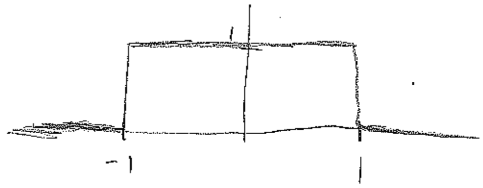
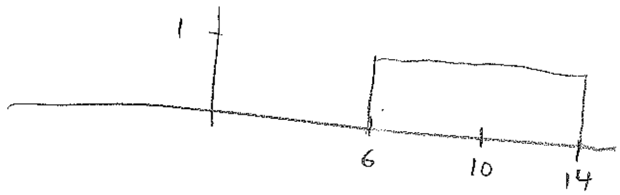
Homework 8 Solutions

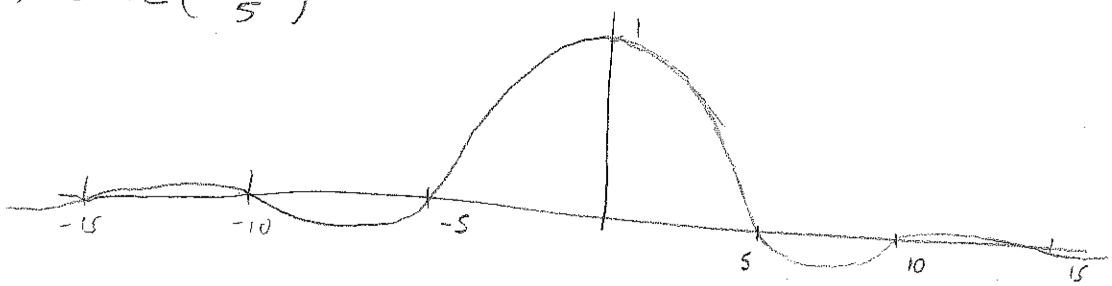
8.1) a) $\tau = 2$



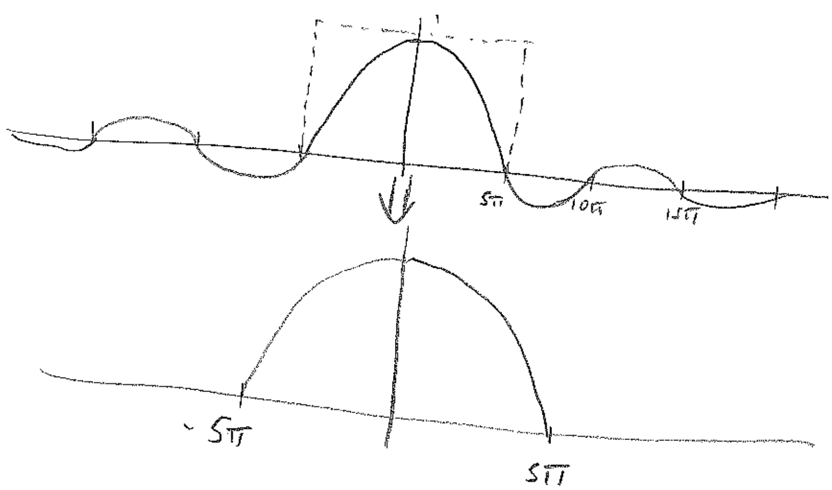
c) $\delta(t-10) * rect\left(\frac{t}{8}\right) = rect\left(\frac{t-10}{8}\right)$



d) $sinc\left(\frac{\pi\omega}{5}\right)$



f)



8.2 a) $F(\omega) = \underbrace{[\delta(\omega-4) + \delta(\omega+4)]}_{X_1(\omega)} * \underbrace{\mathrm{rect}\left(\frac{\omega}{2}\right)}_{X_2(\omega)}$

$\mathcal{F}^{-1}\{X_1(\omega) * X_2(\omega)\} = 2\pi\, x_1(t)\, x_2(t)$

$x_1(t) = \frac{1}{\pi}\cos(4t)$

$x_2(t) = \frac{1}{\pi}\mathrm{sinc}(t)$

$f(t) = \frac{2}{\pi}\cos(4t)\,\mathrm{sinc}(t)$

b) $F(\omega) = [\underbrace{\delta(\omega-4) + \delta(\omega+4)}_{X_1(\omega)}] * \underbrace{\Delta\left(\frac{\omega}{4}\right)}_{X_2(\omega)}$

$\mathcal{F}^{-1}\{X_1(\omega) * X_2(\omega)\} = 2\pi\, x_1(t)\, x_2(t)$

$x_1(t) = \frac{1}{\pi}\cos(4t)$

$x_2(t) = \frac{1}{\pi}\mathrm{sinc}^2(t)$

$f(t) = \frac{2}{\pi}\cos(4t)\,\mathrm{sinc}^2(t)$

8.3 $x(t)$ is real $\quad x(t) \overset{\mathcal{F}}{\Leftrightarrow} X(\omega) \quad x(t) = x^*(t)$

a) $x(t)$ is even $x(t) = x(-t)$

If $x(t)$ is real $X^*(\omega) = X(-\omega)$ Conjugation Property

If $x(t)$ is even $\mathcal{F}\{x(t)\} = \mathcal{F}\{x(-t)\}$ Time-Reversal Property

where $\mathcal{F}\{x(t)\} = X(\omega)$ and $\mathcal{F}\{x(-t)\} = X(-\omega)$

So $X(\omega) = X(-\omega) = X^*(\omega)$

$X(\omega) = X(-\omega) \Longrightarrow X(\omega)$ is even

$X(\omega) = X^*(\omega) \Longrightarrow X(\omega)$ is real //

b) $x(t)$ is odd $x(t) = -x(-t)$

If $x(t)$ is real $\underline{X^*(\omega) = X(-\omega)}$ Conjugation Property

If $x(t)$ is odd $\mathcal{F}\{x(t)\} = -\mathcal{F}\{x(-t)\}$

where $\mathcal{F}\{x(t)\} = X(\omega)$ and $\underbrace{-\mathcal{F}\{x(-t)\} = -X(-\omega)}$

so $X(\omega) = -X(-\omega) = -X^*(\omega)$ Time Reversal Property

$X(\omega) = -X(-\omega) \Longrightarrow X(\omega)$ is odd

$X(\omega) = -X^*(\omega) \Longrightarrow X(\omega)$ is imaginary

8.4 $x(t) = rect\left(\frac{t+T/2}{T}\right) - rect\left(\frac{t-T/2}{T}\right)$

a) $$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-T}^{0} e^{-j\omega t} dt - \int_{0}^{T} e^{-j\omega t} dt$$

$$= \frac{-1}{j\omega} e^{-j\omega t} \Big|_{-T}^{0} - \frac{-1}{j\omega} e^{-j\omega t} \Big|_{0}^{T}$$

$$= \frac{-1}{j\omega}\left(1 - e^{j\omega T}\right) + \frac{1}{j\omega}\left(e^{-j\omega T} - 1\right)$$

$$= \frac{1}{j\omega}\left(e^{j\omega t} + e^{-j\omega t} - 2\right)$$

$$= \frac{2\cos(\omega t) - 2}{j\omega}$$

b) $$x(t - t_0) \overset{\mathcal{F}}{\Longleftrightarrow} X(\omega) e^{-j\omega t_0}$$

$$X(\omega) = T\,\text{sinc}\left(\frac{\omega T}{2}\right) \cdot \left(e^{j\omega T/2} - e^{-j\omega T/2}\right)$$

$$= 2jT\,\text{sinc}\left(\frac{\omega T}{2}\right) \sin\left(\frac{\omega T}{2}\right)$$

$$= \frac{-4\sin^2\left(\frac{\omega T}{2}\right)}{j\omega}$$

$$= \frac{2\cos(\omega t) - 2}{j\omega}$$

c) $\frac{df(t)}{dt} \overset{\mathcal{F}}{\Longleftrightarrow} j\omega F(\omega)$

$$f(t) = u(t+T) - 2u(t) + u(t-T)$$

$$\frac{df(t)}{dt} = \delta(t+T) - 2\delta(t) + \delta(t-T)$$

$$\mathcal{F}\left\{\frac{df(t)}{dt}\right\} = e^{j\omega T} - 2 + e^{-j\omega T}$$

$$F(\omega) = \frac{1}{j\omega}\mathcal{F}\left\{\frac{df(t)}{dt}\right\}$$

$$= \frac{1}{j\omega}\left(e^{j\omega T} + e^{-j\omega T} - 2\right)$$

$$= \frac{(2\cos(\omega T) - 2)}{j\omega}$$

8.5) a) Prove $-jt\,f(t) \Longleftrightarrow \frac{d}{d\omega}F(\omega)$

$$\frac{d}{d\omega}F(\omega) = \frac{d}{d\omega}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} f(t)\left(\frac{d}{d\omega}e^{-j\omega t}\right)dt$$

$$= \int_{-\infty}^{\infty} \underbrace{\left[-jt\,f(t)\right]}_{g(t)}e^{-j\omega t}\,dt$$

$$= \mathcal{F}\{g(t)\}$$

$$\frac{d}{d\omega}F(\omega) \Longleftrightarrow g(t) = -jt\,f(t)$$

b) $f(t) = te^{-at}u(t)$

$$u(t)e^{-at} \overset{\mathcal{F}}{\Longleftrightarrow} \frac{1}{a+j\omega}$$

$$(j)(-jt)e^{-at}u(t) \Longleftrightarrow j\frac{d}{d\omega}\mathcal{F}\{e^{-at}u(t)\}$$

$$= j\frac{d}{d\omega}(a+j\omega)^{-1}$$

$$= \frac{1}{(a+j\omega)^2}$$