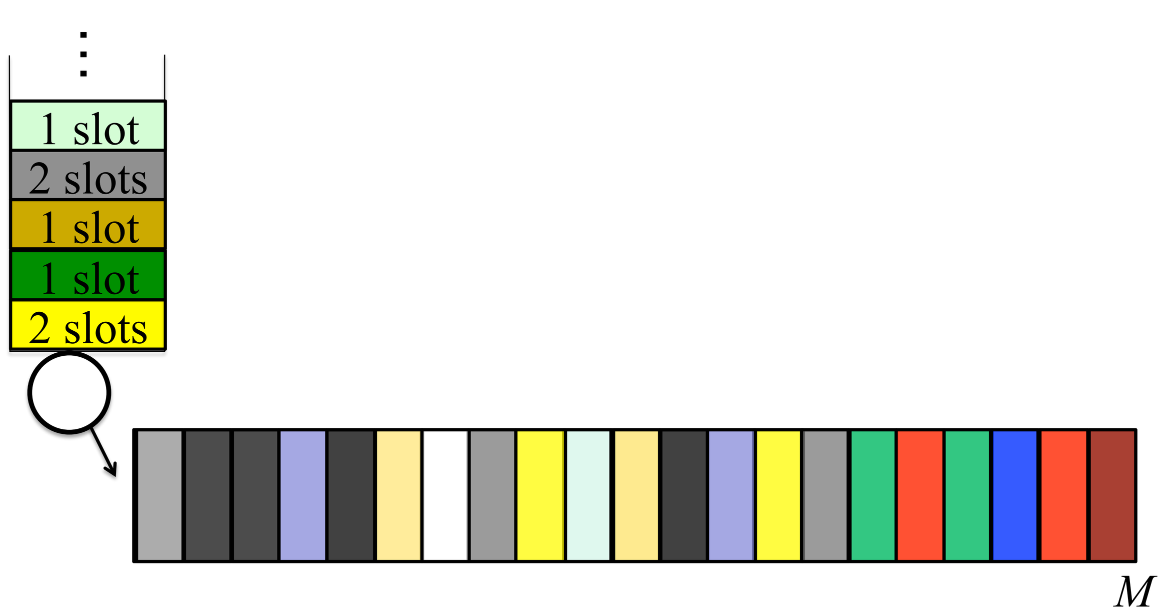
Complete Fragmentation

- An item is completely fragmented when no two of its allocated slots are adjacent.
- Does fragmentation progress to a point where nearly all items are completely fragmented?
- Proofs that the answer is "yes" are in terms of *bonds*; a bond exists between any adjacent pair of empty slots or slots occupied by the same item

Completely Fragmented Structure

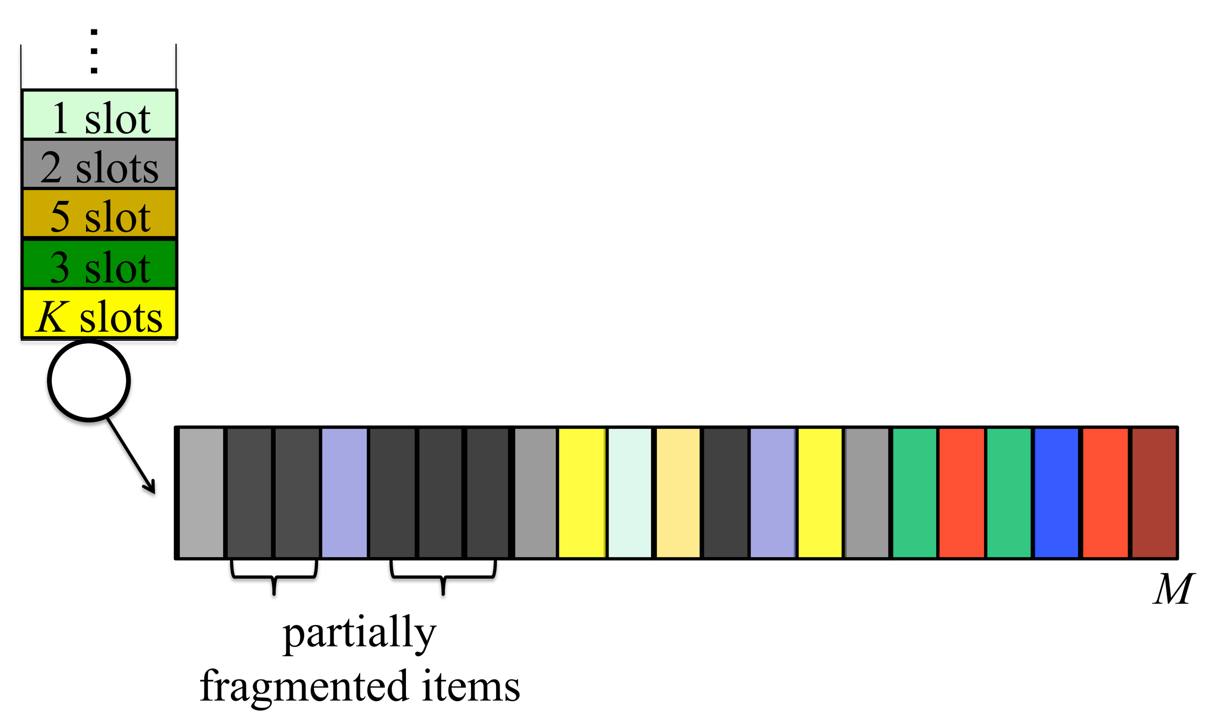


Case 1: Item sizes restricted to {1, 2} ($0 < q_1, q_2 < 1$)



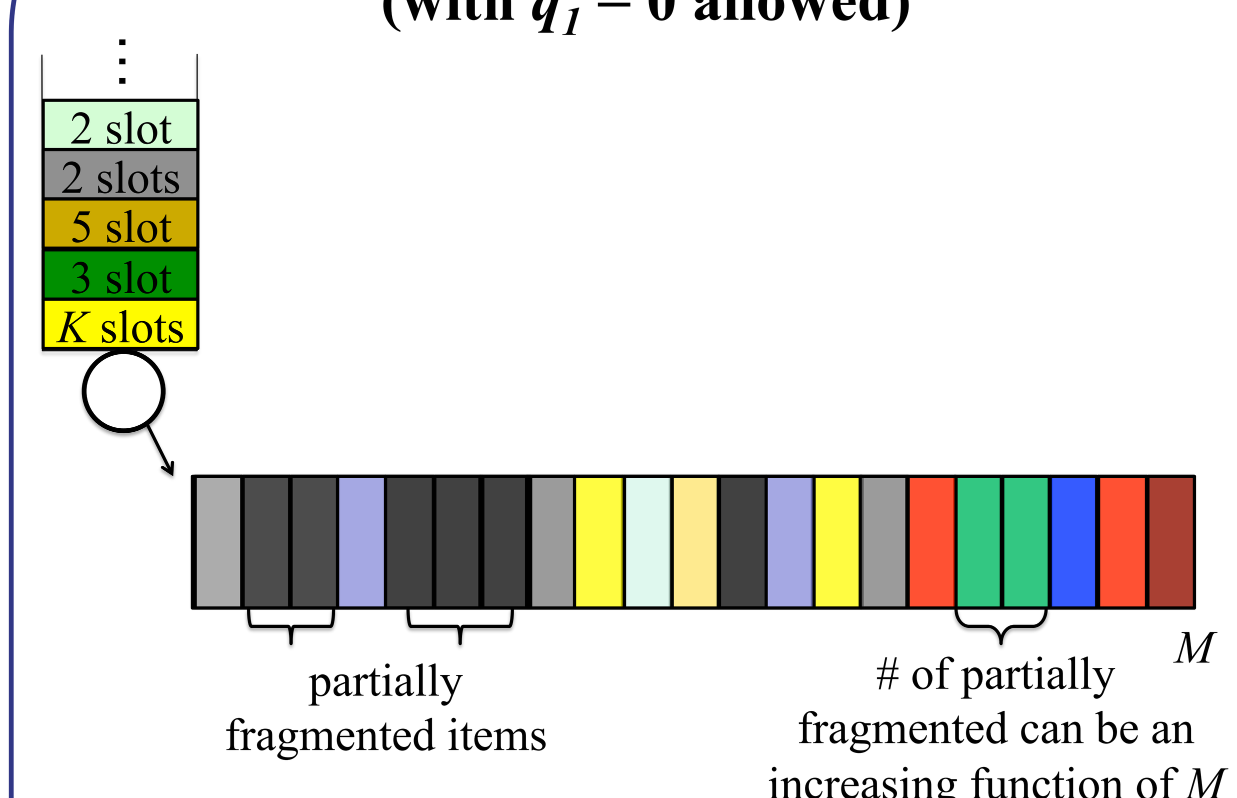
Expected number of bonds (unsplit size-2 items) has the tight upper bound $2(1-q_1)/q_1$

Case 2: Item sizes unrestricted with positive probability of size-1 items ($q_1 > 0$)



The expected number of partially fragmented items has a constant upper bound independent of M.

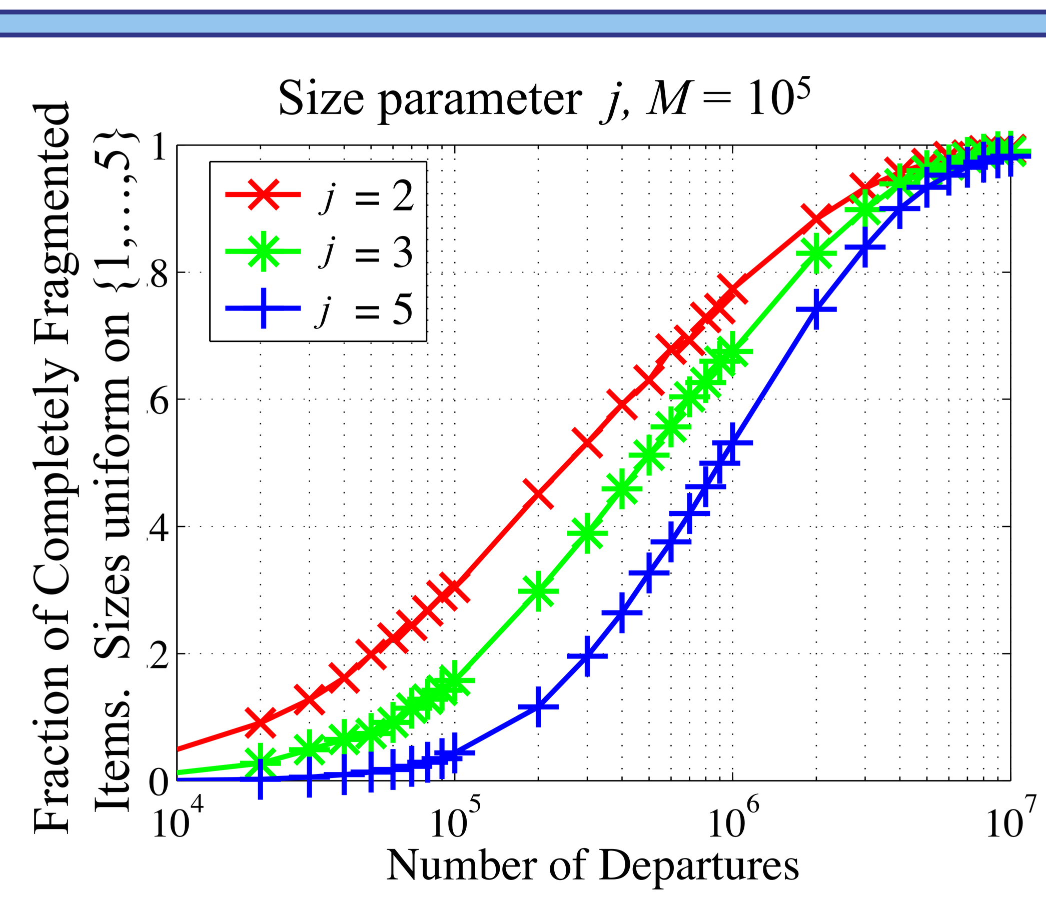
Case 3: Item sizes unrestricted (with $q_1 = 0$ allowed)

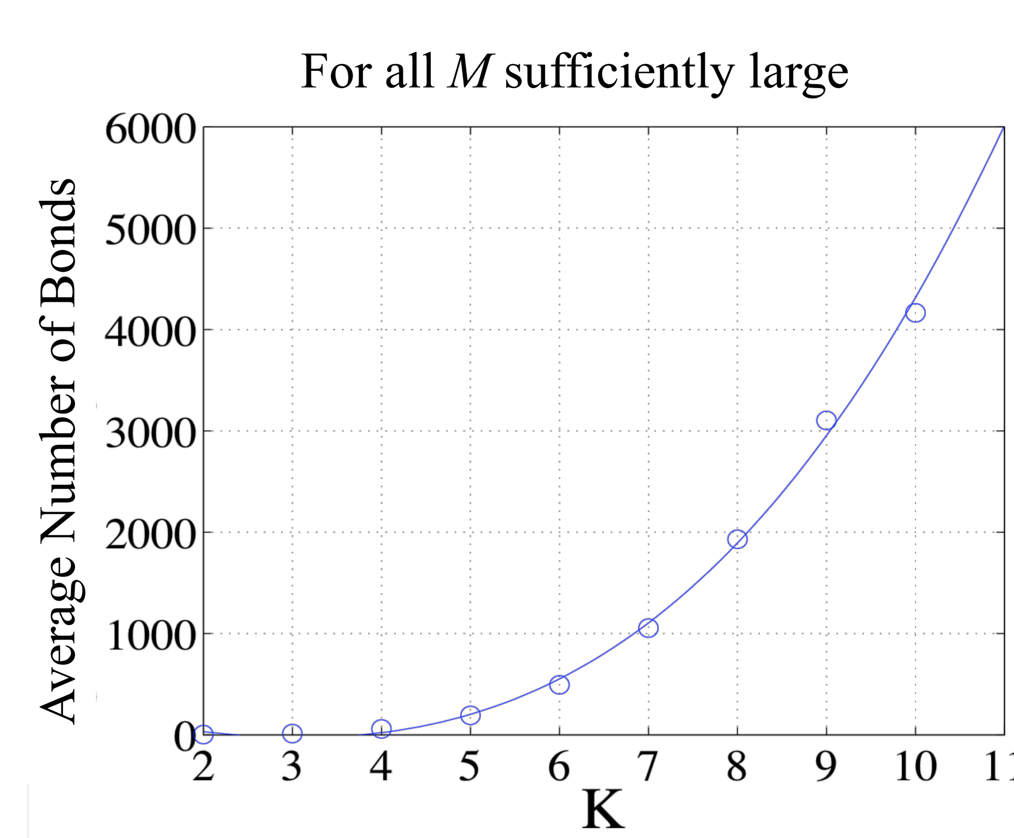


The fraction of items partially fragmented tends to 0 as M tends to infinity.

Observations

- Nearly all items become *completely fragmented* in statistical equilibrium for large structures.
- Proofs for cases 1 and 2 balance the rates at which the number of bonds increases and decreases in equilibrium
- Convergence rates from initially unfragmented states can be surprisingly slow, as shown by experiments with a uniform law for item sizes.





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