

ELEN6887 - Spring 2010

Homework 2

Due Friday February 12th

In this homework we will study the rates of error decay using the sieves developed in class. We consider various classes of functions, including ones that can be used as a reasonable model for images.

Problem 1: Consider the one-dimensional smooth function model we have seen in class:

$$\mathcal{F}(L) = \{f : [0, 1] \rightarrow [0, 1] : |f(x) - f(y)| \leq L|x - y| \forall x, y \in [0, 1]\} .$$

These are called Lipschitz functions. Let $f^* \in \mathcal{F}(L)$, but otherwise unknown, and consider the observation model

$$Y_i = f^*(i/n) + W_i \quad , i \in \{1, \dots, n\},$$

where $E[W_i] = 0$ and $E[W_i^2] \leq \sigma^2 < \infty$.

We have constructed in class an estimator \hat{f}_n , based on the method of sieves, such that $E[\|\hat{f}_n - f^*\|^2] = O(n^{-2/3})$. Let's consider now a different class, the class of *piecewise Lipschitz functions*. These are functions that are composed by a finite number of pieces that are Lipschitz. An example of such a function is $g(t) = f_1(t)\mathbf{1}\{t \in [0, 1/3]\} + f_2(t)\mathbf{1}\{t \in (1/3, 1/2)\} + f_3(t)\mathbf{1}\{t \in (1/2, 1]\}$, where $f_1, f_2, f_3 \in \mathcal{F}(L)$.

Let $\mathcal{G}(M, L, R)$ denote the class of bounded piecewise Lipschitz functions. Each piece belongs to class $\mathcal{F}(L)$, there are at most M pieces, and any function $g \in \mathcal{G}(M, L, R)$ is bounded in the sense that $|g(x)| \leq R$ for all $x \in [0, 1]$. Use the sieve method described in lecture 4 to estimate piecewise Lipschitz functions, and identify the best rate of error decay when using such a procedure.

Problem 2: Suppose you want to denoise an image. An image can be thought of as a function $f : [0, 1]^2 \rightarrow [0, 1]$. Let's suppose it satisfies a 2-dimensional Lipschitz condition

$$|f(x_1, y_1) - f(x_2, y_2)| \leq L \max(|x_1 - x_2|, |y_1 - y_2|) \quad , \forall x_1, y_1, x_2, y_2 \in [0, 1] .$$

1. Do you think this is a good model for images? Why and why not.
2. Assume n , the number of samples you get from the function, is a perfect square, therefore $\sqrt{n} \in \mathbb{N}$. Let f^* be a function in this class and let the observation model be

$$Y_{i,j} = f^*(i/\sqrt{n}, j/\sqrt{n}) + W_{i,j} \quad , i, j \in \{1, \dots, \sqrt{n}\} ,$$

where as before the noise variables are mutually independent and again $E[W_{i,j}] = 0$ and $E[W_{i,j}^2] \leq \sigma^2 < \infty$.

Using a similar approach to the one in class construct an estimator \hat{f}_n for f^* . Using this procedure what is the best rate of convergence attainable when f^* is a 2-dimensional Lipschitz function?

Problem 3: Let's extend the results of question 1 to the two dimensional setting. Consider the space of piecewise 2-d Lipschitz functions, that consist of a finite number of Lipschitz smooth regions separated by 1-d boundary curves.

1. Is this a reasonable model for images? Why or why not?
2. **Extra credit:** Building on answers to the previous questions construct an estimator, and show its rate of convergence. (Note: if you partition the unit square into m^2 equal size boxes a 1-dimensional curve passes through $O(m)$ of these boxes).
3. Implement and experiment with your methods for denoising an image (Use the file `magnolia.mat`). Comment on the results and explain if you would recommend this procedure.