Homework 4 Solutions

1] Because the system is time-invariant

$$f(t) \rightarrow \boxed{T} \rightarrow f(t-t') \rightarrow \boxed{H} \rightarrow y(t-t')$$

$$f(t) \rightarrow \boxed{H} \rightarrow y(t) \rightarrow \boxed{T} \rightarrow y(t-t')$$

$f(t) = a$ so $f(t-t') = a$ and $a \rightarrow \boxed{H} \rightarrow y(t-t')$

Notice that $f(t) = a \rightarrow \boxed{H} \rightarrow y(t)$. This means that for all t', $y(t) = y(t-t')$. Clearly this is a constant and not a function of time. Thus showing $y(t) = y(t-t') = b$.

2] $x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau$

$x(\tau) = x(\tau)$
$g(\tau) = h(-\tau)$ "flip"
$g(\tau - t) = h(t-\tau)$ "shift"

a) $x(\tau) = u(\tau)$
$g(\tau) = e^{\tau} u(-\tau)$
$g(\tau - t) = e^{\tau - t} u(t-\tau)$

$$\int_{\tau=-\infty}^{\infty} u(\tau) u(t-\tau) e^{\tau - t}\, d\tau$$

$u(\tau)$ makes the integral 0 for $\tau < 0$
$u(t-\tau)$ makes the integral 0 for $t > \tau$
so we adjust the bounds

$$\int_{\tau=0}^{t} u(\tau) u(t-\tau) e^{\tau - t}\, d\tau = u(\tau) u(t-\tau) e^{\tau - t} \Big|_{\tau=0}^{t}$$

$$= u(0) u(t) (1 - e^{-t}) = u(t)(1 - e^{-t})$$

b) $x(\tau) = e^{-\tau}u(\tau)$ $\qquad h(\tau) = e^{-\tau}u(\tau)$

$h(t-\tau) = e^{\tau-t}u(t-\tau)$

$$\int_{-\infty}^{\infty} u(\tau)u(t-\tau)e^{-\tau}e^{\tau-t}\,d\tau = e^{-t}\int_{0}^{t} u(\tau)u(t-\tau)\,d\tau$$

where the bounds were set by $u(\tau)$ = and $u(t-\tau)$

$$= e^{-t}\left[u(\tau)u(t-\tau)\right]\Big|_{\tau=0}^{t} = u(t)\cdot te^{-t}$$

c) $x(\tau) = \sin(3\tau)u(\tau)$ $\qquad h(\tau) = e^{-\tau}u(\tau)$

$h(t-\tau) = e^{\tau-t}u(t-\tau)$

$$\int_{-\infty}^{\infty} u(\tau)u(t-\tau)e^{\tau-t}\sin(3\tau)\,d\tau = \int_{\tau=0} u(\tau)u(t-\tau)e^{\tau-t}\sin(3\tau)\,d\tau$$

$u = \sin(3\tau)$ $\qquad v = e^{\tau-t}$

$du = 3\cos(3\tau)d\tau$ $\qquad dv = e^{\tau-t}d\tau$

$$= u(t)\left[\left(\sin(3\tau)e^{\tau-t}\right)\Big|_{\tau=0}^{t} - \int_{\tau=0}^{t} 3\cos(3\tau)e^{\tau-t}\,d\tau\right]$$

$u = 3\cos(3\tau)$ $\qquad v = e^{\tau-t}$

$du = -9\sin(3\tau)$ $\qquad dv = e^{\tau-t}d\tau$

$$u(t)\int_{\tau=0}^{t}\sin(3\tau)e^{t-\tau}\,d\tau = u(t)\sin(3t) - u(t)\left(3\cos(3\tau)e^{\tau-t}\right)\Big|_{\tau=0}^{t}$$

$$-u(t)\int_{\tau=0}^{t} 9\sin(3\tau)e^{\tau-t}\,d\tau$$ ← Add this term to both sides and divide by 10.

$$u(t)\int_{\tau=0}^{t}\sin(3\tau)e^{t-\tau}\,d\tau = \frac{1}{10}u(t)\left(\sin(3t) - 3\cos(3t) + 3e^{-t}\right)$$

3] $p(t) = u(t+1) - u(t-1)$

First consider $u(t-a) * u(t-b) = \left(\delta(t-a) * u(t)\right) * \left(\delta(t-b) * u(t)\right)$

$$= \left(u(t) * u(t)\right) * \left(\delta(t-a) * \delta(t-b)\right) = \left(u(t) * u(t)\right) * \delta(t-a-b)$$

$$u(t) * u(t) = \int_{-\infty}^{\infty} u(\tau) u(t-\tau)\, d\tau \qquad \begin{matrix} \tau > 0 \\ t < \tau \end{matrix}$$

$$= \int_{\tau=0}^{t} u(\tau) u(t-\tau)\, d\tau = u(t)t$$

$$u(t-a) * u(t-b) = \left(u(t)t\right) * \delta(t-a-b)$$

$$= u(t-a-b)(t-a-b)$$

a) $p(t) * p(t) = [u(t+1) - u(t-1)] * [u(t+1) - u(t-1)]$

$$= u(t+1) * u(t+1) - u(t+1) * u(t-1)$$
$$- u(t-1) * u(t+1) + u(t-1) * u(t-1)$$
$$= u(t+1) * u(t+1) - 2u(t+1) * u(t-1) + u(t-1) * u(t-1)$$
$$= u(t+2)(t+2) - 2u(t)t + u(t-2)(t-2)$$

$p(t) * p(t)$

b) We know $u(t-a) * u(t-b) = u(t-a-b)(t-a-b)$

Now we need to know $u(t-a)*u(t-b)*u(t-c)$ since all terms in $p(t)*p(t)*p(t)$ can be expressed that way

$$
\begin{aligned}
u(t-a)*u(t-b)*u(t-c) &= \big(u(t)*u(t)*u(t)\big)*\big(\delta(t-a)*\delta(t-b)*\delta(t-c)\big) \\
&= \big(u(t)*u(t)*u(t)\big)*\delta(t-a-b-c) \\
&= \big(tu(t)*u(t)\big)*\delta(t-a-b-c)
\end{aligned}
$$

where the last line comes from our earlier result.

$x(\tau) = \tau u(\tau)$

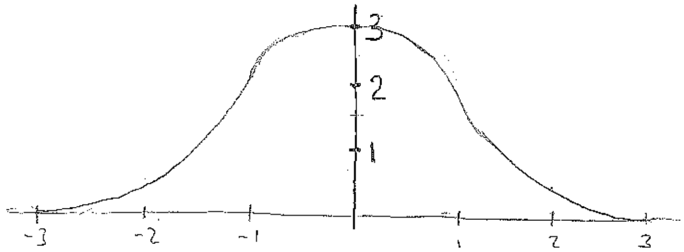
$h(t-\tau) = u(t-\tau)$

$$\int_{-\infty}^{\infty} \tau u(\tau)u(t-\tau)\,d\tau = \int_{\tau=0}^{t} \tau u(\tau)u(t-\tau)\,d\tau$$

$$= \tfrac{1}{2}t^2 u(t)$$

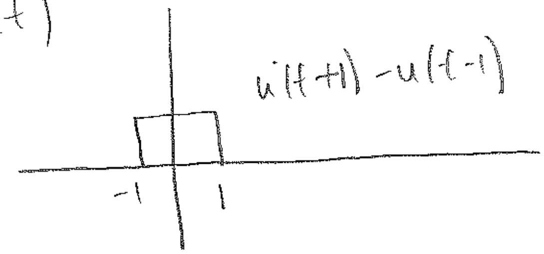
$$
\begin{aligned}
u(t-a)*u(t-b)*u(t-c) &= \left(\tfrac{1}{2}t^2u(t)\right)*\delta(t-a-b-c) \\
&= \tfrac{1}{2}(t-a-b-c)^2\,u(t-a-b-c)
\end{aligned}
$$

$$
\begin{aligned}
p(t)*p(t)*p(t) &= [u(t+1)-u(t-1)]*[u(t+1)-u(t-1)]*[u(t+1)-u(t-1)] \\
&= u(t+1)*u(t+1)*u(t+1) - u(t+1)*u(t+1)*u(t-1) \\
&\quad - u(t+1)*u(t-1)*u(t+1) + u(t+1)*u(t-1)*u(t-1) \\
&\quad - u(t+1)*u(t+1)*u(t+1) + u(t-1)*u(t+1)*u(t-1) \\
&\quad + u(t-1)*u(t-1)*u(t+1) - u(t-1)*u(t-1)*u(t-1) \\
&= \tfrac{1}{2}\Big[(t+3)^2u(t+3) - (t+1)^2u(t+1) - (t+1)^2u(t+1) + (t-1)^2u(t-1) \\
&\quad - (t+1)^2u(t+1) + (t-1)^2u(t-1) + (t-1)^2u(t-1) - (t-3)^2u(t-3)\Big] \\
&= \tfrac{1}{2}(t+3)^2u(t+3) - \tfrac{3}{2}(t+1)^2u(t+1) + \tfrac{3}{2}(t-1)^2u(t-1) - \tfrac{1}{2}(t-3)^2u(t-3)
\end{aligned}
$$



4.3) $p(t) = u(t+1) - u(t-1)$

p(t)

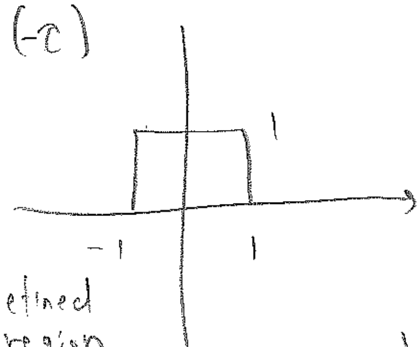


$y(t) = (p * p)\, t$

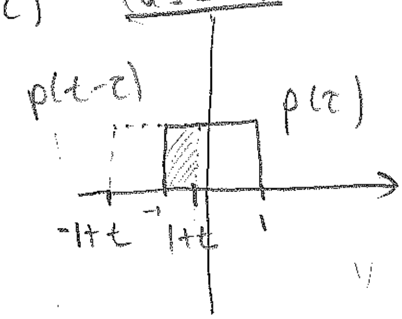
$$y(t) = \int_{-\infty}^{\infty} p(\tau)\, p(t-\tau)\, d\tau$$

we need to find the limit of integration where this integral exists.

$p(-\tau)$



$p(t-\tau)$ <u>case 1</u>



$1+t \leq 1$

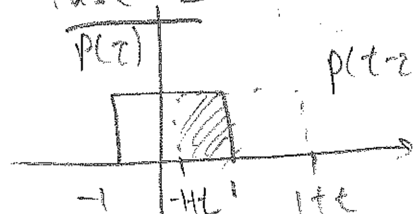
$t \leq 0$

$p(\tau) = 1$
$p(t-\tau) = 1$ in defined region

$$y(t) = \int_{-1}^{1+t} d\tau \qquad \underline{\text{case 1}} \quad -2 \leq t \leq 0$$

$$= \int_{-1+t}^{1} d\tau \qquad \underline{\text{case 2}} \quad 2 \geq t \geq 0$$

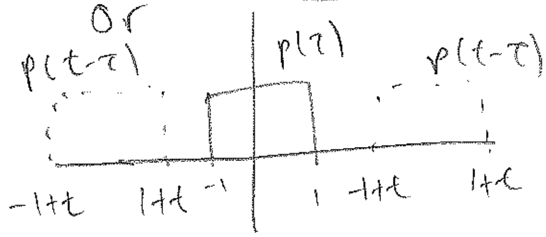
or <u>case 2</u>



$-1+t \geq -1$

$t \geq 0$

or <u>case 3</u>



$1+t \leq -1$ $\quad$ $-1+t \geq 1$

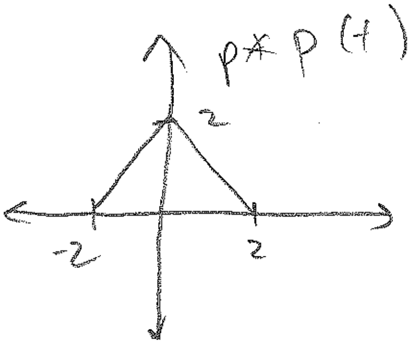
$t \leq -2$ $\quad$ $t \geq 2$

$\Rightarrow$ $1+t+1 = \underline{t+2}$ $\quad -2 \leq t \leq 0$

$1-(-1+t) = \underline{-t+2}$ $\quad 2 \geq t \geq 0$

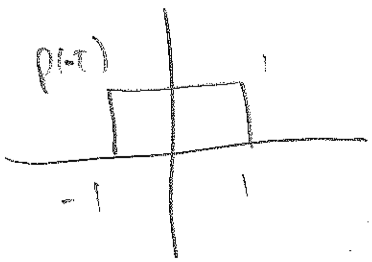
<u>case 3</u> $\quad 0 \qquad t \leq -2$

$\qquad t \geq 2$

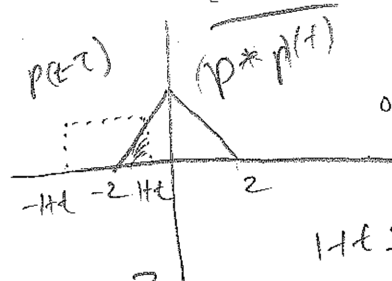


two boxes convoluted yields a triangle of width 2× width of original and height 2× height of original.

$y(t)\ (p*p*p)\ t$



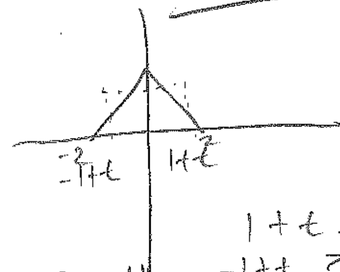
case 1



$1+t \le 0$

or

Case 2

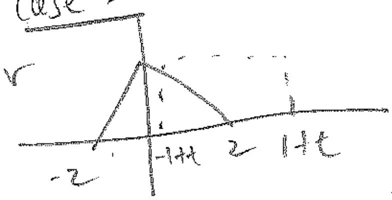


$1+t \le 2$

$-1+t \ge -2$

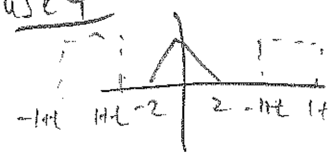
Case 3

Or



$-1+t \ge 0$

Case 4



$1+t \le -2$

$-1+t \ge 2$

$p(t-\tau) = 1$

$(p*p)(t) = t+2 \quad 2 \ge t \ge 0$

$= \quad -2 \le t \le 0$

case 1

$$y(t) = \int_{-2}^{1+t} \tau + 2 \, d\tau$$

$$= \frac{\tau^2}{2} + 2\tau \Big|_{-2}^{1+t} = \frac{(1+t)^2}{2} + 2(1+t) - \left(\frac{(-2)^2}{2} + (-4)\right)$$

$$= \frac{t^2}{2} + \frac{2t}{2} + \frac{1}{2} + 2 + 2t + 2$$

$$= \tfrac{1}{2}t^2 + 3t + \tfrac{9}{2} \qquad -3 \le t \le -1$$

Case 2

$$y(t) = \int_{-1+t}^{0} \tau + 2 \, d\tau + \int_{0}^{1+t} -\tau + 2 \, d\tau$$

$$= \left(\frac{\tau^2}{2} + 2\tau\right)\Big|_{-1+t}^{0} + \left(-\frac{\tau^2}{2} + 2\tau\right)\Big|_{0}^{1+t}$$

$$y(t) = -\left(\frac{(-1+t)^2}{2} + 2(-1+t)\right) + \frac{-(1+t)^2}{2} + 2(1+t)$$

$$= -\left(\frac{t^2}{2} - \frac{2t}{2} + \frac{1}{2} - 2 + 2t\right) + -\left(\frac{t^2}{2} + \frac{2t}{2} + \frac{1}{2}\right) + 2 + 2t$$

$$= -t^2 + 3 \qquad -1 \leq t \leq 1$$

Case 3

$$y(t) = \int_{-1+t}^{2} -\tau + 2 \, d\tau$$

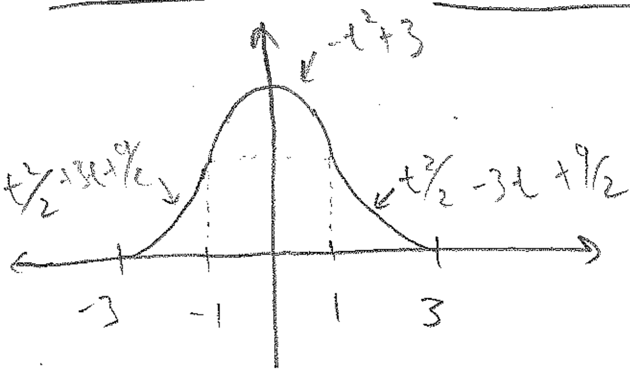
$$= -\frac{\tau^2}{2} + 2\tau \Big|_{-1+t}^{2}$$

$$= -2 + 4 - \left(-\frac{(-1+t)^2}{2} + 2(-1+t)\right)$$

$$= 2 - \left(-\frac{t^2}{2} + \frac{2t}{2} - \frac{1}{2} + 2t - 2\right)$$

$$= \frac{t^2}{2} - 3t + 9/2 \qquad 1 \leq t \leq 3$$

Case 4 $\qquad 0 \qquad t \; \forall$ else



triangle and a box yields parabolas.

4.4 $h(t) = -\delta(t) + 2e^{-t}u(t) \quad x(t) = e^{t}u(-t) \quad y(t) = h(t) * x(t)$

$$y(t) = \int_{-\infty}^{\infty} [-\delta(t-\tau) + 2e^{\tau-t}u(t-\tau)]e^{\tau}u(-\tau)d\tau = \underbrace{\int_{-\infty}^{\infty} -\delta(t-\tau)e^{\tau}u(-\tau)d\tau}_{y_1} + \underbrace{\int_{-\infty}^{\infty} 2e^{\tau-t}e^{\tau}u(-\tau)u(t-\tau)d\tau}_{\triangleq y_2}$$

We solve this in two parts
$y(t) = y_1(t) + y_2(t)$

$$y_1(t) = -\int_{-\infty}^{\infty} \delta(t-\tau)e^{\tau}u(-\tau)d\tau = -e^{t}u(-t)\int_{-\infty}^{\infty}\delta(t-\tau)d\tau = -e^{t}u(-t)$$

$$y_2(t) = \int_{-\infty}^{\infty} 2e^{2\tau-t}u(-\tau)u(t-\tau)d\tau = 2e^{-t}\int_{-\infty}^{\infty} e^{2\tau}u(-\tau)u(t-\tau)d\tau$$

$u(-\tau) \Rightarrow \tau < 0$ } For $t<0$ $u(-\tau)u(t-\tau) \equiv u(t-\tau)u(-t^{+})$

$u(t-\tau) \Rightarrow \tau < t$ } For $t \geq 0$ $u(-\tau)u(t-\tau) \equiv u(-\tau)u(t^{-})$

$$y_2(t) = 2e^{-t}u(-t)\int_{-\infty}^{\infty} \underbrace{u(t-\tau)}_{\tau<t}e^{2\tau}d\tau + 2e^{-t}u(t)\int_{-\infty}^{\infty}\underbrace{u(-\tau)}_{\tau<0}e^{2\tau}d\tau$$

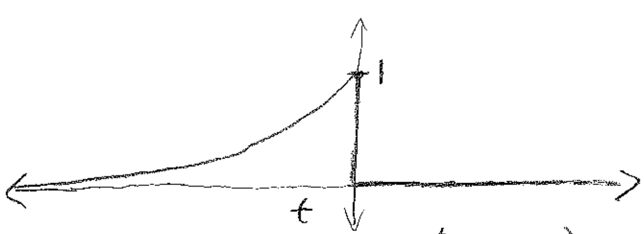
$$= 2e^{-t}u(-t)\int_{-\infty}^{t} u(t-\tau)e^{2\tau}d\tau + 2e^{-t}u(t)\int_{-\infty}^{0} u(-\tau)e^{2\tau}d\tau$$

$$= 2e^{-t}u(-t)\left(u(t-\tau)\frac{e^{2\tau}}{2}\Big|_{-\infty}^{t}\right) + 2e^{-t}u(t)\left(u(-\tau)\frac{e^{2\tau}}{2}\Big|_{-\infty}^{0}\right)$$

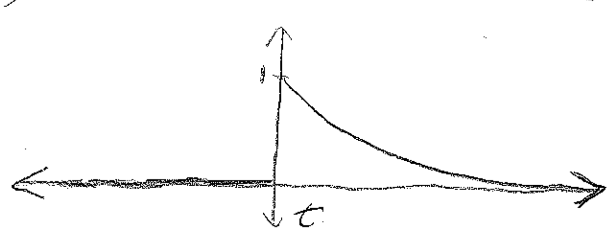
$$= e^{-t}u(-t)\left(u(0)e^{2t} - u(\infty)\cdot 0\right) + e^{-t}u(t)\left(u(0)\cdot 1 - u(\infty)\cdot 0\right)$$

$$= e^{t}u(-t) + e^{-t}u(t)$$

$$y(t) = y_1(t) + y_2(t) = -e^{t}\cancel{u(-t)} + e^{t}\cancel{u(-t)} + e^{-t}u(t) = e^{-t}u(t)$$



$X(t) = e^{t}u(-t)$



$Y(t) = e^{-t}u(t)$

4.5 Prove $f_1(t) * (f_2(t) * f_3(t)) = (f_1(t) * f_2(t)) * f_3(t)$

Define $f_{23}(t) = f_2(t) * f_3(t)$ and $f_{12}(t) = f_1(t) * f_2(t)$

$$f_{23}(t) = \int_{-\infty}^{\infty} f_2(\tau) f_3(t-\tau)\, d\tau \qquad \begin{matrix} \alpha = t-\tau \\ d\alpha = -d\tau \end{matrix} \qquad f_{23}(t) = \int_{-\infty}^{\infty} f_2(t-\alpha) f_3(\alpha)\, d\alpha$$

$$f_1(t) * f_{23}(t) = \int_{-\infty}^{\infty} f_1(\tau) f_{23}(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} f_1(\tau) \int_{-\infty}^{\infty} f_2(t-\tau-\alpha) f_3(\alpha)\, d\alpha\, d\tau$$

$$= \int_{\tau=-\infty}^{\infty} \int_{\alpha=-\infty}^{\infty} f_1(\tau) f_2(t-\tau-\alpha) f_3(\alpha)\, d\alpha\, d\tau$$

$$f_{12}(t) = \int_{\tau=-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\, d\tau$$

$$f_{12}(t) * f_3(t) = \int_{\alpha=-\infty}^{\infty} \int_{\tau=-\infty}^{\infty} f_1(\tau) f_2(t-\alpha-\tau)\, d\tau\, f_3(\alpha)\, d\alpha$$

$$= \int_{\alpha=-\infty}^{\infty} \int_{\tau=-\infty}^{\infty} f_1(\tau) f_2(t-\tau-\alpha) f_3(\alpha)\, d\alpha\, d\tau$$

=

Problem 6.a)

```
function y=f(t)
%This function computes the value of -t(u(t+1)-u(t))
y=-t.*((t+1>=0)-(t>=0));
```

Problem 6.b)

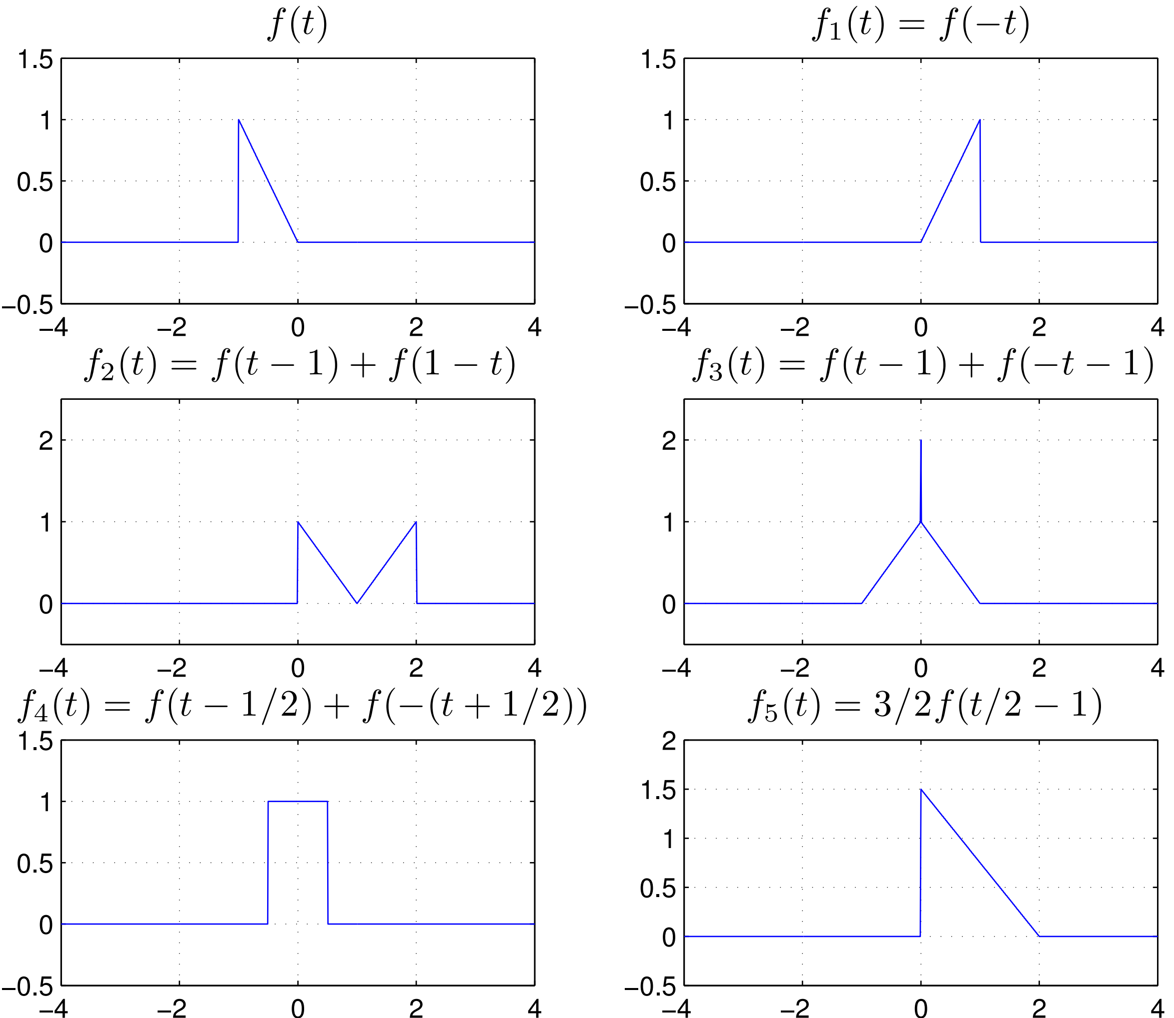
```
t= -4:.01:4;

subplot(3,2,1)
plot(t,f(t))
title('$f(t)$','interpreter','latex','fontsize',14)
grid on
ylim([-1/2 3/2])
subplot(3,2,2)
plot(t,f(-t))
title('$f_1(t)=f(-t)$','interpreter','latex','fontsize',14)
grid on
ylim([-1/2 3/2])
subplot(3,2,3)
plot(t,f(t-1)+f(1-t))
title('$f_2(t) = f(t-1) + f(1-t)$','interpreter','latex','fontsize',14)
grid on
ylim([-1/2 5/2])
subplot(3,2,4)
plot(t,f(t-1)+f(-t-1))
title('$f_3(t) = f(t-1)+f(-t-1)$','interpreter','latex','fontsize',14)
grid on
ylim([-1/2 5/2])
subplot(3,2,5)
plot(t,f(t-1/2)+f(-(t+1/2)))
title('$f_4(t) = f(t-1/2)+f(-(t+1/2))$','interpreter','latex','fontsize',14)
grid on
ylim([-1/2 3/2])
subplot(3,2,6)
plot(t, 3/2*f(t/2-1) )
title('$f_5(t) = 3/2f(t/2-1)$','interpreter','latex','fontsize',14)
grid on
ylim([-1/2 2])
```

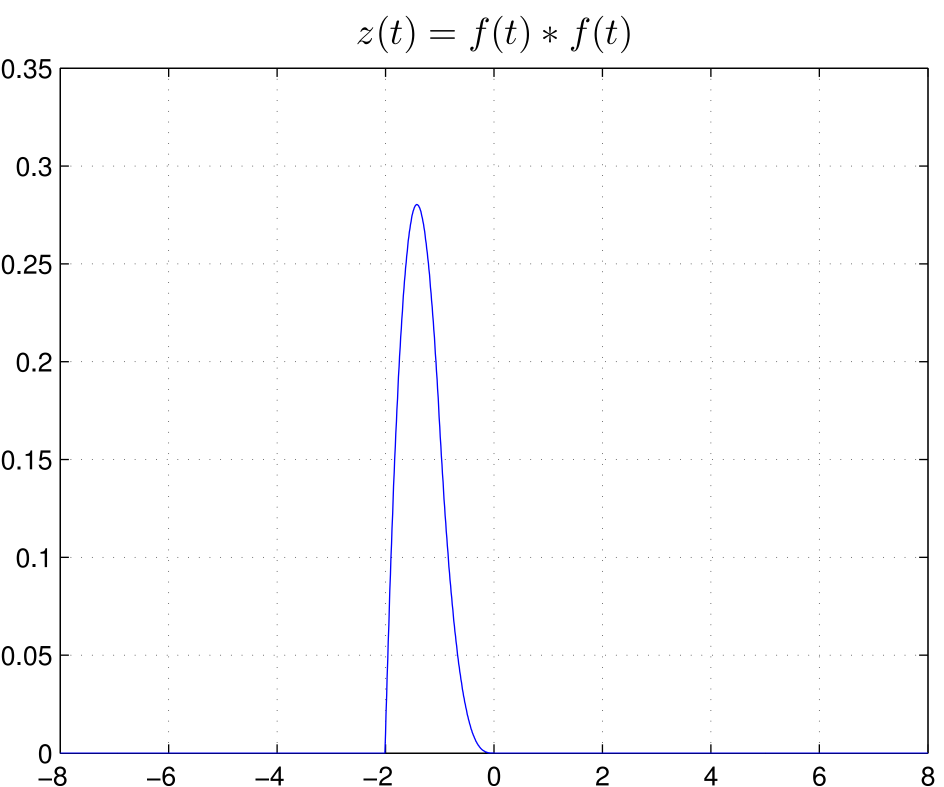
Problem 6.c)

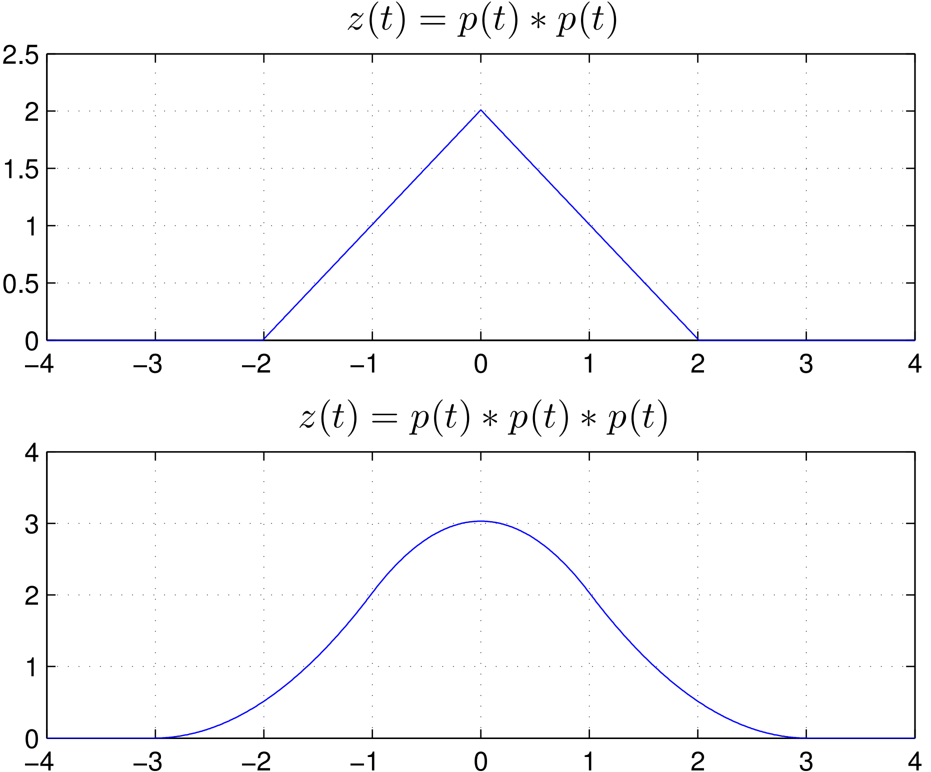
```
t = -4:.01:4;
z=0.01*conv(f(t),f(t));

figure
plot(-8:0.01:8,z)
title('$z(t)=f(t) * f(t)$','interpreter','latex','fontsize',14)
grid on
```

Problem 6.d)

```
function question6d
% convolves and plots two and three pulses, respectively.

figure;
% convolve two pulses
t = -4:.01:4;
[tp y] = conv_pulses(t, 2);

% plot two convolved pulses
subplot(2,1,1)
plot(tp,y)
title('$z(t)=p(t) * p(t)$','interpreter','latex','fontsize',14)
xlim([t(1) t(end)])
grid on

% convolve three pulses
t = -4:.01:4;
[tp y] = conv_pulses(t, 3);

% plot three convolved pulses
subplot(2,1,2)
plot(tp,y)
title('$z(t)=p(t) * p(t) * p(t)$','interpreter','latex','fontsize',14)
xlim([t(1) t(end)])
grid on
end

function [tp y] = conv_pulses(t, n)
% convolves n pulses together. pulses defined by p(t)

dt = t(2)-t(1);
y = p(t);
for i = 1:n-1
    y = conv(y,p(t))*dt;
end
tp = n*t(1):.01:n*t(end);

end

function z = p(t)
% definition of pulse
    z = f( (t-1)/2 ) + f( -(t+1)/2 );
end
```


f(t)
f_1(t) = f(−t)
f_2(t) = f(t − 1) + f(1 − t)
f_3(t) = f(t − 1) + f(−t − 1)
f_4(t) = f(t − 1/2) + f(−(t + 1/2))
f_5(t) = 3/2f(t/2 − 1)


$z(t) = f(t) * f(t)$
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
–8
–6
–4
–2
0
2
4
6
8


$z(t) = p(t) * p(t)$
2.5
2
1.5
1
0.5
0
–4
–3
–2
–1
0
1
2
3
4
$z(t) = p(t) * p(t) * p(t)$
4
3
2
1
0
–4
–3
–2
–1
0
1
2
3
4