1] a) We know $\delta(t) \rightarrow \boxed{H} \rightarrow h(t)$ so to test if the given system response is correct we plug $y(t)=h(t)$ and see if $f(t)=\delta(t)$:

$$h''(t)+2h'(t)+10h(t) \stackrel{?}{=} \delta(t)$$

$$h(t)=\frac{1}{3}e^{-t}\sin 3t\, u(t)$$

$$h'(t)=-\frac{1}{3}e^{-t}\sin 3t\, u(t)+e^{-t}\cos 3t\, u(t)+\cancel{\frac{1}{3}e^{-t}\sin 3t\,\delta(t)}^{\,0}$$

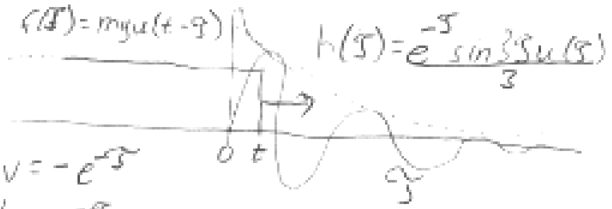
Plug in $t=0$, term goes to zero

$$=\frac{-1}{3}e^{-t}\sin 3t\, u(t)+e^{-t}\cos 3t\, u(t)$$

$$h''(t)=\frac{1}{3}e^{-t}\sin 3t\, u(t)-e^{-t}\cos 3t\, u(t)-\frac{1}{3}e^{-t}\sin 3t\,\delta(t)$$
$$-e^{-t}\cos 3t\, u(t)-3e^{-t}\sin 3t\, u(t)+e^{-t}\cos 3t\,\delta(t)$$
$$=\frac{-8}{3}e^{-t}\sin 3t\, u(t)-2e^{-t}\cos 3t\, u(t)-\frac{1}{3}e^{-t}\sin 3t\,\delta(t)$$
$$+e^{-t}\cos 3t\,\delta(t)$$

$$h''(t)+2h'(t)+10h(t)$$
$$=\frac{-8}{3}e^{-t}\cancel{\sin 3t}\, u(t)-2e^{-t}\cancel{\cos 3t}\, u(t)-\frac{1}{3}e^{-t}\sin 3t\,\delta(t)+e^{-t}\cos 3t\,\delta(t)$$
$$+2\left(\frac{-1}{3}e^{-t}\cancel{\sin 3t}\, u(t)+e^{-t}\cancel{\cos 3t}\, u(t)\right)+\frac{10}{3}e^{-t}\cancel{\sin 3t}\, u(t)$$
$$=e^{-t}\cos 3t\,\delta(t)$$
$$=\delta(t)$$

b) If initial conditions $y(0) = \dot{y}(0) = 0$ then the response of the system to input $f(t) = mg\,u(t)$ is $y(t) = h(t) * f(t)$



$$y(t) = \int_{-\infty}^{\infty} \tfrac{1}{3} e^{-\xi} \sin 3\xi\, u(\xi)\, mg\, u(t-\xi)\, d\xi$$

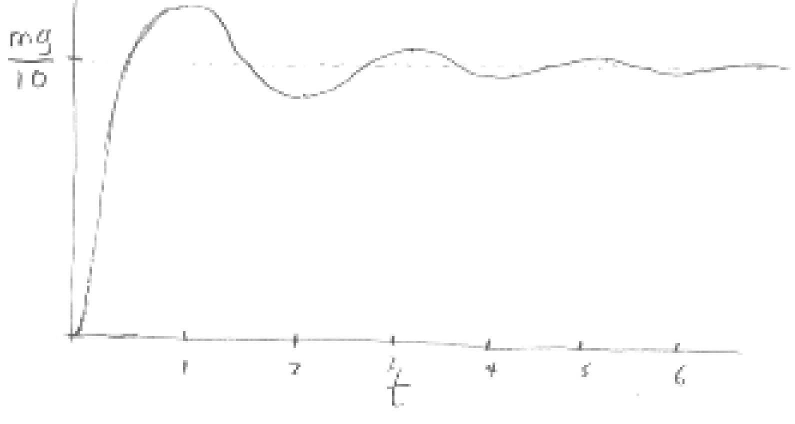
$$= \frac{mg}{3} \int_0^t e^{-\xi} \sin 3\xi\, d\xi \cdot u(t) \qquad \begin{matrix} u = \sin 3\xi & v = -e^{-\xi} \\ du = 3\cos 3\xi\, d\xi & dv = e^{-\xi} \end{matrix}$$

$$= \frac{mg}{3}\left[-e^{-\xi} \sin 3\xi \Big|_{\xi=0}^{t} + 3\int_0^t e^{-\xi} \cos 3\xi\, d\xi \right] u(t) \qquad \begin{matrix} u = \cos 3\xi & v = -e^{-\xi} \\ du = -3\sin 3\xi\, d\xi & dv = e^{-\xi} \end{matrix}$$

$$= \frac{mg}{3}\left[-e^{-t} \sin 3t - 3e^{-\xi} \cos 3\xi \Big|_{\xi=0}^{t} - 9\int_{\xi=0}^{t} e^{-\xi} \sin 3\xi\, d\xi \right] u(t)$$

$$= \frac{mg}{3}\left[-e^{-t} \sin 3t - 3e^{-t} \cos 3t + 3 - 9\, y(t) \cdot \frac{3}{mg} \right] u(t)$$

$$y(t) = \frac{mg}{30}\left[-e^{-t} \sin 3t - 3e^{-t} \cos 3t + 3 \right] u(t)$$



A weight is put on the board, it oscillates for a while and settles down to a steady state value of $\frac{mg}{10}$, which is clearly proportional to the weight put on the ramp. Makes sense

c) Recall $y''(t) + 2y'(t) + 10y(t) = f(t)$

To check if the given $y(t)$ is a solution, plug into above equation and set $f(t) = 0$. $y(t) = c_1 e^{-t}\sin 3t + c_2 e^{-t}\cos 3t$

$$y'(t) = -c_1 e^{-t}\sin 3t + 3c_1 e^{-t}\cos 3t - c_2 e^{-t}\cos 3t - 3c_2 e^{-t}\sin 3t$$

$$= -(c_1 + 3c_2)e^{-t}\sin 3t + (3c_1 - c_2)e^{-t}\cos 3t$$

$$y''(t) = (c_1 + 3c_2)e^{-t}\sin 3t - 3(c_1 + 3c_2)e^{-t}\cos 3t$$
$$-(3c_1 - c_2)e^{-t}\cos 3t - 3(3c_1 - c_2)e^{-t}\sin 3t$$

$$= (-8c_1 + 6c_2)e^{-t}\sin 3t + (-6c_1 - 8c_2)e^{-t}\cos 3t$$

$$y''(t) + 2y'(t) + 10y(t)$$
$$= (-8c_1 + 6c_2)e^{-t}\sin 3t + (-6c_1 - 8c_2)e^{-t}\cos 3t$$
$$- 2(c_1 + 3c_2)e^{-t}\sin 3t + 2(3c_1 - c_2)e^{-t}\cos 3t$$
$$+ 10c_1 e^{-t}\sin 3t + 10c_2 e^{-t}\cos 3t$$
$$= 0 = f(t)$$

So $y(t) = c_1 e^{-t}\sin 3t + c_2 e^{-t}\cos 3t$ is a solution.

d) $y(t) = c_1 e^{-t}\sin 3t + c_2 e^{-t}\cos 3t$ w/ $y(0^-)=0$, $\dot{y}(0^-) = -6$

To satisfy $y(0^-)=0$, plug 0 in for t into $y(t)$

$$y(0^-) = 0 = c_1 \cdot 0 + c_2 \implies \underline{c_2 = 0}$$

To satisfy $\dot{y}(0^-) = -6$ plug 0 in for t into $\dot{y}(t)$ from (b)

$$\dot{y}(0^-) = -6 = -(c_1 + 3c_2)e^{-t}\sin 3t + (3c_1 - c_2)e^{-t}\cos 3t$$

plug $c_2 = 0$ in

$$= -c_1 \cdot 0 + 3c_1 \implies 3c_1 = -6 \implies \underline{c_1 = -2}$$

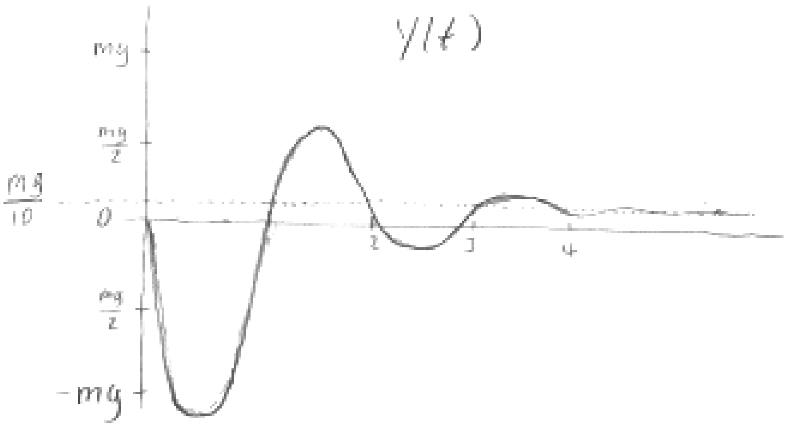
$y(t) = -2e^{-t}\sin 3t$ for $t \geq 0$. Undefined for $t < 0$.

e) $\underbrace{\text{Total Response}}_{y(t)} = \underbrace{\text{Zero-input Response}}_{y_0(t)} + \underbrace{\text{zero-state Response}}_{y_f(t)}$

We just found the zero-input response in (d): $y_0(t) = -2e^{-t}\sin 3t$ for $t \geq 0$

We found the zero-state response in part (b): $y_f(t) = \frac{mg}{30}\left[e^{-t}\sin 3t - 3e^{-t}\cos 3t + 3\right]u(t)$

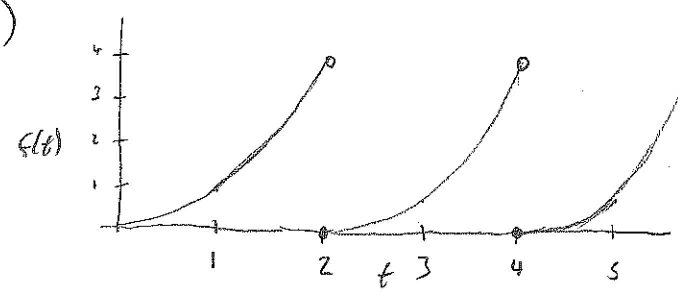
$$y(t) = \left[\underbrace{\frac{mg}{10}}_{\text{Steady State}} + \underbrace{\left(\frac{mg}{30} - 2\right)e^{-t}\sin 3t - \frac{mg}{10}e^{-t}\cos 3t}_{\text{Transient}}\right]$$ for $t \geq 0$. Undefined for $t < 0$.



The trajectory is at first down because $\dot{y}(0) = -6$ and then the transient part decays and leaves only the steady state solution of be fore.

2] $f(t) = t^2$ for $t \in [0, 2]$ $T = 2$

a)



b) $\varphi(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_o t) + b_n \sin(n\omega_o t)$ $\omega_o = \frac{2\pi}{T} = \pi$

$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi t) + b_n \sin(n\pi t)$

$a_0 = \frac{1}{T_0}\int_0^{T_0} f(t)\,dt = \frac{1}{2}\int_0^2 t^2\,dt = \frac{1}{6}t^3\Big|_0^2 = \frac{4}{3}$

$a_n = \frac{2}{T_0}\int_0^{T_0} f(t)\cos(n\omega_o t)\,dt = \int_0^2 t^2\cos(n\pi t)\,dt$ $u = t^2$ $du = 2t\,dt$ $v = \frac{1}{n\pi}\sin(n\pi t)$ $dv = \cos(n\pi t)$

$= \frac{1}{n\pi}t^2 \cancelto{0}{\sin(n\pi t)}\Big|_0^2 - \frac{2}{n\pi}\int_0^2 t\sin(n\pi t)\,dt$ $u = t$ $du = dt$ $v = \frac{-1}{n\pi}\cos(n\pi t)$ $dv = \sin(n\pi t)$

$= \frac{2}{n^2\pi^2}t\cos(n\pi t)\Big|_0^2 - \frac{2}{n^2\pi^2}\int_0^2 \cos(n\pi t)\,dt$

$= \frac{4}{n^2\pi^2} - \frac{2}{n^3\pi^3}\sin(n\pi t)\Big|_0^2 = \frac{4}{n^2\pi^2}$

$b_n = \frac{2}{T_0}\int_0^{T_0} f(t)\sin(n\omega_o t) = \int_0^2 t^2\sin(n\pi t)\,dt$ $u = t^2$ $du = 2t\,dt$ $v = \frac{-1}{n\pi}\cos(n\pi t)$ $dv = \sin(n\pi t)$

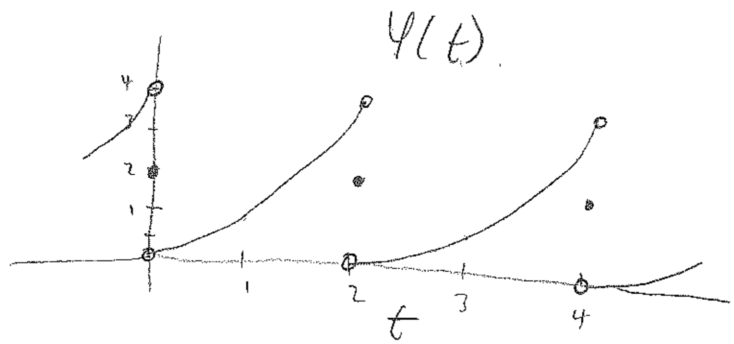
$= \frac{-1}{n\pi}t^2\cos(n\pi t)\Big|_0^2 + \frac{2}{n\pi}\int_0^2 t\cos(n\pi t)\,dt$ $u = t$ $du = dt$ $v = \frac{1}{n\pi}\sin(n\pi t)$ $dv = \cos(n\pi t)$

$= \frac{-4}{n\pi} + \frac{2}{n^2\pi^2}t\cancelto{0}{\sin(n\pi t)}\Big|_0^2 - \frac{2}{n^2\pi^2}\int_0^2 \sin(n\pi t)\,dt$

$$= \frac{-4}{n\pi} + \frac{2}{n^3\pi^3}\cos(n\pi t)\Big|_0^2$$

$$= \frac{-4}{n\pi}$$

$$\varphi(t) = \frac{4}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\cos(n\pi t) - \frac{4}{n\pi}\sin(n\pi t)$$



4.31 a) As N gets larger the approximation gets better. Gets very "wiggly" around end points: 0, 2.

b) As N gets larger, $\varphi(2) \to 2$.

To see why consider $t = T = 2$

$$\varphi(2) = \frac{4}{3} + \sum_{n=1}^{N} \frac{4}{n^2\pi^2}\cos(2\pi n) - \frac{4}{n\pi}\sin(2\pi n)$$

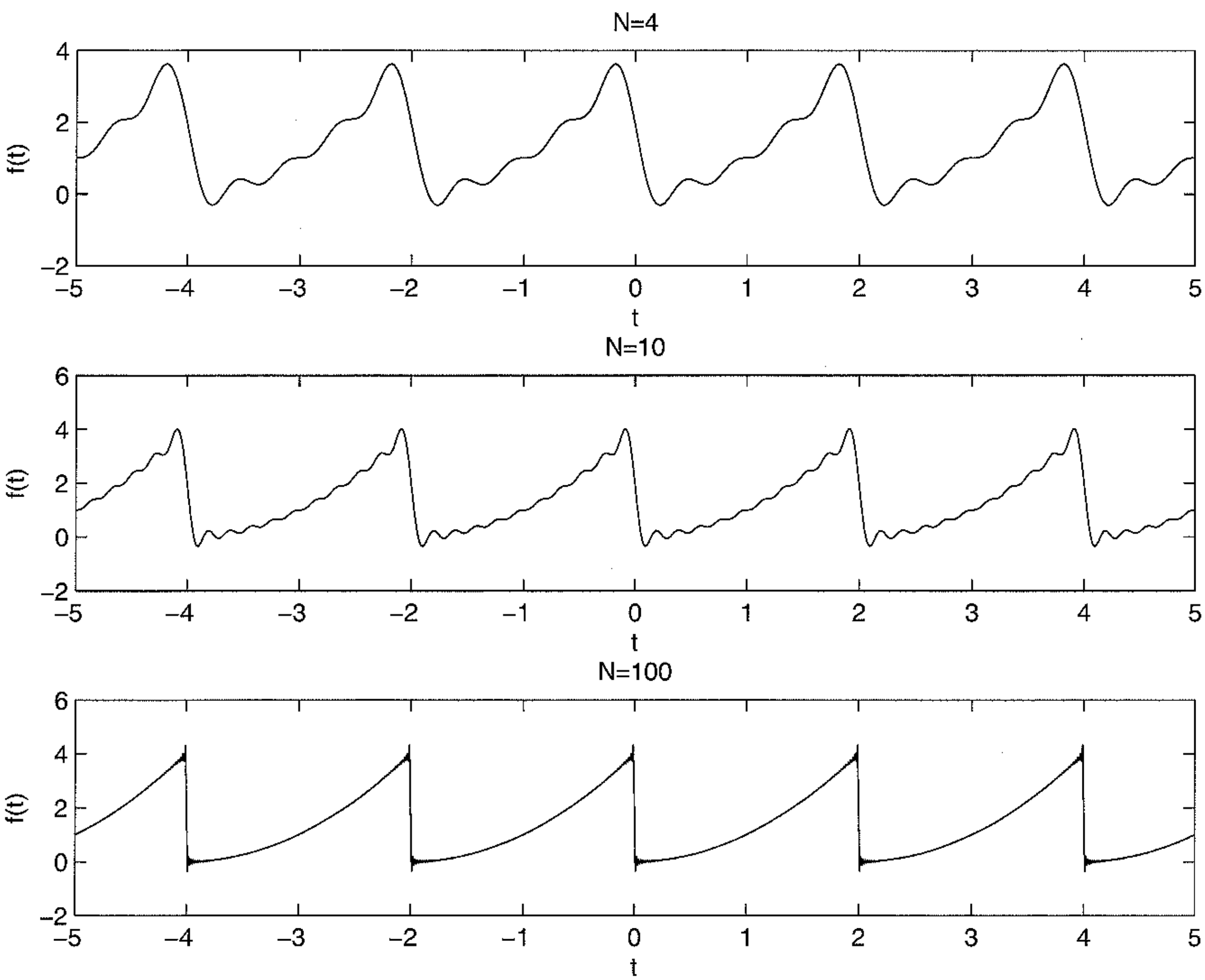
$$= \frac{4}{3} + \sum_{n=1}^{N} \frac{4}{n^2\pi^2}$$

$$= \frac{4}{3} + \frac{4}{\pi^2}\underbrace{\sum_{n=1}^{N}\frac{1}{n^2}}_{=\pi^2/6 \text{ As } N\to\infty}$$

$N = 4$ $\varphi(2) = 1.9103$

$N = 10$ $\varphi(2) = 1.9614$

$N = 100$ $\varphi(2) = 1.9960$


N=4
f(t)
4
2
0
−2
−5
−4
−3
−2
−1
0
1
2
3
4
5
t
N=10
f(t)
6
4
2
0
−2
−5
−4
−3
−2
−1
0
1
2
3
4
5
t
N=100
f(t)
6
4
2
0
−2
−5
−4
−3
−2
−1
0
1
2
3
4
5
t

```
function[x1]=trigft(t,N)
x1=4/3;
for k=1:N;
   x1=x1+4./(k.*pi).^2.*cos(k.*pi.*t)-4./(k.*pi).*sin(k.*pi.*t);
end
```

4) a) Periodic functions satisfy the property $f(t) = f(t+T)$ for some T

$f(t) = 2\cos(3\pi t - \pi/3) - \sin(2\pi t)$

$f(t+T) = 2\cos(3\pi t + 3\pi T - \pi/3) - \sin(2\pi t + 2\pi T)$

$\cos(3\pi t - \pi/3) = \cos(3\pi t + 3\pi T - \pi/3)$ when $3\pi T = 2\pi k \Rightarrow T = \frac{2}{3}k$ for $k = 0, 1, \dots$

$\sin(2\pi t) = \sin(2\pi t + 2\pi T)$ when $2\pi T = 2\pi l \Rightarrow T = l$ for $l = 0, 1, \dots$

$T = \frac{2}{3}k = l \Rightarrow T = LCM[2/3, 1] = \underline{2}$ $\quad \omega_0 = \frac{2\pi}{T_0} = \pi$

b) $f(t) = 2\cos(3\pi t - \pi/3) - \sin(2\pi t)$

$= 2\cos(\pi/3)\cos(3\pi t) + 2\sin(\pi/3)\sin(3\pi t) - \sin(2\pi t)$

$$\phi(t) = a_0 + a_1\cos(\pi t) + a_2\cos(2\pi t) + a_3\cos(3\pi t) + a_4\cos(4\pi t) + \dots$$
$$+ b_1\sin(\pi t) + a_2\sin(2\pi t) + a_3\sin(3\pi t) + \dots$$

We can simply read off the coefficients:

$a_k = 0$ for all k except $k = 3: a_3 = 2\cos(\pi/3) = 1$

$b_k = 0$ for all k except $k = 2, 3: b_2 = -1$

$b_3 = 2\sin(\pi/3) = \sqrt{3}$

c) $\phi(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} D_n e^{jn\pi t}$

$= \dots + D_{-4}e^{-4\pi j t} + D_{-3}e^{-3\pi j t} + D_{-2}e^{-2\pi j t} + D_{-1}e^{-\pi j t} + D_0 + D_1 e^{\pi j t} + D_2 e^{2\pi j t} + D_3 e^{3\pi j t} + \dots$

$= \dots + (D_{-3}e^{-3\pi j t} + D_3 e^{3\pi j t}) + (D_{-2}e^{-2\pi j t} + D_2 e^{2\pi j t}) + (D_{-1}e^{-\pi j t} + D_1 e^{\pi j t}) + \dots$

$f(t) = \cos(3\pi t) + \sqrt{3}\sin(3\pi t) - \sin(2\pi t)$

$= \frac{1}{2}e^{3\pi j t} + \frac{1}{2}e^{-3\pi j t} + \frac{\sqrt{3}}{2j}e^{3\pi j t} - \frac{\sqrt{3}}{2j}e^{-3\pi j t} - \frac{1}{2j}e^{2\pi j t} + \frac{1}{2j}e^{-2\pi j t}$

$D_n = 0$ except $D_{-3} = \frac{1+\sqrt{3}j}{2}$ $\quad D_3 = \frac{1-\sqrt{3}j}{2}$ $\quad D_{-2} = \frac{-j}{2}$ $\quad D_2 = \frac{j}{2}$