Homework 7 Solutions

7.11 $F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt$

a) $f(t) = e^{-at}[u(t) - u(t-T)]$

$$F(\omega) = \int_0^T e^{-at} e^{-j\omega t}\, dt$$

$$= \int_0^T e^{-(a+j\omega)t}\, dt$$

$$= \frac{-1}{a+j\omega} e^{-(a+j\omega)t}\Big|_0^T$$

$$= \frac{1 - e^{-(a+j\omega)T}}{a+j\omega}$$

b) $f(t) = e^{at}[u(t) - u(t-T)]$

$$F(\omega) = \int_0^T e^{at} e^{-j\omega t}\, dt$$

$$= \int_0^T e^{-(j\omega - a)t}\, dt$$

$$= \frac{1 - e^{-(j\omega - a)T}}{-a + j\omega}$$

7.2 a) $F(\omega) = \omega^2$ for $-\omega_0 \le \omega \le \omega_0$ else 0

$$f(t) = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} \omega^2 e^{j\omega t}\,d\omega \qquad u = \omega^2 \quad v = \frac{1}{jt} e^{j\omega t} \qquad du = 2\omega\,d\omega \quad dv = e^{j\omega t}$$

$$= \frac{\omega^2 e^{j\omega t}}{2\pi j t}\Big|_{-\omega_0}^{\omega_0} - \int_{-\omega_0}^{\omega_0} \frac{\omega}{\pi j t} e^{j\omega t}\,d\omega \qquad u = \omega \quad v = \frac{-1}{\pi t^2} e^{j\omega t} \qquad du = d\omega \quad dv = \frac{1}{\pi j t} e^{j\omega t}$$

$$= \frac{1}{2\pi j t}\left(\omega_0^2 e^{j\omega_0 t} - \omega_0^2 e^{-j\omega_0 t}\right) + \frac{\omega e^{j\omega t}}{\pi t^2}\Big|_{-\omega_0}^{\omega_0} - \frac{1}{\pi t^2}\int_{-\omega_0}^{\omega_0} e^{j\omega t}\,d\omega$$

$$= \frac{\omega_0^2}{\pi t}\sin(\omega_0 t) + \frac{\omega_0}{\pi t^2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) - \frac{1}{j\pi t^3} e^{j\omega t}\Big|_{-\omega_0}^{\omega_0}$$

$$= \frac{\omega_0^2}{\pi t}\sin(\omega_0 t) + \frac{2\omega_0}{\pi t^2}\cos(\omega_0 t) - \frac{2}{\pi t^3}\sin(\omega_0 t)$$

$$= \frac{2\omega_0}{\pi t^2}\cos(\omega_0 t) + \left(\frac{\omega_0^2}{\pi t} - \frac{2}{\pi t^3}\right)\sin(\omega_0 t)$$

b) $F(\omega) = u(\omega+2) - u(\omega-2) + u(\omega+1) - u(\omega-1)$

$$f(t) = \frac{1}{2\pi}\int_{-2}^{2} e^{j\omega t}\,d\omega + \frac{1}{2\pi}\int_{-1}^{1} e^{j\omega t}\,d\omega$$

$$= \frac{e^{j\omega t}}{2\pi j t}\Big|_{-2}^{2} + \frac{e^{j\omega t}}{2\pi j t}\Big|_{-1}^{1}$$

$$= \frac{e^{2jt} - e^{-2jt}}{2\pi j t} + \frac{e^{jt} - e^{-jt}}{2\pi j t}$$

$$= \frac{\sin(2t)}{\pi t} + \frac{\sin(t)}{\pi t}$$

$$= \frac{2}{\pi}\operatorname{sinc}(t) + \frac{1}{\pi}\operatorname{sinc}(t)$$

7.3 a) $F(\omega) = e^{-j\omega t_0}\left[u(\omega+\omega_0) - u(\omega-\omega_0)\right]$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{-j\omega t_0}\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{(t-t_0)j\omega}\, d\omega$$

$$= \frac{1}{2\pi(t-t_0)j}\, e^{(t-t_0)j\omega}\Big|_{-\omega_0}^{\omega_0}$$

$$= \frac{e^{(t-t_0)j\omega_0} - e^{-(t-t_0)j\omega_0}}{2\pi(t-t_0)j}$$

$$= \frac{\sin((t-t_0)\omega_0)}{2\pi(t-t_0)}$$

$$= \frac{\omega_0}{2\pi}\,\mathrm{sinc}\left((t-t_0)\omega_0\right)$$

b) $F(\omega) = e^{\frac{\pi}{2}j}\left[u(\omega+\omega_o) - u(\omega)\right] + e^{-\frac{\pi}{2}j}\left[u(\omega) - u(\omega-\omega_o)\right]$

$$f(t) = \frac{1}{2\pi}\int_{-\omega_o}^{0} e^{\frac{\pi}{2}j} e^{j\omega t}\, d\omega + \frac{1}{2\pi}\int_{0}^{\omega_o} e^{-\frac{\pi}{2}j} e^{j\omega t}\, d\omega$$

$$= \frac{j}{2\pi}\int_{-\omega_o}^{0} e^{j\omega t} d\omega + \frac{-j}{2\pi}\int_{0}^{\omega_o} e^{j\omega t}\, d\omega$$

$$= \frac{\left(1 - e^{-j\omega_o t}\right)}{2\pi t} - \frac{\left(e^{j\omega_o t} - 1\right)}{2\pi t}$$

$$= \frac{1}{2\pi t}\left(2 - e^{j\omega_o t} - e^{-j\omega_o t}\right)$$

$$= \frac{1}{\pi t} - \frac{\cos(\omega_o t)}{\pi t}$$

7.4) Show $f(t+T)+f(t-T) \overset{\mathcal{F}}{\Longleftrightarrow} 2F(\omega)\cos(T\omega)$ when $f(t) \overset{\mathcal{F}}{\Leftrightarrow} F(\omega)$

$$F_T = \int_{-\infty}^{\infty} f(t+T)e^{-j\omega t}\,dt \qquad \begin{array}{l}\lambda = t+T \\ d\lambda = dt\end{array}$$

$$= \int_{-\infty}^{\infty} f(\lambda)e^{-j\omega(\lambda-T)}\,d\lambda$$

$$= e^{j\omega T}\int_{-\infty}^{\infty} f(\lambda)e^{-j\omega\lambda}\,d\lambda$$

$$= e^{j\omega T}F(\omega)$$

$$F_{-T} = \int_{-\infty}^{\infty} f(t-T)e^{-j\omega t}\,dt \qquad \begin{array}{l}\lambda = t-T \\ d\lambda = dt\end{array}$$

$$= \int_{-\infty}^{\infty} f(\lambda)e^{-j\omega(\lambda+T)}\,d\lambda$$

$$= e^{-j\omega T}\int_{-\infty}^{\infty} f(\lambda)e^{-j\omega\lambda}\,d\lambda$$

$$= e^{-j\omega T}F(\omega)$$

$$\mathcal{F}\{f(t+T)+f(t-T)\} = \mathcal{F}\{f(t+T)\} + \mathcal{F}\{f(t-T)\}$$

$$= e^{j\omega T}F(\omega) + e^{-j\omega T}F(\omega)$$

$$= \frac{2F(\omega)}{2}\left(e^{j\omega T} + e^{-j\omega T}\right)$$

$$= 2F(\omega)\cos(\omega T)$$

//

a) $f(t) = \text{rect}\left(\frac{t}{2}\right) = \begin{cases} 1 & |t| < 2 \\ 0 & \text{otherwise} \end{cases}$ $\quad T = 3$

$$F(\omega) = 2\,\text{sinc}(\omega)$$

$$\mathcal{F}\{f(t+T) + f(t-T)\} = 4\,\text{sinc}(\omega)\cos(3\omega)$$

b) $f(t) = \Delta\left(\frac{t}{2}\right) = \begin{cases} 0 & \text{for } |t| < 1 \\ 1 - |t| & \text{otherwise} \end{cases}$

$$F(\omega) = \text{sinc}^2\left(\frac{\omega}{2}\right)$$

$$\mathcal{F}\{f(t+T) + f(t-T)\} = 2\,\text{sinc}^2\left(\frac{\omega}{2}\right)\cos(3\omega)$$

Problem 7.5 part a,b,c

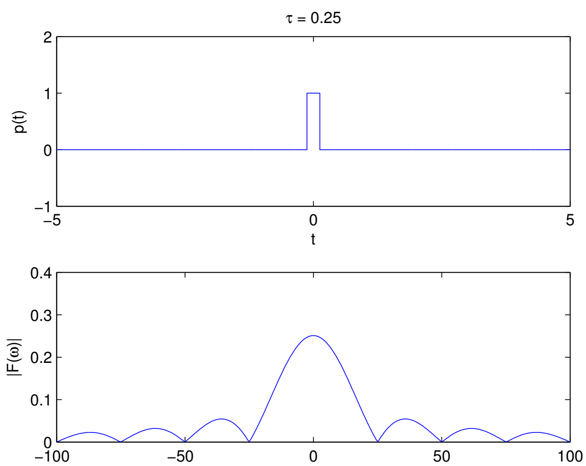
The Fourier transform has magnitude 0 at integer multiples of 2π other than $\omega = 0$. Because

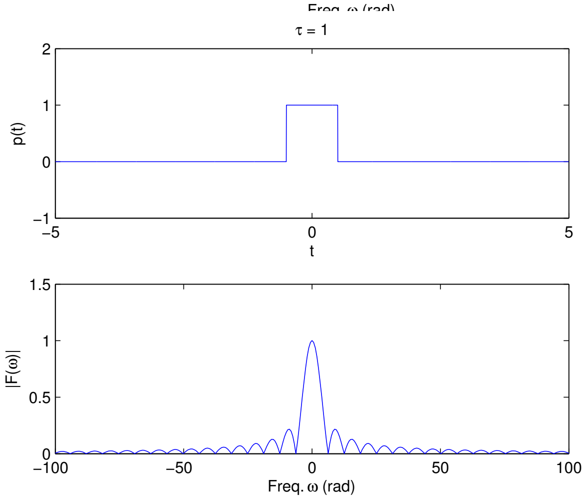
$$\text{rect}\left(\frac{t}{\tau}\right) \iff \tau \text{sinc}\left(\frac{\omega\tau}{2}\right) = 2\frac{\sin\left(\frac{\omega\tau}{2}\right)}{\omega}$$

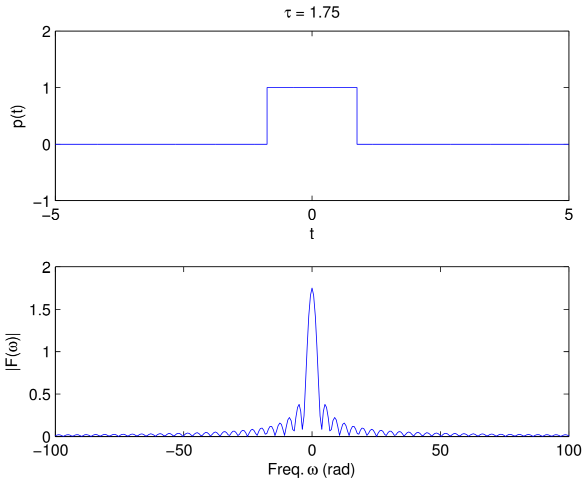
and $\sin\left(\frac{\omega\tau}{2}\right) = 0$ for $\omega = \frac{2\pi}{\tau}k$ for all integer k. For $\omega = 0$:

$$\begin{aligned}
\tau \text{sinc}\left(\frac{\omega\tau}{2}\right)\bigg|_{\omega=0} &= 2\frac{\sin\left(\frac{\omega\tau}{2}\right)}{\omega}\bigg|_{\omega=0} \\
&= \frac{2}{\omega}\left(\left(\frac{\omega\tau}{2}\right) - \frac{\left(\frac{\omega\tau}{2}\right)^3}{3!} + \frac{\left(\frac{\omega\tau}{2}\right)^5}{5!} - \dots\right)\bigg|_{\omega=0} \\
&= \tau - \frac{\tau\left(\frac{\omega\tau}{2}\right)^2}{3!} + \frac{\tau\left(\frac{\omega\tau}{2}\right)^4}{5!} - \dots\bigg|_{\omega=0} \\
&= \tau
\end{aligned}$$

So yes, this makes sense.
Plots for various values of τ:







Problem 7.5 part d

I chose $f0 = 300$Hz and $f1 = 600$Hz. As you kill the higher frequencies (decreasing $f1$) the speech becomes less clear and the “t” sound gets less pronounced because its very short in time, meaning very wide in freqeuncy. When you kill a lot of frequencies, you can’t reconstruct this short sound accurately.