Homework 6 Solutions

1) $$D_k = \frac{1}{T_0}\int_{T_0} f(t)e^{-jk\omega_0 t}\,dt \qquad f(t) = \begin{cases} f(t) & 0 \le t < T_0/2 \\ -f(t) & T_0/2 \le t < T_0 \end{cases}$$

$$= \frac{1}{T_0}\int_0^{T_0/2} f(t)e^{-jk\omega_0 t}\,dt + \frac{1}{T_0}\int_{T_0/2}^{T_0} f(t)e^{-jk\omega_0 t}\,dt \qquad \begin{matrix} \tau = t + T_0/2 \\ d\tau = dt \end{matrix}$$

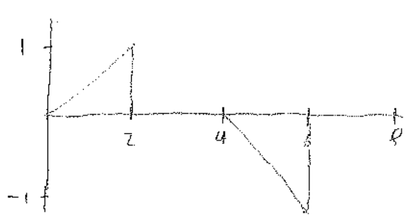
$$= \frac{1}{T_0}\int_0^{T_0/2} f(t)e^{-jk\omega_0 t}\,dt + \frac{1}{T_0}\int_{T_0}^{3T_0/2} f(\tau - T_0/2)e^{-jk\omega_0(\tau - T_0/2)}\,d\tau$$

$$= \frac{1}{T_0}\int_0^{T_0/2} f(t)e^{-jk\omega_0 t}\,dt - \frac{e^{jk\omega_0 \frac{T_0}{2}}}{T_0}\int_0^{T_0/2} f(t)e^{-jk\omega_0 t}\,dt \qquad \omega_0 = \frac{2\pi}{T_0}$$

$$= \frac{(1 - e^{jk\pi})}{T_0}\int_0^{T/2} f(t)e^{-jk\omega_0 t}\,dt$$

$$= \frac{(1 - (-1)^k)}{T_0}\int_0^{T_0/2} f(t)e^{-jk\omega_0 t}\,dt$$

$$= \begin{cases} 0 & k = 0,\ \text{even} \\ \frac{2}{T_0}\int_0^{T_0/2} f(t)e^{-jk\omega_0 t}\,dt & k\ \text{odd} \end{cases}$$

a) 

$T_0 = 8$

$\omega_0 = \frac{\pi}{4}$

$$D_k = \frac{1}{4}\int_0^2 \left(\frac{t}{2}\right)e^{-jk\frac{\pi}{4}t}\,dt$$

$$= \frac{1}{8}\int_0^2 t e^{-jk\frac{\pi}{4}t}\,dt$$

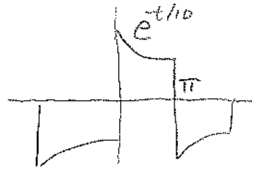
$u = t \qquad v = \frac{-1}{jk\pi/4}e^{-jk\frac{\pi}{4}t}$

$du = dt \qquad dv = e^{-jk\frac{\pi}{4}t}$

$$D_k = \frac{1}{8}\left[\frac{-t}{jk\pi/4}e^{-jk\frac{\pi}{4}t}\Big|_0^2 + \frac{1}{jk\pi/4}\int_0^2 e^{-jk\frac{\pi}{4}t}\,dt\right]$$

$$D_k = \frac{1}{8}\left[\frac{4j}{k\pi}\left(2j^{-k}\right) + \frac{16}{k^2\pi^2}e^{-jk\frac{\pi}{4}t}\Big|_0^2\right] = \frac{j^{(-k+1)}}{k\pi} + \frac{2}{k^2\pi^2}\left(j^{(-k)} - 1\right) \quad \text{for odd } k \text{ only}$$

b)



$T_0 = 2\pi$ $\quad \omega_0 = 1$

$$D_n = \begin{cases} 0 & n=0,\ \text{even} \\ \dfrac{2}{T_0}\displaystyle\int_0^{T_0/2} f(t)\, e^{-jn\omega_0 t}\, dt \end{cases}$$

$$D_k = \frac{1}{\pi}\int_0^{\pi} e^{-t/10} e^{-jkt}\, dt = \frac{1}{\pi}\int_0^{\pi} e^{-(jk+\frac{1}{10})t}\, dt = \frac{-1}{\pi(jk+1/10)} e^{-(jk+1/10)t}\Big|_0^{\pi}$$

$$= \frac{-1}{\pi(jk+1/10)}\left[e^{-jk\pi} e^{-\pi/10} - 1\right] = \frac{1 - e^{-\pi/10}(-1)^k}{\pi(jk+1/10)} \quad \text{for } k \text{ odd}$$

6.2/ $f(t) = t^2$ on $0 \le t \le 1$

a) $\omega_0 = \pi/2$ $\quad T_0 = 4$ $\quad$ only cosine terms

To have a series w/ just cosines, it must be even.

$f(t)$ is even on interval $-2 \le t \le 2$

$$a_0 = \frac{1}{4}\int_{-2}^{2} t^2\, dt = \frac{1}{2}\int_0^2 t^2\, dt = \frac{t^3}{6}\Big|_0^2 = \frac{4}{3}$$

$$a_n = \frac{1}{2}\int_{-2}^{2} t^2 \cos(n\tfrac{\pi}{2}t)\, dt = \int_0^2 t^2 \cos(n\tfrac{\pi}{2}t)\, dt \qquad \begin{matrix} u = t^2 & v = \frac{2}{n\pi}\sin(\frac{n\pi}{2}t) \\ du = 2t & dv = \cos(\frac{n\pi}{2}t) \end{matrix}$$

$$= \frac{2t^2}{n\pi}\cancelto{0}{\sin(\tfrac{n\pi}{2}t)}\Big|_0^2 - \frac{4}{n\pi}\int_0^2 t\sin(\tfrac{n\pi}{2}t)\, dt \qquad \begin{matrix} u = t & v = \frac{-2}{n\pi}\cos(\frac{n\pi}{2}t) \\ du = dt & dv = \sin(\frac{n\pi}{2}t) \end{matrix}$$

$$= \frac{8t}{n^2\pi^2}\cos(\tfrac{n\pi}{2}t)\Big|_0^2 - \frac{8}{n^2\pi^2}\int_0^2 \cancelto{0}{\cos(\tfrac{n\pi}{2}t)\, dt}$$

$$= \frac{16}{n^2\pi^2}(-1)^n$$

$$\phi(t) = \frac{4}{3} + \sum_{n=1}^{\infty} \frac{16}{n^2\pi^2}(-1)^n \cos(\tfrac{n\pi}{2}t)$$

b) $\omega_0 = 2 \quad T_0 = \pi$ only sine terms

To have only sine terms, series must be odd

$$f(t) = \begin{cases} -t^2 & -\pi/2 \le t < 0 \\ t^2 & 0 \le t \le \pi/2 \end{cases}$$

$$b_n = \frac{2}{\pi}\int_{-\pi/2}^{0} -t^2 \sin(2nt)\,dt + \frac{2}{\pi}\int_{0}^{\pi/2} t^2 \sin(2nt)\,dt$$

$$= \frac{4}{\pi}\int_0^{\pi/2} t^2 \sin(2nt)\,dt \qquad \begin{matrix} u = t^2 & v = \frac{-1}{2n}\cos(2nt) \\ du = 2t\,dt & dv = \sin 2nt \end{matrix}$$

$$= \frac{-4t^2}{2\pi n}\cos(2nt)\Big|_0^{\pi/2} + \frac{4}{n\pi}\int_0^{\pi/2} t\cos(2nt)\,dt \qquad \begin{matrix} u = t & v = \frac{1}{2n}\sin(2nt) \\ du = dt & dv = \cos(2nt) \end{matrix}$$

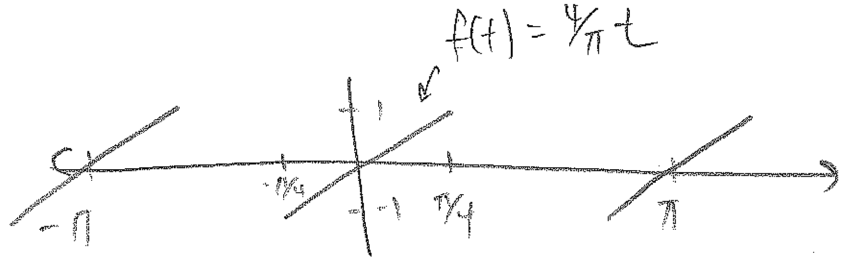
$$= \frac{-\pi}{2n}(-1)^n + \cancel{\frac{2t}{n^2\pi}\sin(2nt)\Big|_0^{\pi/2}}^{\,0} - \frac{2}{n^2\pi}\int_0^{\pi/2} \sin(2nt)\,dt$$

$$= \frac{-\pi}{2n}(-1)^n + \frac{1}{n^3\pi}\cos(2nt)\Big|_0^{\pi/2}$$

$$= \frac{-\pi}{2n}(-1)^n + \frac{1}{n^3\pi}\left[(-1)^n - 1\right]$$

$$\phi(t) = \sum_{n=1}^{\infty}\left[\frac{-\pi}{2n}(-1)^n + \frac{1}{n^3\pi}\left((-1)^n - 1\right)\right]\sin(2nt)$$

6.3) d.)



repeats every π

$T_0 = \pi \quad \omega_0 = 2\pi/T_0$

$\omega_0 = 2$

$$D_0 = \frac{1}{\pi}\int_{T_0} f(t)\,dt$$

$$= \frac{1}{\pi}\int_{-\pi/4}^{\pi/4} \frac{4}{\pi} t\,dt$$

$$= \frac{4}{\pi^2}\left(t^2\right)\Big|_{-\pi/4}^{\pi/4} = 0$$

Since $f(t)$ is average is 0 we can see why $D_0 = 0$.

$$D_k = \frac{1}{T_0}\int_{T_0} f(t)\, e^{-jk\omega_0 t}\,dt$$

$$= \frac{1}{\pi}\int_{-\pi/4}^{\pi/4} \frac{4}{\pi} t\, e^{-jk2t}\,dt$$

$$= \frac{4}{\pi^2}\int_{-\pi/4}^{\pi/4} t e^{-jk2t}\,dt \qquad u = t \quad dv = e^{-jk2t}dt$$

$$du = dt \quad v = \frac{e^{-jk2t}}{-jk2}$$

$$= \frac{4}{\pi^2}\left(\frac{te^{-jk2t}}{-jk2}\right)\Big|_{-\pi/4}^{\pi/4} - \frac{4}{\pi^2}\int_{-\pi/4}^{\pi/4}\frac{e^{-jk2t}}{-jk2}\,dt$$

$$= \frac{4}{-jk2\pi^2}\left(\frac{\pi}{4}e^{-jk\pi/2} + \frac{\pi}{4}e^{jk\pi/2}\right) - \frac{4}{(-jk2\pi)^2}\left(e^{-jk2t}\right)\Big|_{-\pi/4}^{\pi/4}$$

$$D_k = \frac{4j}{2k\pi^2}\left(\frac{\pi}{4}\left(e^{-j\pi/2 k} + e^{j\pi/2 k}\right)\right) + \frac{4}{(2k\pi)^2}\left(e^{-j\pi/2 k} - e^{j\pi/2 k}\right)$$

$$= \frac{j}{2k\pi}\left(e^{-j\pi/2 k} + e^{j\pi/2 k}\right) + \frac{4}{(2k\pi)^2}\left(e^{-j\pi/2 k} - e^{j\pi/2 k}\right)$$

$$\left(\frac{j}{2k\pi} + \frac{4}{(2k\pi)^2}\right)\left(e^{-j\pi/2 k}\right) + \left(\frac{j}{2k\pi} - \frac{4}{(2k\pi)^2}\right)\left(e^{j\pi/2 k}\right)$$

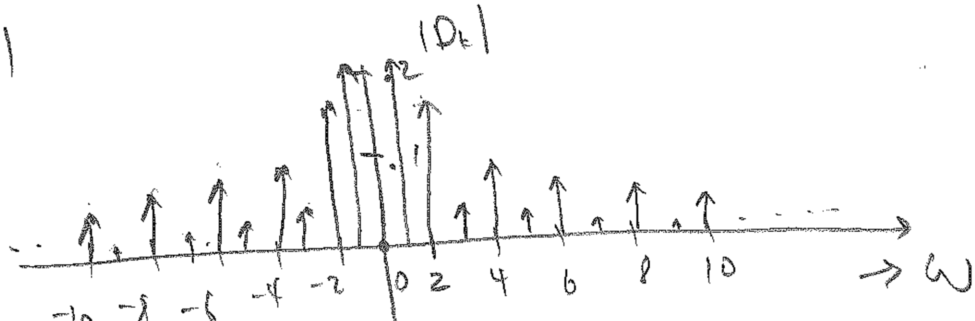
$$f(t) = \sum_{-\infty}^{1}\left(\left(\frac{j}{2k\pi} + \frac{4}{(2k\pi)^2}\right)e^{-j\pi/2 k} + \left(\frac{j}{2k\pi} - \frac{4}{(2k\pi)^2}\right)e^{j\pi/2 k}\right)e^{j2kt}$$

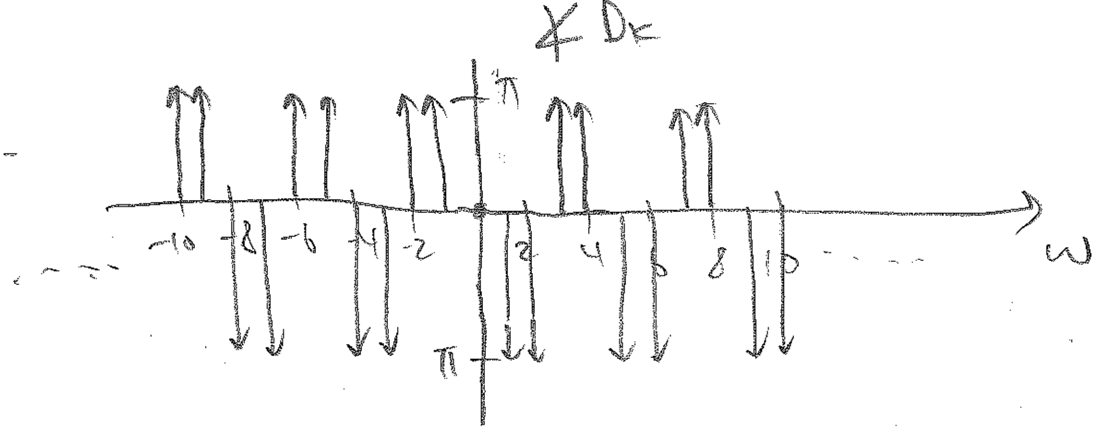
$$+ \sum_{1}^{\infty}\left(\left(\frac{j}{2k\pi} + \frac{4}{(2k\pi)^2}\right)e^{-j\pi/2 k} + \left(\frac{j}{2k\pi} - \frac{4}{(2k\pi)^2}\right)e^{j\pi/2 k}\right)e^{j2kt}$$

$$e^{-j\pi/2 k} = j^{-k} \qquad e^{j\pi/2 k} = j^{k}$$

$$\Rightarrow \left(\frac{j^{(-k+1)}}{2k\pi} + \frac{j^{-k}}{(k\pi)^2}\right) + \left(\frac{j^{k+1}}{2k\pi} - \frac{j^{k}}{(k\pi)^2}\right)$$ coefficients of e^{j2kt}

Exponential Spectra with matlab:





6.4 a) $$c = \frac{\int_0^1 x(t) f^*(t)\,dt}{\int_0^1 |x(t)|^2\,dt} = \int_0^1 t\,dt = \tfrac{1}{2}t^2\Big|_0^1 = \tfrac{1}{2}$$

b) $e(t) = f(t) - c\,x(t) = t - \tfrac{1}{2}$ for $0 \le t \le 1$

$$E_e = \int_0^1 |e(t)|^2 dt = \int_0^1 e(t)e(t)^* dt = \int_0^1 (t-\tfrac{1}{2})(t-\tfrac{1}{2})^* dt$$

$$= \int_0^1 (t^2 - t + \tfrac{1}{4})\,dt = \left(\tfrac{1}{3}t^3 - \tfrac{1}{2}t^2 + \tfrac{1}{4}t\right)\Big|_0^1 = \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{4} = \boxed{\tfrac{1}{12}}$$

Show $e(t)$ is orthogonal to $x(t)$:

$$\frac{1}{\sqrt{E_e}\sqrt{E_x}} \int_0^1 e(t)\,x(t)^* dt = \frac{1}{\sqrt{E_e E_x}} \int_0^1 (t-\tfrac{1}{2})\,dt = \frac{1}{\sqrt{E_e E_x}} \left(\tfrac{1}{2}t^2 - \tfrac{1}{2}t\right)\Big|_{t=0}^1 = \boxed{0}$$

$$E_x = \int_0^1 |x(t)|^2 dt = \int_0^1 dt = 1 \qquad E_e = \tfrac{1}{12}$$

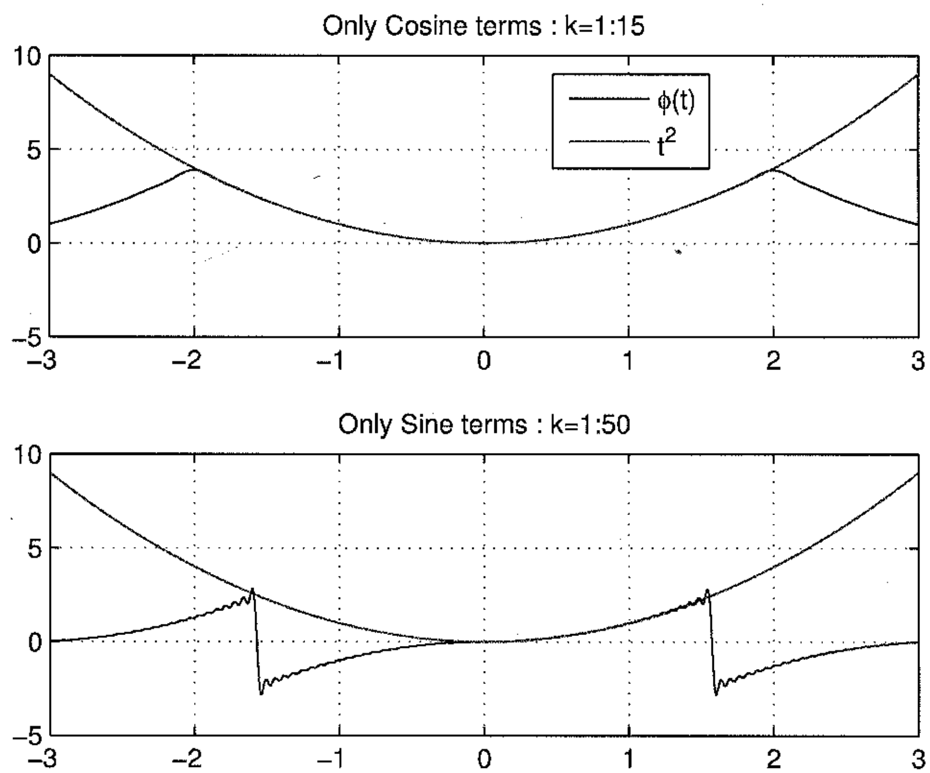
$$E_f = \int_0^1 |f(t)|^2 dt = \int_0^1 t^2 dt = \tfrac{1}{3}t^3\Big|_0^1 = \tfrac{1}{3}$$

Show $E_f = c^2 E_x + E_e$

$$E_f = \tfrac{1}{3} = (\tfrac{1}{2})^2(1) + \tfrac{1}{12} = \boxed{\tfrac{1}{3}} \checkmark$$

If x and y are orthogonal vectors
$$|x+y|^2 = |x|^2 + |y|^2$$

So energy of a signal can be interpretted as the "norm" of the signal and if these signals are orthogonal then Pythagorean's Theorem holds.


Only Cosine terms : k=1:15
φ(t)
t2
10
5
0
−5
−3
−2
−1
0
1
2
3
Only Sine terms : k=1:50
10
5
0
−5
−3
−2
−1
0
1
2
3