Prob I:

a) FALSE: if $f(t)$ is odd then $f(t) = -f(-t)$ which means $-f(-t)$ is also odd

b) FALSE: Linearity and time-invariance are two distinct concepts. for example the system of question VI is linear but not time-invariant

c) FALSE: For most signals (the ones that satisfy DiRichelet conditions) the FS rep. is equal <u>except</u> at signal discontinuities

d) TRUE: $x(t)$ is the sum of two periodic signals, with periods respectively $\sqrt{2}$ and $\frac{\sqrt{2}}{2}$. Since $\text{lcm}\left(\sqrt{2}, \frac{\sqrt{2}}{2}\right) = \sqrt{2}$ the signal $x(t)$ is periodic with period $\sqrt{2}$.

e) FALSE: Most periodic signals have finite, but non-zero power, and therefore, since these are everlasting signals they have infinite energy.

Prob II:

$$
\begin{aligned}
f(t) &= -2\cos(\omega_0 t) + 2\sqrt{3}\sin(\omega_0 t + \pi/3) \\
&= -2\cos(\omega_0 t) + 2\sqrt{3}\sin(\omega_0 t)\cos\left(\tfrac{\pi}{3}\right) + 2\sqrt{3}\cos(\omega_0 t)\sin\left(\tfrac{\pi}{3}\right) \\
&= -2\cos(\omega_0 t) + 2\sqrt{3}\sin(\omega_0 t)\,\tfrac{1}{2} + 2\sqrt{3}\cos(\omega_0 t)\,\tfrac{\sqrt{3}}{2} \\
&= -2\cos(\omega_0 t) + \sqrt{3}\sin(\omega_0 t) + 3\cos(\omega_0 t) \\
&= \cos(\omega_0 t) + \sqrt{3}\sin(\omega_0 t) \\
&= a\cos(\omega_0 t) + b\sin(\omega_0 t) \qquad \text{, where } a = 1,\ b = \sqrt{3} \\
&= C\cos(\omega_0 t + \theta) \qquad \text{, where } C = \sqrt{a^2+b^2} = \sqrt{1+3} = 2 \\
&\qquad\qquad\qquad\qquad\qquad\quad\ \ \theta = \tan^{-1}\left(-\tfrac{b}{a}\right) = -\tan^{-1}(\sqrt{3}) = -\tfrac{\pi}{3} \\
&= 2\cos\left(\omega_0 t - \tfrac{\pi}{3}\right)
\end{aligned}
$$

Prob III:

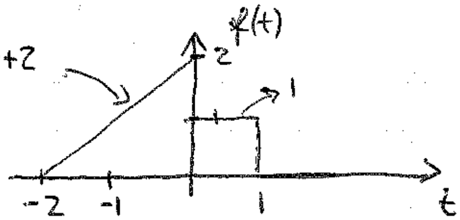
a) $$\delta(t-3)\,\frac{4-jt^2}{2t} = \delta(t-3)\,\frac{4-j3^2}{2.3} = \delta(t-3)\cdot\frac{4-j9}{6} = \left(\frac{2}{3}-j\frac{3}{2}\right)\delta(t-3)$$

b) $$\int_{-\infty}^{\infty}\sqrt{t}\cos(3\pi t)\,\delta(1-t)\,dt = \int_{-\infty}^{\infty}\sqrt{t}\cos(3\pi t)\,\delta(t-1)\,dt$$

(sampling property) $$= \sqrt{1}\cos(3\pi) = \cos(+\pi) = -1$$

Prob IV:

a)



so $f(t) = (t+2)\big(u(t+2)-u(t)\big) + 1.\big(u(t)-u(t-1)\big)$

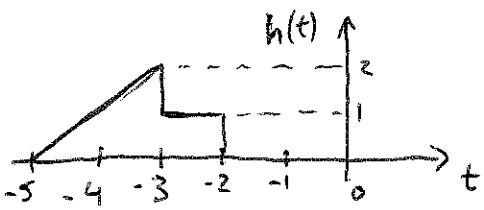
$= (t+2)\,u(t+2) - (t+1)\,u(t) - u(t-1)$

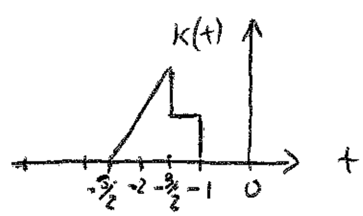
b) Let's do it in steps:

(time-shift) $h(t) = f(t+3)$

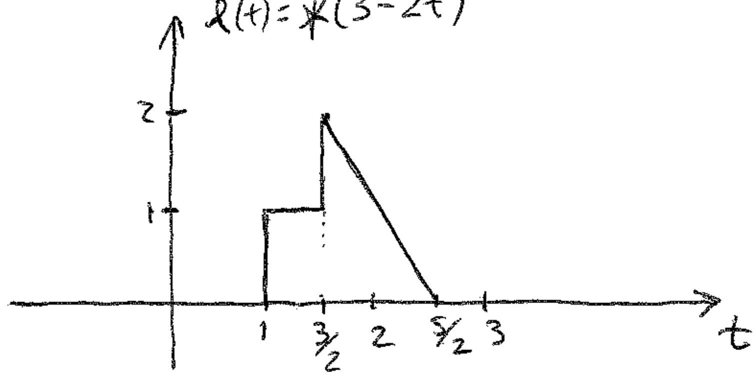
(time-scaling) $k(t) = h(2t) = f\big((2t)+3\big) = f(2t+3)$

(time-reversal) $l(t) = k(-t) = f\big(2(-t)+3\big) = f(-2t+3)$





$l(t) = f(3-2t)$



Prob V: a) $x(t)$ is clearly the sum of two pulses

$$x(t) = \big(u(t) - u(t-1)\big) + 2\big(u(t-1) - u(t-2)\big)$$
$$= p(t) + 2p(t-1)$$

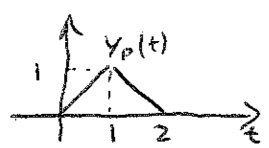
therefore $x(t) = a\,p(t) + b\,p(t-t_0)$ with $a=1,\ b=2,\ t_0=1$

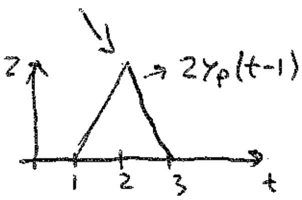
b) Since the system is linear and time invariant we can use these properties to compute the response based on $y_p(t)$.

$$H\{x(t)\} = H\{a\,p(t) + b\,p(t-t_0)\}$$

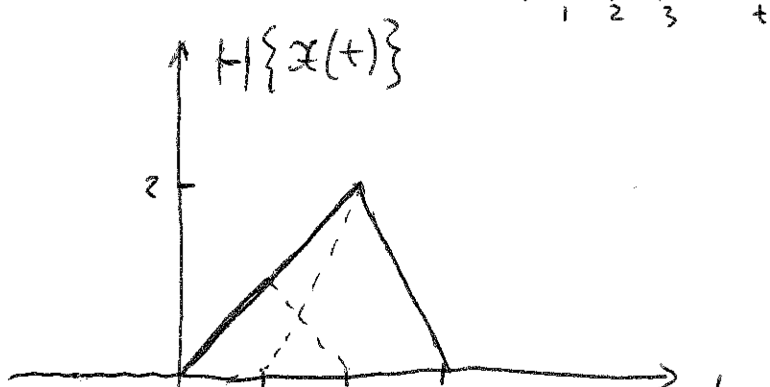
(linearity) $$= a\,H\{p(t)\} + b\,H\{p(t-t_0)\}$$

(TI) $$= \underbrace{a\,y_p(t)}\ + \underbrace{b\,y_p(t-t_0)}$$





so



$$H\{x(t)\} = t\big(u(t) - u(t-2)\big) + (6-2t)\big(u(t-2) - u(t-3)\big)$$

Prob VI: To check if the system is linear we need to check if for any inputs $f_1(t)$ & $f_2(t)$ and scalars a, b we have $H\{a f_1(t) + b f_2(t)\} = a H\{f_1(t)\} + b H\{f_2(t)\}$

Let f_1, f_2 be two inputs, and y_1, y_2 be the corresponding outputs, respectively. This means that

$$y_1(t) = H\{f_1(t)\} \qquad y_1(t) - \tfrac{1}{2} y_1(t-1) = t f_1(t) \quad \forall t$$

$$y_2(t) = H\{f_2(t)\} \qquad y_2(t) - \tfrac{1}{2} y_2(t-1) = t f_2(t) \quad \forall t$$

Let $f(t) = a f_1(t) + b f_2(t)$ if the system is linear then

$$H\{f(t)\} = y(t) = a y_1(t) + b y_2(t)$$

that is $y(t) - \frac{1}{2} y(t-1) = t f(t) \quad \forall t$

Let's check this:

$$y(t) - \tfrac{1}{2} y(t-1) = t f(t) \iff a y_1(t) + b y_2(t) - \tfrac{1}{2}\left(a y_1(t-1) + b y_2(t-1)\right) = t\,(a f_1(t) + b f_2(t))$$

$$\iff a y_1(t) - a \tfrac{1}{2} y_1(t-1) + b y_2(t) - b \tfrac{1}{2} y_2(t-1) = a t f_1(t) + b t f_2(t)$$

$$\iff a\left(y_1(t) - \tfrac{1}{2} y_1(t-1)\right) + b\left(y_2(t) - \tfrac{1}{2} y_2(t-1)\right) = a t f_1(t) + b t f_2(t)$$

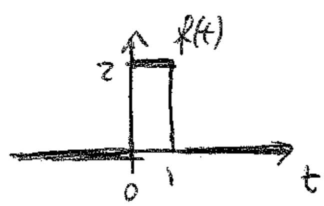
$$\iff a\Big(\underbrace{y_1(t) - \tfrac{1}{2} y_1(t-1) - t f_1(t)}_{=0 \ \forall t}\Big) + b\Big(\underbrace{y_2(t) - \tfrac{1}{2} y_2(t-1) - t f_2(t)}_{=0 \ \forall t}\Big) = 0$$

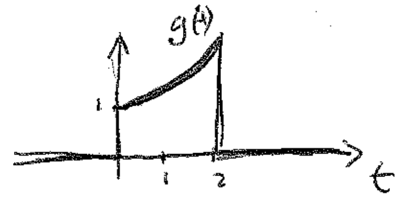
since $y_1(t) = H\{f_1(t)\}$ $\qquad y_2(t) = H\{f_2(t)\}$

So the statement is true for all t, therefore the system is linear.
It is easy to check that the system is time-varying as well.

Prob VII:

a)



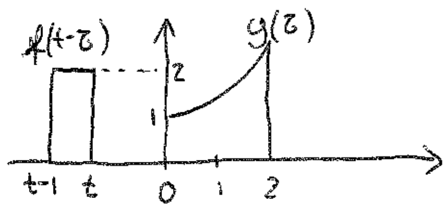


b) let's use the graphical method.

$$(f*g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau = (g*f)(t) = \int_{-\infty}^{\infty} g(\tau)f(t-\tau)d\tau$$

(f is easier to "flip around")

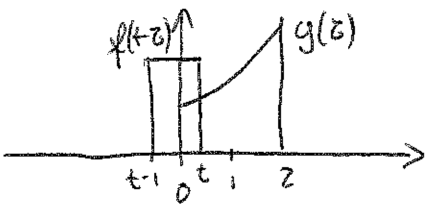
Case 1)



$t<0 \Rightarrow$

$(f*g)(t)=0$

Case 2)



$t\geq 0$ & $t-1<0 \iff$

$0\leq t<1$

$$(f*g)(t) = \int_0^t g(\tau)f(t-\tau)d\tau =$$

$$= \int_0^t e^{\tau}.2d\tau = \left[2e^{\tau}\right]_0^t = 2e^t-2 = 2(e^t-1)$$

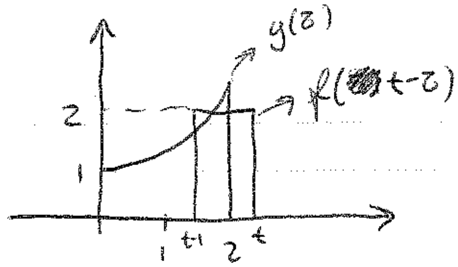
Case 3)

$t-1\geq 0$ & $t<2 \iff$

$1\leq t<2$

$$(f*g)(t) = \int_{t-1}^{t} e^{\tau}.2d\tau = \left[2e^{\tau}\right]_{t-1}^{t} = 2e^t - 2e^{t-1} = 2e^t(1-e^{-1})$$

Case 4)



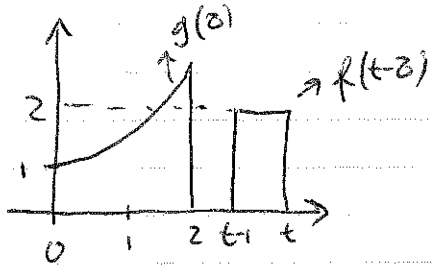
$t \geq 2$ & $t-1 < 2 \iff$

$2 \leq t < 3$

$$(f*g)(t) = \int_{t-1}^{2} 2e^{\tau} d\tau = \left[2e^{\tau}\right]_{t-1}^{2} = 2e^{2} - 2e^{t-1}$$

$$= 2\left(e^{2} - e^{t-1}\right)$$

Case 5)



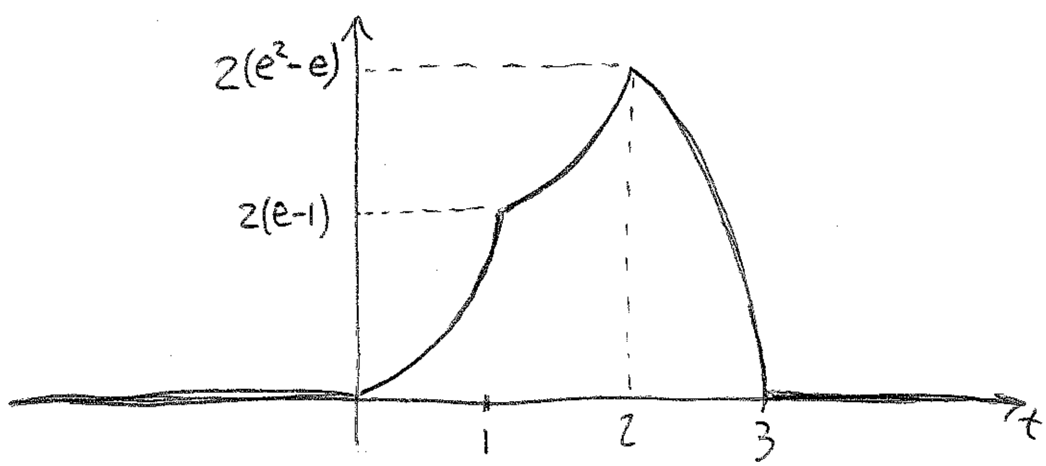
$t-1 \geq 2 \iff t \geq 3$

$(f*g)(t) = 0$

Putting all these togthether

$$(f*g)(t) = \begin{cases} 0 & , \ t<0 \\ 2(e^{t}-1) & , \ 0 \leq t < 1 \\ 2e^{t}(1-e^{-1}) & , \ 1 \leq t < 2 \\ 2(e^{2}-e^{t-1}) & , \ 2 \leq t < 3 \\ 0 & , \ t \geq 3 \end{cases}$$

Although it was not asked we can sketch this signal



Prob VIII: a) Just by inspection we see that $\omega_0 = 1 \Rightarrow T_0 = 2\pi$

Another way is to notice the $x(t)$ is the sum of periodic signals with period $\frac{2\pi}{3}$ and $\frac{2\pi}{2}$, therefore it is periodic with period

$$\text{l.c.m.}\left(\frac{2\pi}{3}, \frac{2\pi}{2}\right) = 2\pi$$

period $T_0 = 2\pi$, fundamental freq. $\omega_0 = 1$

b) Note that $x(t)$ is already written in the exponential Fourier ~~[illegible]~~ series form

$$x(t) = \sum_{k=-\infty}^{\infty} D_k e^{j\omega_0 k t}$$

where $D_{-3} = 2+2j$, $D_{-2} = -3j$, $D_0 = 5$, $D_2 = 3j$, $D_3 = 2-2j$ and $D_k = 0$ for all other k. Now notice $D_k = D_{-k}^*$ for all k implying the signal is real. To get the trigonometric FS representation we can either use the formulas in class, or do it directly

$$x(t) = (2+2j)e^{-j3t} + (2-2j)e^{j3t} + (-3j)e^{-j2t} + 3je^{j2t} + 5$$

$$= 2\sqrt{2}e^{j\frac{\pi}{4}}e^{-j3t} + 2\sqrt{2}e^{-j\frac{\pi}{4}}e^{j3t} + 3e^{-j\frac{\pi}{2}}e^{-j2t} + 3e^{j\frac{\pi}{2}}e^{j2t} + 5$$

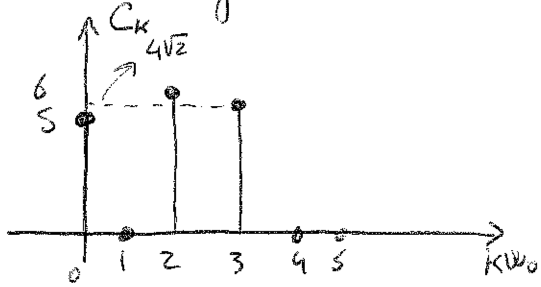
$$= 2\sqrt{2}\left(e^{j(3t-\frac{\pi}{4})} + e^{-j(3t-\frac{\pi}{4})}\right) + 3\left(e^{j(2t+\frac{\pi}{2})} + e^{-j(2t+\frac{\pi}{2})}\right) + 5$$

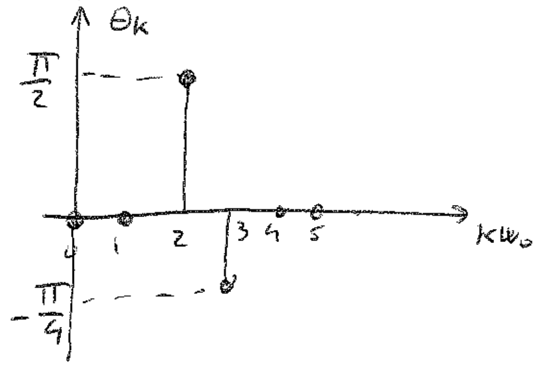
$$= 2\sqrt{2}.2\cos\left(3t-\frac{\pi}{4}\right) + 3.2.\cos\left(2t+\frac{\pi}{2}\right) + 5$$

$$= 4\sqrt{2}\cos\left(3t-\frac{\pi}{4}\right) + 6\cos\left(2t+\frac{\pi}{2}\right) + 5$$

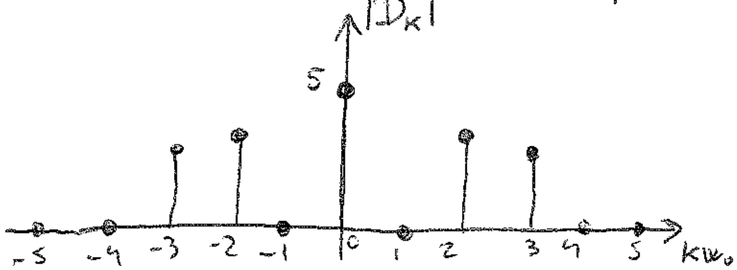
Therefore $$x(t) = 5 + 6\cos\left(2t+\frac{\pi}{2}\right) + 4\sqrt{2}\cos\left(3t-\frac{\pi}{4}\right)$$

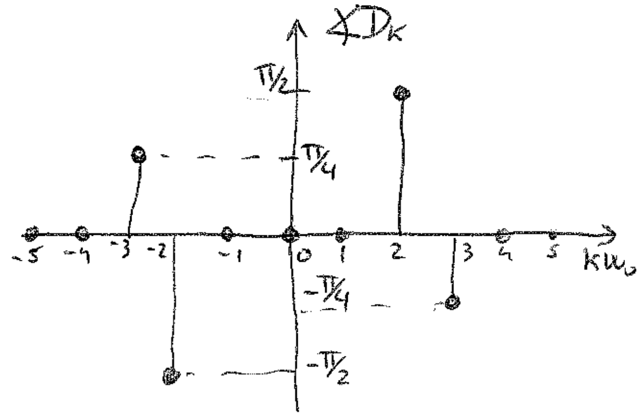
c) Trigonometric Fourier spectra





Exp. Fourier spectra





d) $P_x = \sum_{k=-\infty}^{\infty} |D_k|^2$ by Parseval's theorem, therefore

$$P_x = 5^2 + 2\left(\sqrt{2^2+2^2}\right)^2 + 2\, 3^2 = 25 + 16 + 18 = 59$$