1) $E_f = \int_{-\infty}^{\infty} |f(t)|^2 dt$ $\qquad g(t) = c f(at-b)$

$$E_g = \int_{-\infty}^{\infty} |c f(at-b)|^2 dt$$

$$= c^2 \int_{-\infty}^{\infty} |f(at-b)|^2 dt \qquad \begin{array}{l} u = at-b \\ du = a dt \end{array}$$

$$= \frac{c^2}{|a|} \int_{-\infty}^{\infty} |f(u)|^2 du$$

$$= \frac{c^2}{|a|} E_f$$

$-f(t)$ is $g(t)$ w/ $\begin{array}{l} a=1 \\ b=0 \\ c=-1 \end{array}$ $\quad E_g = E_f$

$f(-t)$ is $g(t)$ w/ $\begin{array}{l} a=-1 \\ b=0 \\ c=1 \end{array}$ $\quad E_g = E_f$

$f(t-T)$ is $g(t)$ w/ $\begin{array}{l} a=1 \\ b=T \\ c=1 \end{array}$ $\quad E_g = E_f$

2] a) $f(t) = [u(t) - u(t-1)] - [u(t-1) - u(t-3)] + \delta(t-3)$

$$= u(t) - 2u(t-1) + u(t-3) + \delta(t-3)$$

$$\int_{x=-\infty}^{t} [u(x) - 2u(x-1) + u(x-3) + \delta(x-3)]dx$$

$$= \int_{-\infty}^{t} u(x)dx - 2\int_{-\infty}^{t} u(x-1)dx + \int_{-\infty}^{t} u(x-3)dx + \int_{-\infty}^{t} \delta(x-3)dx$$

$$\int_{-\infty}^{t} u(x)dx = \begin{cases} 0 & t<0 \\ \int_{x=0}^{t} dx & t \geq 0 \end{cases} \Rightarrow tu(t)$$

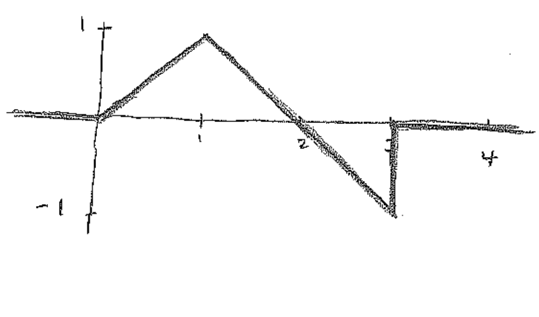
$$-2\int_{-\infty}^{t} u(x-1)dx = \begin{cases} 0 & t<1 \\ -2\int_{x=1}^{t} dx & t \geq 1 \end{cases} \Rightarrow -2(t-1)u(t-1)$$

$$\int_{-\infty}^{t} u(x-3)dx = \begin{cases} 0 & t<3 \\ \int_{x=3}^{t} dx & t \geq 3 \end{cases} \Rightarrow (t-3)u(t-3)$$

$$\int_{-\infty}^{t} \delta(t-3)dx = \begin{cases} 0 & t<3 \\ \int_{3^-}^{t} \delta(x-3)dx & t \geq 3 \end{cases} \Rightarrow u(t-3)$$

$$\int_{x=-\infty}^{t} f(x)dx = (u(t) - 2(t-1)u(t-1) + (t-3)u(t-3) + u(t-3))$$

$$= tu(t) - 2(t-1)u(t-1) + (t-2)u(t-3)$$

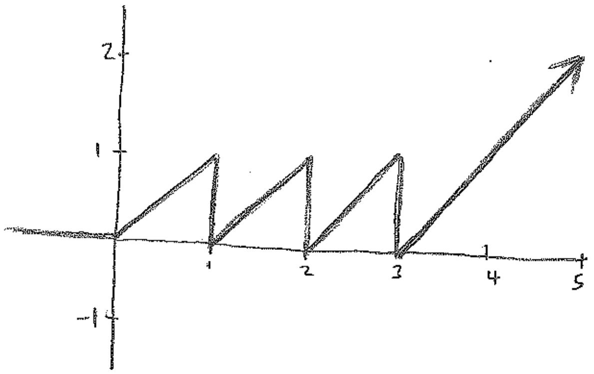


b) $f(t) = u(t) - \delta(t-1) - \delta(t-2) - \delta(t-3)$

$$\int_{x=-\infty}^{t} [u(x) - \delta(x-1) - \delta(x-2) - \delta(x-3)]\,dx$$

$$= \int_{-\infty}^{t} u(x)\,dx - \int_{-\infty}^{t} \delta(x-1)\,dx - \int_{-\infty}^{t} \delta(x-2)\,dx - \int_{-\infty}^{t} \delta(x-3)\,dx$$

$$= tu(t) - u(t-1) - u(t-2) - u(t-3)$$




3) To show that a system $H$ is linear we must show

$$k_1 x_1 + k_2 x_2 \longrightarrow \boxed{H} \longrightarrow k_1 y_1 + k_2 y_2$$

a) $\frac{dy(t)}{dt} + 3t\,y(t) = t^2 f(t)$

$\frac{dy_1(t)}{dt} + 3t\,y_1(t) = t^2 f_1(t)$ Multiply both sides by $k_1$ $\Rightarrow$ $k_1 \frac{dy_1(t)}{dt} + 3k_1 t\,y_1(t) = k_1 t^2 f_1(t)$

$\frac{dy_2(t)}{dt} + 3t\,y_2(t) = t^2 f_2(t)$ " " $k_2$ $\Rightarrow$ $k_2 \frac{dy_2(t)}{dt} + 3k_2 t\,y_2(t) = k_2 t^2 f_2(t)$

Add them together

$$\left(k_1 \frac{dy_1(t)}{dt} + k_2 \frac{dy_2(t)}{dt}\right) + 3t\underbrace{\left(k_1 y_1(t) + k_2 y_2(t)\right)}_{y(t)} = t^2 \underbrace{\left(k_1 f_1(t) + k_2 f_2(t)\right)}_{f(t)}$$

Which shows that this system is <u>linear</u>.

b) $3y(t)+2=f(t)$

$$3k_1 y_1(t)+2k_1=f_1(t)$$
$$+\quad 3k_2 y_2(t)+2k_2=f_2(t)$$

$$3\underbrace{\left(k_1 y_1(t)+k_2 y_2(t)\right)}_{y(t)}+2\underbrace{(k_1+k_2)}_{\neq 1}=\underbrace{\left(k_1 f_1(t)+k_2 f_2(t)\right)}_{f(t)}$$

Because of the $(k_1+k_2)$ term, this system is <u>nonlinear</u>.

$$f(t)=0 \Rightarrow y(t)=\frac{-2}{3}$$

c) $y(t)=\int_{-\infty}^{t} f(\tau)\,d\tau$

$$k_1 y_1(t)=\int_{-\infty}^{t} f_1(\tau)\,d\tau$$
$$k_2 y_2(t)=\int_{-\infty}^{t} f_2(\tau)\,d\tau \xrightarrow{+} \underbrace{\left(k_1 y_1(t)+k_2 y_2(t)\right)}_{y(t)}=\int_{-\infty}^{t}\underbrace{\left(k_1 f_1(\tau)+k_2 f_2(\tau)\right)}_{f(t)}d\tau$$

System is <u>linear</u>.

d) $\frac{dy(t)}{dt} + 2y(t) = f(t)\frac{df(t)}{dt}$

If this system is denoted as $H$ then

$$f(t) \longrightarrow \boxed{H} \longrightarrow y(t)$$

$$k_1 f_1(t) \longrightarrow \boxed{H} \longrightarrow k_1 y_1(t)$$ If system is linear.

Set $y(t) = k_1 y_1(t)$:

$$\frac{dy(t)}{dt} + 2y(t) = \left(\frac{d}{dt}+2\right)y(t) = \left(\frac{d}{dt}+2\right)k_1 y_1(t) \overset{(a)}{=} k_1 f_1(t)\frac{df_1(t)}{dt}$$

Set $f(t) = k_1 f_1(t)$:

$$f(t)\frac{df(t)}{dt} \overset{(b)}{=} k_1^2 f_1(t)\frac{df_1(t)}{dt}$$

Clearly, the equalities a and b are not equal so this system is <u>nonlinear.</u>

Ex. Let $f(t) = t$ and $g(t)$ be the corresponding output:

$$t \longrightarrow \boxed{H} \longrightarrow g(t).$$

Then to be linear, $H\{-t\} = -g(t)$ but

$$\frac{dg(t)}{dt} + 2g(t) = (t)\frac{d}{dt}(t) = t \quad \text{w/ } f(t) = t$$

$$\frac{d(-g(t))}{dt} + 2(-g(t)) \neq (-t)\frac{d}{dt}(-t) = t \quad \text{w/ } f(t) = -t$$

Using input $f(t) = t$ or $f(t) = -t$ produces the same non-zero output $g(t)$ so this system is nonlinear.

4) Same thing as problem 3. Need to show that if

$x[n] = k_1 x_1[n] + k_2 x_2[n]$ then $y[n] = k_1 y_1[n] + k_2 y_2[n]$.

a) $y[n] = \cos(\pi n) + x[n-4]$

$k_1 y_1[n] = k_1 \cos(\pi n) + x_1[n-4]$ $\quad k_2 y_2[n] = k_2 \cos(\pi n) + k_2 x_2[n-4]$

$$\underbrace{\left(k_1 y_1[n] + k_2 y_2[n]\right)}_{y[n]} = \underbrace{(k_1 + k_2)}_{\neq 1} \cos(\pi n) + \underbrace{\left(k_1 x_1[n-4] + k_2 x_2[n-4]\right)}_{x[n]}$$

Because of the $(k_1 + k_2)$ out front of the cosine term, this system is <u>nonlinear.</u> If $x[n] = 0 \ \forall n$, $y[n] = \cos(\pi n)$.

b) $y[n] = y[n-1] + x[n]$

$k_1 y_1[n] = k_1 y_1[n-1] + k_1 x_1[n]$ $\quad k_2 y_2[n] = k_2 y_2[n-1] + k_2 x_2[n]$

$$\underbrace{\left(k_1 y_1[n] + k_2 y_2[n]\right)}_{y[n]} = \underbrace{\left(k_1 y_1[n-1] + k_2 y_2[n-1]\right)}_{y[n-1]} + \underbrace{\left(k_1 x_1[n] + k_2 x_2[n]\right)}_{x[n]}$$

System is <u>linear.</u>

c) $y[n] = \sin(\pi n) + x[n-4] = x[n-4]$ since $\sin(\pi n) = 0 \ \forall n$

$k_1 y_1[n] = k_1 x_1[n-4]$ $\quad k_2 y_2[n] = k_2 x_2[n-4]$

$$\underbrace{\left(k_1 y_1[n] + k_2 y_2[n]\right)}_{y[n]} = \underbrace{\left(k_1 x_1[n] + k_2 x_2[n]\right)}_{x[n]}$$

System is <u>linear.</u>

d) $y[n] = \sum_{i=-\infty}^{n} x[n]$

$$k_1 y_1[n] = \sum_{i=-\infty}^{n} k_1 x_1[n] \qquad k_2 y_2[n] = \sum_{i=-\infty}^{n} k_2 x_2[n]$$

$$\underbrace{(k_1 y_1[n] + k_2 y_2[n])}_{y[n]} = \left(\sum_{i=-\infty}^{n} k_1 x_1[n] + \sum_{i=-\infty}^{n} k_2 x_2[n]\right)$$

$$= \sum_{i=-\infty}^{n} \underbrace{(k_1 x_1[n] + k_2 x_2[n])}_{x[n]}$$

System is <u>linear.</u>

3.5. -

If a system $H$ is time-invariant it means that if the system's response to input $f(t)$ is $y(t) = H\{f(t)\}$ then the system's response to input $g(t) = f(t-T)$ for any $T \in \mathbb{R}$ is $H\{g(t)\} = y(t-T)$

a) Suppose the system is time-invariant, and let $f(t)$ & $y(t)$ be a valid input/output pair. Then

$$y(t) = f(-t)$$

Now what is the systems response to $g(t) = f(t-T)$?

$$H\{g(t)\} = g(-t) = f(-t-T)$$

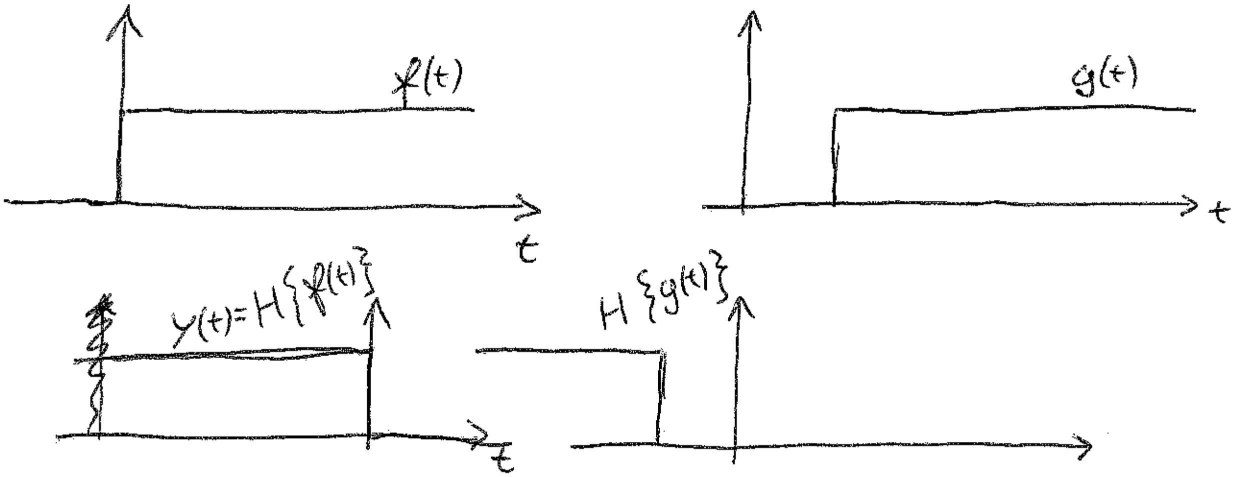
If the system is time-invariant then $H\{g(t)\} = y(t-T)$

$$= f(-(t-T)) = f(-t+T)$$

since in general $f(-t+T) \neq f(-t-T)$ the system is time-varying.

Let's see in particular how the system behaves when the inputs are $f(t) = u(t)$ and $g(t) = u(t-1)$

$H\{f(t)\} = u(-t) = y(t)$ $\qquad$ $H\{g(t)\} = u(-t-1)$




so clearly $H\{g(t)\}$ is not a shifted-to-the-right by 1 version of $y(t)$, and so the system is TV.

b) Note that the output of the system only depends on the behavior of the input for a fixed time-interval. This gives you a clue that it might be TV.

Let $f(t) = \delta(t)$, then $H\{f(t)\} = \int_{-3}^{3} \delta(t)\,dt = 1 = y(t)$

Now let $g(t) = f(t-6)$, then $H\{g(t)\} = \int_{-3}^{3} \delta(t-6)\,dt = 0$

therefore $H\{g(t)\} \neq y(t-6)$, and so the system is TV.

c) Let $y(t) = H\{f(t)\} = f(3t)$ and let's see what is the system's output to $g(t) = f(t-T)$.

$$H\{g(t)\} = g(3t) = f(3t - T)$$

Now $y(t-T) = f(3(t-T)) = f(3t - 3T)$

so in general ~~#~~ $y(t-T) \neq H\{g(t)\}$, and so the system is TV.

For example, if $f(t) = u(t)$ then $y(t) = u(t)$ (check this)

and $H\{f(t-1)\} = u(t - \frac{1}{3}) \neq y(t-1) = u(t-1)$

d) The system is ~~###~~ time-varying. As you have seen in probem 4.1, the response of a TI to a constant input must be constant, therefore if $x[n] = 1$ and the system is TI $H\{x[n]\}$ should be a constant, but in this case $y[n] = 2n$.

Of course you can solve the problem as before as well.

e) The system's response to an arbitrary input $x[n]$ is

$$H\{x[n]\} = (x[n] - x[n-1])^2 = y[n]$$

Let's see what is the output when the input is $z[n] = x[n-N]$.

$$H\{z[n]\} = (z[n] - z[n-1])^2 = (x[n-N] - x[(n-1)-N])^2$$
$$= (x[n-N] - x[n-N-1])^2$$

Now note that $y[n-N] = (x[n-N] - x[(n-N)-1])^2$
$$= (x[n-N] - x[n-N-1])^2$$

and so $y[n-N] = H\{z[n]\}$, since $x[\cdot]$ and $N$ were arbitrary the system is TI.

f) In this case we don't have an explicit input-output relation, and this often causes a lot of confusion.

Let's do it carefully. Suppose $x[n]$ and $y[n]$ are a valid input output pair, meaning that $y[n] = y[n-1] + x[n]$ for all $n \in \mathbb{N}$

Now let $x'[n] = x[n-N]$, a shifted version of $x[\cdot]$. If the system is TI then $H\{x'[n]\} = y'[n] = y[n-N]$

Let's check this is indeed the case, that is $x'[n]$ & $y'[n]$ are valid input-output pairs of signals.

~~If~~ so they must satisfy the input-output relation and therefore

$$y'[n] = y'[n-1] + x'[n] \qquad \forall n \in \mathbb{N}$$

$$\Leftrightarrow \quad y[n-N] = y[(n-1)-N] + x[n-N] \qquad \forall n \in \mathbb{N}$$

$$\Leftrightarrow \quad y[\underbrace{n-N}_{=m}] = y[n-N-1] + x[n-N] \qquad \forall n \in \mathbb{N}$$

But since we know that $y[n] = y[n-1] + x[n] \;\; \forall n \in \mathbb{N} \;\; \Leftrightarrow$

$$y[m] = y[m-1] + x[m] \quad \forall m \in \mathbb{N}$$

~~then~~ we see that indeed $H\{x'[n]\} = y[n-N]$

showing that the system is TI.

6)

$$y[n] = y[n-1] + x[n]$$
$$= y[n-1] + \delta[n]$$
$$y[n] = u[n]$$

| $n$ | $x[n]$ | $y[n-1]$ | $y[n]$ |
|---|---|---|---|
| $-1$ | 0 | 0 | 0 |
| 0 | 1 | 0 | 1 |
| 1 | 0 | 1 | 1 |
| 2 | 0 | 1 | 1 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $\infty$ | 0 | 1 | 1 |

You may also notice

$$y[n] = y[n-1] + x[n]$$
$$= y[n-2] + x[n] + x[n-1]$$
$$= y[n-3] + x[n] + x[n-1] + x[n-2]$$
$$= y[n-N] + \sum_{m=0}^{N-1} x[n-m]$$
$$= y[n-N] + \sum_{m=0}^{N-1} \delta(n-m)$$

The sum is equal to 1 if $n-N+1 \leq 0 \leq n$ which can be rewritten as $u[n] \cdot u[N-n-1]$

$$= y[n-N] + u[n]u[N-n-1]$$

Choose $N = n+1$ since it is arbitrary

$$= y[-1] + u[n]u[0]$$

where $y[-1]$ can be taken from the table above.

$$= u[n]$$