ELEN3801 - Fall 2009
Midterm 2

November 24, 2009

INSTRUCTIONS:

- Carry only notes on one side of a $8\frac{1}{2}'' \times 11''$ piece of paper, and a pencil and/or pen with you.
- The exam is closed-book, closed-notes. No calculator or other electronic devices are allowed, apart from a dedicated timing device (cell-phones in particular are not allowed).
- You have 75 minutes to complete the exam.
- A correct answer does not guarantee full credit, and a wrong answer does not guarantee loss of credit. You should clearly and concisely indicate your reasoning and **show all relevant work**. Your grade on each problem will be based on our best assessment of your level of understanding as reflected by what you have written. **JUSTIFY** your answers and be **CRITICAL** of your results.
- Please be organized in your write-up – we can't grade what we can't decipher!
- Write all your answers in the question sheet provided - Ask if you need extra paper. Hand in your notes sheet with the exam. Remember to **IDENTIFY YOUR HANDOUT**.

Problem	Points
I	/15
II	/15
III	/25
IV	/20
V	/25
Total	/100

Name: Solutions

Problem I [15pts]

State whether each of the following statements is TRUE or FALSE.

3 a) A band-limited signal with positive energy is also time-limited.

3 b) Let $h(t)$ be a real signal with finite energy. The convolution of $h(t)$ with $\cos(2t)$ is always equal to $A\cos(4t-\theta)$ for some A and θ. **Hint:** the answer is straightforward if you recall what is the response of an LTI system to a sinusoidal input.

3 c) Let $f(t)$ be a bandlimited signal with bandwidth 300Hz. The signal $f(2t)$ is band-limited with bandwidth 150Hz.

3 d) If $x(t)$ is a real signal then the magnitude of the Fourier transform $\mathcal{F}\{x(t)\}$ is an even function.

3 e) The Fourier transform of $f(t) = e^{-2t}u(t)$ is given by $F(\omega) = \frac{1}{2+j\omega}$.

a) False: A signal cannot be simultaneously time-limited and band-limited. In particular the Paley-Wiener criterion tells us that if a signal's FT is zero over some band of frequencies then the signal is time-unlimited.

b) False: The convolution of $h(t)$ with $\cos(\omega_0 t)$ corresponds to the output of a system with impulse response $h(t)$ and input $\cos(\omega_0 t)$. We saw in class that $h(t) * \cos(\omega_0 t) = |H(\omega_0)| \cos(\omega_0 t + \measuredangle H(\omega_0))$, where H is the FT of $h(t)$. Therefore the statement is false, but if instead it said $A\cos(2t-\theta)$ the statement would be true.

c) False: $f(2t)$ is a compression of the signal in time, therefore a dilation in frequency. Recall that $\mathcal{F}\{f(at)\} = \frac{1}{|a|}F\left(\frac{\omega}{a}\right)$ therefore the FT of $f(2t)$ is $\frac{1}{2}F\left(\frac{\omega}{2}\right)$, where $F(\omega) = \mathcal{F}\{f(t)\}$ and so all we can say is that the signal has bandwidth of most 600Hz.

d) True: We proved this in class using the fact that $\mathcal{F}\{x^*(t)\} = X^*(-\omega)$ where $X(\omega) = \mathcal{F}\{f(t)\}$. Since $x(t)$ is real $x(t) = x^*(t)$ and so $X(\omega) = X^*(-\omega)$ therefore $|X(\omega)| = |X^*(-\omega)| \Leftrightarrow |X(\omega)| = |X(-\omega)|$, so $|X(\omega)|$ is even (but $X(\omega)$ is in general not even)

e) True: We proved in class that $\mathcal{F}\{e^{-at}u(t)\} = \frac{1}{j\omega + a}$, $a > 0$

Problem II [15pts]

Let $f(t)$ be a signal, and let $F(\omega) = \mathcal{F}\{f(t)\}$ be its Fourier transform. Find the Fourier transform of the signal

$$g(t) = f(a(t-b)) ,$$

where $a \neq 0$ and $b \in \mathbb{R}$.

Note that $g(t)$ can be obtained from $f(t)$ by performing a time scaling and shifting operation.

Define

$$h(t) = f(at)$$

note that $h(t-b) = f(a(t-b)) = g(t)$

Now let $F(w) = \mathcal{F}\{f(t)\}$, $H(w) = \mathcal{F}\{h(t)\}$, $G(w) = \mathcal{F}\{g(t)\}$

by the time scaling property $H(w) = \frac{1}{|a|} F\left(\frac{w}{a}\right)$

by the time shifting property $G(w) = H(w) e^{-jwb}$

Therefore $G(w) = \frac{1}{|a|} F\left(\frac{w}{a}\right) e^{-jwb}$

<u>Alternate ~~way~~</u>: You can do the shifting operation first. Note that $f(a(t-b)) = f(at-ab)$

Let $h(t) = f(t-ab)$ and note that

$$h(at) = f(at-ab) = f(a(t-b)) = g(t)$$

Therefore $$H(\omega) = F(\omega)\, e^{-j\omega ab}$$

and $$G(\omega) = \frac{1}{|a|} H\left(\frac{\omega}{a}\right)$$

pluggin in $H\left(\frac{\omega}{a}\right)$ yields

$$G(\omega) = \frac{1}{|a|} F\left(\frac{\omega}{a}\right) e^{-j\left(\frac{\omega}{a}\right)ab} = \frac{1}{|a|} F\left(\frac{\omega}{a}\right) e^{-j\omega b}$$

<u>Alternate Way II</u>: Just use the definition

$$G(\omega) = \int_{-\infty}^{\infty} f(a(t-b))\, e^{-j\omega t}\, dt$$

and do a change of variables $u = a(t-b)$ $\left(t = \frac{u}{a} + b,\ du = a\,dt\right)$

case 1) $a > 0$

$$G(\omega) = \int_{a(-\infty-b)}^{a(+\infty-b)} f(u)\, e^{-j\omega\left(\frac{u}{a}+b\right)} \frac{1}{a}\, du$$

$$= \int_{-\infty}^{\infty} f(u)\, e^{-j\frac{\omega}{a}u} \cdot e^{-j\omega b} \cdot \frac{1}{a}\, du = e^{-j\omega b} \cdot \frac{1}{a} \int_{-\infty}^{\infty} f(u)\, e^{-j\left(\frac{\omega}{a}\right)u}\, du$$

$$= \frac{e^{-j\omega b}}{a} F\left(\frac{\omega}{a}\right)$$

case 2) $a < 0$

$$G(\omega) = \int_{a(-\infty-b)}^{a(+\infty-b)} f(u)\, e^{-j\frac{\omega}{a}u} e^{-j\omega b} \frac{1}{a}\, du =$$

$$= \int_{+\infty}^{-\infty} f(u)\, e^{-j\left(\frac{\omega}{a}\right)u}\, du \cdot \frac{e^{-j\omega b}}{a}$$

$$= -\frac{e^{-j\omega b}}{a} \int_{-\infty}^{\infty} f(u)\, e^{-j\left(\frac{\omega}{a}\right)u}\, du = -\frac{1}{a} F(\omega/a)\, e^{-j\omega b}$$

Problem III [~~20~~ 25 pts]

Let $\mathcal{H}$ be a Linear Time-Invariant (LTI) system with impulse response $h(t)$ and such that the transfer function of the system is

$$H(\omega) = \mathcal{F}\{h(t)\} = \frac{2}{2 + j2\omega} e^{j\omega} .$$

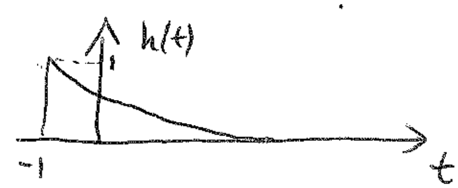
(10) a) Find and sketch the system's impulse response $h(t)$.

(5) b) Is the system practically realizable? Why or why not?

(10) c) Let $g(t) = \cos(t)$ be the input of the system. What is $G(\omega) = \mathcal{F}\{g(t)\}$? What is the system output $y(t) = (h * g)(t)$? **Hint:** we have derived the answer to this question in class, and you can find it very easily by using the convolution property of the Fourier transform.

a) $H(\omega) = \frac{1}{1+j\omega} \underbrace{e^{j\omega}}_{\text{time-shift by 1 to the left}}$. Since $\mathcal{F}^{-1}\left\{\frac{1}{1+j\omega}\right\} = e^{-t}u(t)$

we have $h(t) = e^{-(t+1)} u(t+1)$



b) h(t) is not causal, hence not practically realizable.

c) $G(\omega) = \pi\delta(\omega+1) + \pi\delta(\omega-1)$ as we seen in class (recall that $\mathcal{F}\{e^{j\omega_0 t}\} = 2\pi\delta(\omega-\omega_0)$) (*)

Now let $Y(\omega) = \mathcal{F}\{y(t)\}$, and note that

$$Y(\omega) = H(\omega)G(\omega) = H(\omega)\left(\pi\delta(\omega+1) + \pi\delta(\omega-1)\right)$$

$$= \pi H(-1)\,\delta(\omega+1) + \pi H(1)\,\delta(\omega-1)$$

$$= \pi \frac{1}{1-j} e^{-j} \delta(\omega+1) + \pi \frac{1}{1+j} e^{j} \delta(\omega-1)$$

using (*) again we have that

$$y(t) = \frac{1}{2}\frac{1}{1-j} e^{-j} e^{-jt} + \frac{1}{2}\frac{1}{1+j} e^{j} e^{jt}$$

$$= \frac{1}{2}\left(\frac{1+j}{2} e^{-j(t+1)} + \frac{1-j}{2} e^{j(t+1)}\right)$$

Finally, noting that $\frac{1+j}{2} = \frac{\sqrt{2}}{2} e^{j\frac{\pi}{4}}$, and $\frac{1-j}{2} = \frac{\sqrt{2}}{2} e^{-j\frac{\pi}{4}}$ we have

$$y(t) = \frac{1}{2}\left(\frac{\sqrt{2}}{2} e^{j\frac{\pi}{4}} e^{-j(t+1)} + \frac{\sqrt{2}}{2} e^{-j\frac{\pi}{4}} e^{j(t+1)} \right)$$

$$= \frac{1}{2}\left(e^{-j(t+1-\pi/4)} + e^{j(t+1-\frac{\pi}{4})} \right) \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2} \cos\left(t+1-\frac{\pi}{4}\right)$$

Alternatively: recall the derivation done in class (see Prob I b))

$$y(t) = |H(1)| \cos\left(t + \measuredangle H(1)\right)$$

Now $H(1) = \frac{1}{1+j} e^{j} = \frac{1-j}{2} e^{j} = \frac{\sqrt{2}}{2} e^{-j\frac{\pi}{4}} e^{j} = \frac{\sqrt{2}}{2} e^{j(1-\frac{\pi}{4})}$

therefore $|H(1)| = \frac{\sqrt{2}}{2}$, and $\measuredangle H(1) = 1 - \frac{\pi}{4}$, and so

$$y(t) = \frac{\sqrt{2}}{2} \cos\left(t+1-\frac{\pi}{4}\right)$$

Problem IV [20pts]

Using the definition of the Fourier transform compute the the following:

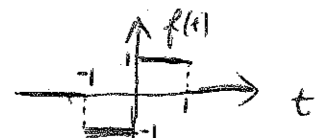
10 a) $\mathcal{F}\{e^{at}u(-t))\}$, where $a > 0$.

10 b) $\mathcal{F}\{-u(t+1) + 2u(t) - u(t-1)\}$.

a) $$\mathcal{F}\{e^{at}u(-t)\} = \int_{-\infty}^{\infty} e^{at}u(-t)e^{-j\omega t}dt = \int_{-\infty}^{0} e^{at}e^{-j\omega t}dt = \int_{-\infty}^{0} e^{(a-j\omega)t}dt$$

$$= \left[\frac{e^{(a-j\omega)t}}{a-j\omega}\right]_{-\infty}^{0} = \frac{1-0}{a-j\omega} = \frac{1}{a-j\omega}$$

b) Let $f(t) = -u(t+1) + 2u(t) - u(t-1)$



$$\mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt = \int_{-1}^{0}(-1)e^{-j\omega t}dt + \int_{0}^{1}(1)e^{-j\omega t}dt$$

case 1) $\boxed{\omega = 0}$ $$\mathcal{F}\{f(t)\} = \int_{-1}^{0} -1dt + \int_{0}^{1} 1\,dt = 0$$

case 2) $\boxed{\omega \neq 0}$ $$\mathcal{F}\{f(t)\} = -\left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-1}^{0} + \left[\frac{e^{-j\omega t}}{-j\omega}\right]_{0}^{1}$$

$$= -\frac{1-e^{+j\omega}}{-j\omega} + \frac{e^{-j\omega}-1}{-j\omega} = \frac{1-e^{j\omega}+1-e^{-j\omega}}{j\omega}$$

$$= \frac{2-2\cos(\omega)}{j\omega}$$

If you note that $\cos(2x) = \cos^2 x - \sin^2(x) = 1 - 2\sin^2 x$ we can replace $\cos(\omega) = \cos\left(2\frac{\omega}{2}\right) = 1 - 2\sin^2\frac{\omega}{2}$ and get

$$\mathcal{F}\{f(t)\} = \frac{2-2\cos(\omega)}{j\omega} = \frac{2-2\left(1-\sin^2\left(\frac{\omega}{2}\right)\right)}{j\omega} = \frac{2\sin^2\left(\frac{\omega}{2}\right)}{j\omega}$$

$$= \frac{\omega/2}{j} \; \frac{\sin^2(\omega/2)}{(\omega/2)^2} = \frac{1}{j}\,\frac{\omega}{2}\;\mathrm{sinc}^2\left(\frac{\omega}{2}\right) = -j\,\frac{\omega}{2}\,\mathrm{sinc}^2\left(\frac{\omega}{2}\right)$$

Problem V [25pts]

Let $f(t)$ be a signal with spectrum depicted below.

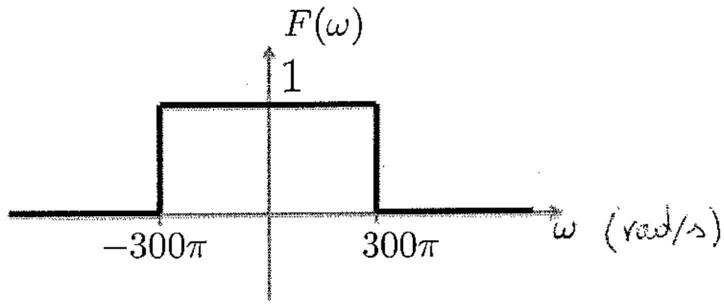


Figure 1: The spectrum of signal $f(t)$.

(8) a) What is the Nyquist sampling frequency (in Hz) of signal $f(t)$?

(5)+(5) b) Suppose the signal is sampled using a Dirac train as we did in class. The procedure is depicted in Figure 2.

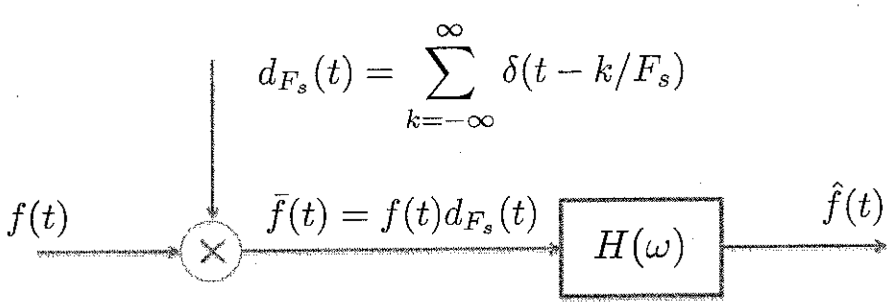


Figure 2: Sampling and reconstruction of signal $f(t)$.

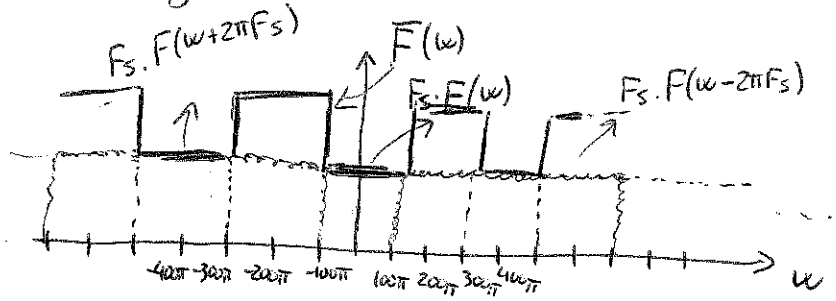
Consider two different scenarios for the sampling frequency: (i) $F_s = 200$Hz and (ii) $F_s = 400$Hz. The sampled signal $\bar{f}(t)$ is then passed through a **unit gain** ideal low-pass filter with bandwidth $F_s/2$ Hz. For each of the cases sketch the spectrum of the output signal $\hat{f}(t)$. Comment your results, and state if you can recover the original signal from the samples in each case.

(7) c) For case (i) in question b) compute the energy of the reconstructed signal $\hat{f}(t)$.

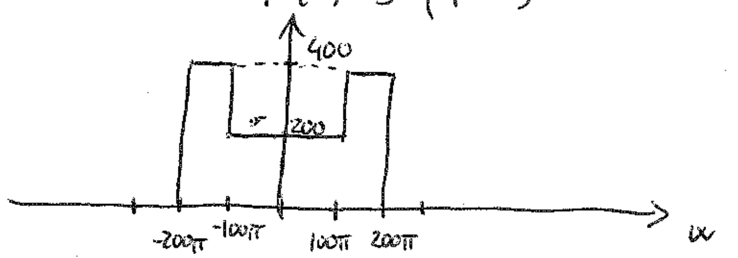
a) The signal bandwidth is 300π rad/s $= 150$ Hz, so the Nyquist frequency is $2 \times 150\,Hz = 300\,Hz = 600\pi$ rad/s

b) Recall that $\bar{F}(\omega) = F_S \sum_{k=-\infty}^{\infty} F(\omega - 2\pi F_S k)$, where $\bar{F}(\omega) = \mathcal{F}\{\bar{f}(t)\}$, and $F_S = \frac{1}{T_S}$.

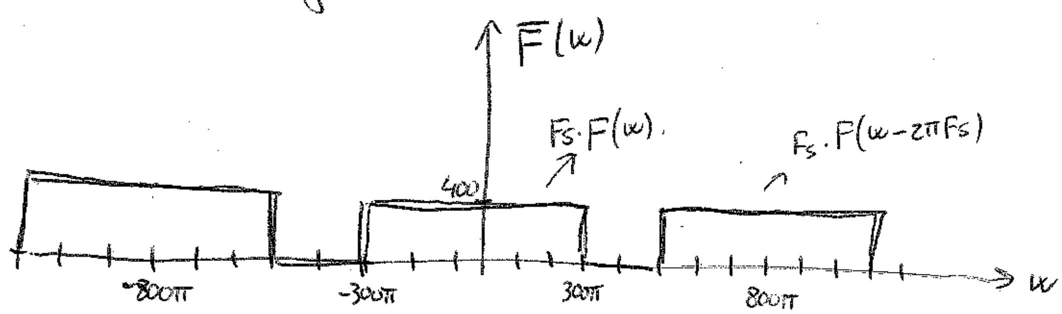
(i) $F_S = 200\,Hz = 400\pi$ rad/s

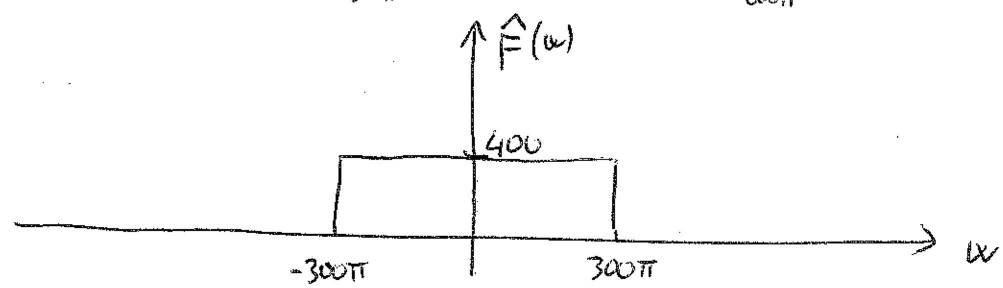


therefore $\hat{F}(\omega) = \mathcal{F}\{\hat{f}(t)\}$



(ii) $F_S = 400\,Hz = 800\pi$ rad/s





(iii) Recall Parseval's theorem

$$E_{\hat{f}} = \int_{-\infty}^{\infty} |\hat{f}(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{F}(\omega)|^2 d\omega =$$

$$= \frac{1}{2\pi}\left(400^2 \times 100\pi + 200^2 \times 200\pi + 400^2 \times 100\pi\right)$$

$$= \frac{200\pi}{2\pi}\left(400^2 + 200^2\right) = 100\left(160000 + 40000\right)$$

$$= 20000000 = 2 \times 10^7$$