SPACE FILLING AND DEPLETION

Yuliy Baryshnikov* E. G. Coffman, Jr.** Predrag Jelenković**

*Department of Mathematics
University of Versailles–Saint–Quentin–en–Yvelines
45 Avenue des Etats–Unis
78035 Versailles Cedex, France

**Department of Electrical Engineering
Columbia University
New York, NY 10027, USA

Abstract. For a given $k \geq 1$, subintervals of a given interval $[0, X]$ arrive at random and are accepted (allocated) so long as they overlap fewer than k subintervals already accepted. Subintervals not accepted are cleared, while accepted subintervals remain allocated for random retention times before they are released and made available to subsequent arrivals. Thus, the system operates as a generalized many-server queue under a loss protocol. We study a discretized version of this system with many applications, including communication networks, surface adsorption/desorption processes, and reservation systems. Our primary interest is in steady-state estimates of the wasted space, i.e., the total length of available subintervals $kX - \sum \ell_i$, where the ℓ_i are the lengths of the subintervals currently allocated. We obtain explicit results for $k = 1$ and for general k with all retention times equal to 2, the classical *dimer* case. Our focus is on the asymptotic regime of large retention times.

1 Generalities

1.1 Configuration space.

Consider the interval $[0, X]$, X an integer, subdivided by the integers into *slots* of length 1. Call this interval the *book*, and fix an integer k, to be interpreted as the number of *channels* (or more generally the number of copies of a resource). Our terminology is motivated by certain applications to be mentioned later. An *interval* in this work is always composed of consecutive slots. A configuration $\mathbf{C}$ of intervals is simply a finite set of intervals in the book. For given k, a *packing* of the configuration $\mathbf{C}$ is a function on $\mathbf{C}$ with values in the set of k channels such that the intervals mapped into the same channel do not overlap; they can have common endpoints but no common interior points. We say that a configuration is *admissible* if there exists a packing of the configuration, i.e., if the intervals of the configuration can be placed over k channels with no overlapping.

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The *counting function* of configuration $\mathbf{C}$ counts the number of intervals containing a given non-integer point in the book:

$$n_{\mathbf{C}}(x) = \#\{I \in \mathbf{C} : x \in I\},\ x \in [0, X] - \mathbb{N};$$

the counting function is not defined at integer points. Clearly, an admissible configuration $\mathbf{C}$ satisfies

$$n_{\mathbf{C}} \leq k \text{ on } [0, X] - \mathbb{N}. \tag{1}$$

Analysis of a simple greedy algorithm, e.g., one that processes intervals in order of increasing left endpoint, shows that this condition is also sufficient.

Proposition 1.1. *If condition (1) is satisfied, the configuration* $\mathbf{C}$ *is admissible.*

Denote the set of admissible configurations on the interval $[0, X]$ by $\mathcal{C}_X$. The asymmetry in parameters reflects the point of view here; specifically, for fixed k, the interest is in large X asymptotics.

1.3 Dynamics.

Consider now the following Markov chain on the configuration space $\mathcal{C}_X$. Assume that for any integer point i an interval of length ℓ with the left endpoint i arrives at rate λ_ℓ; the arrivals of intervals at different points and of different lengths are independent.

A newly arrived interval is included in the configuration if the resulting configuration is admissible; otherwise the interval is rejected. As there is (a.s.) at most one candidate, this definition of the filling process is unambiguous.

The depletion process, i.e., the departure of intervals from configurations, has a similar description: the flow of "killing" signals for intervals of length ℓ arrive at each integer i at rate μ_ℓ. If at the time such a signal arrives, there is at least one interval of length ℓ with its left endpoint at i in the configuration, then one of them leaves.

It is convenient to assume that the arrival rates λ_ℓ vanish for all but a finite number of lengths ℓ, say $\lambda_\ell > 0,\ 1 \leq \ell \leq L$, and $\lambda_\ell = 0$ otherwise.

1.4 Background.

The above Markov chain finds diverse applications, although most of the known results are restricted to the pure filling model with $\mu_i = 0$ for all i. In his study of one-dimensional communication networks, Kelly [4] points out that the Markov chain models bus-connected local area networks. In this application, intervals correspond to the circuits connecting communicating parties and $[0, X]$ represents the bus. Kelly also mentions similar applications in computer interconnection networks and data base structures.

In operations research and engineering, applications to reservation systems have motivated considerable research, especially for the pure filling case [1,2]. In such models, intervals are the times during which copies of a resource (e.g., hotel rooms, communication channels, etc.) are reserved. The book is then the reservation book, and departures model the cancellation of reservations.

In the physics and chemistry literature, one finds applications to surface adsorption/desorption, granular densification, and polymer chain processes [3,6]. Evans [3] gives further applications and a great many references. We mention Renyi's [5] space filling problem, better known as the car parking problem, as a landmark early contribution that spawned a great deal of research (see [1] for many references).

To date, the best results are those of Kelly, who studies a model very similar to the one defined here, and of Ziedins [7], who studies the continuous relaxation of Kelly's model. Kelly's main results apply to the case $k = 1$ and to the case of general k with interval lengths governed by a geometric law. For $k = 1$, Kelly defines an alternating renewal process $\{\xi_j\}$ on the integers with values in $\{0, 1\}$; sequences of 1's between consecutive 0's represent accepted intervals. Explicit formulas are derived for computing the parameters of $\{\xi_j\}$. The probability distributions describing the finite process on $[0, X]$ are proved to be just those of $\{\xi_j, 0 \leq j \leq X\}$ conditioned on $\xi_{-1} = \xi_{X+1} = 0$. The vacant intervals of $\{\xi_j\}$, i.e., the strings of 0's separating consecutive 1's, are shown to have a geometric distribution.

The focus here is on space utilization, so the results here add to those of Kelly in two principal ways. First, detailed calculations are worked out for total vacant space. Special consideration is given to small-μ asymptotics, for behavior in this limit is quite different from that seen in the "jamming" limit (absorbing state) of Renyi's pure filling model, where $\mu = 0$ (see [6] for the results of simulations supporting this fact). Second, the domino case where all intervals have length 2 (the important *dimer* case in chemical applications [3]) is worked out in detail.

1.5 Stationary distribution.

Let $q_\ell = \lambda_\ell/\mu_\ell$, and note immediately that the Markov chain defined on $\mathcal{C}_X$ is reversible. Its stationary probabilities are given by

$$\pi(\mathbf{C}) = Z_X^{-1} \prod_{\ell: q_\ell > 0} q_\ell^{n_\ell(\mathbf{C})}, \tag{2}$$

where $n_\ell(\mathbf{C})$ is the number of intervals of length ℓ in the configuration $\mathbf{C}$, and Z_X is the partition function (normalizing constant)

$$Z_X(\overline{q}) = \sum_{\mathbf{C} \in \mathcal{C}_X} \prod_{\ell: q_\ell > 0} q_\ell^{n_\ell(\mathbf{C})} = \sum_{\overline{n}} N(\overline{n}) \overline{q}^{\overline{n}}. \tag{3}$$

where $\overline{q}^{\overline{n}} = \prod_{\ell: q_\ell > 0} q_\ell^{n_\ell(\mathbf{C})}$, and where $N(\overline{n}), \overline{n} = (n_1, n_2, \ldots, n_L)$, is the number of configurations having n_1 intervals of length 1, n_2 intervals of length 2, $\ldots$.

Let $u(\mathbf{C}) = \sum_\ell \ell n_\ell(\mathbf{C})$ be the total used space and $v(\mathbf{C}) = kX - u(\mathbf{C})$ the total wasted space in configuration $\mathbf{C}$, and extend the partition function to the following polynomial in a formal variable x:

$$Z_X(x; \overline{q}) = \sum_{\mathbf{C} \in \mathcal{C}_X} x^{v(\mathbf{C})} \prod_{\ell: q_\ell > 0} q_\ell^{n_\ell(\mathbf{C})}. \tag{4}$$

In words, $Z_X(x, \overline{q})$ is the generating polynomial for the wasted space in admissible configurations on $[0, X]$.

To wrap up all the data uniformly, form the generating function for all book lengths X,

$$Z(x, y; \overline{q}) = \sum_X Z_X(x; \overline{q}) y^X.$$

Hereafter, we suppress the dependence on $\overline{q}$ in the generating function notation.

1.6 Wasted space.

We find that the average wasted space over admissible configurations in a book of length X is given by

$$\langle v \rangle_X = Z_X^{-1} \frac{\partial Z_X}{\partial x} \bigg|_{x=1} \tag{5}$$

where $Z_X = Z_X(x)$ is given by [4], and where the subscript in $\langle v \rangle_X$ denotes averaging over the stationary probabilities of the Markov chain on $\mathcal{C}_X$. To find the two terms on the right-hand side, we use the residue method. That is, we compute

$$Z_X|_{x=1} = \mathrm{Res}_0\, y^{-X-1} Z(x, y; \overline{q})\big|_{x=1} \tag{6}$$

and

$$\frac{\partial Z_X}{\partial x}\bigg|_{x=1} = \mathrm{Res}_0 y^{-X-1} \frac{\partial}{\partial x} Z(x, y; \overline{q})\bigg|_{x=1}. \tag{7}$$

We are interested chiefly in the asymptotic analysis of the expected wasted space for large X. If the partition function is rational, $Z = P/Q$, then, in a standard way, one can use the residue theorem to pinpoint the leading terms of the asymptotics. Indeed, fix x and let $y_1(x), \ldots, y_n(x)$ be the roots of the polynomial Q (some of them can, of course, collide or escape to infinity as x varies). For $X \gg 1$ the contour integral of (6) or (7) over circles of large radius tends to zero, whence

$$Z_X|_{x=1} = -\sum_m \mathrm{Res}_{y_m} \frac{y^{-X-1} P}{Q} \tag{8}$$

and

$$\frac{\partial Z_X}{\partial x}\bigg|_{x=1} = -\sum_m \mathrm{Res}_{y_m}\, y^{-X-1} \frac{\partial}{\partial x} \frac{P}{Q}\bigg|_{x=1}. \tag{9}$$

It is readily seen that (8) and (9) are dominated by the residues at the roots closest to the origin.

Further, if $y_m(x)$ is a root of Q of multiplicity α, then the contribution of the corresponding residue to (8) is a polynomial in X of degree $(m-1)$ with the leading coefficient being

$$y_m^{-X-\alpha} \frac{P(y_m)}{\tilde{Q}(y_m)},$$

where we just write y_m for $y_m(1)$, and $\tilde{Q} = \frac{1}{m!}Q^{(m)}$.

Similarly, the leading coefficient for (9) is

$$y_m^{-X-\alpha-1} \left.\frac{d}{dx} y_m(x)\right|_{x=1} \frac{P(y_m)}{\tilde{Q}(y_m)}.$$

Therefore, one can deduce the following useful expression for $\frac{\partial Z_X}{\partial x}/Z_X$.

Proposition 1.2. *Assume that there is a single root y_m of Q with the least absolute value. Then the ratio $\frac{\partial Z_X}{\partial x}/Z_X$ is given by*

$$X y_m^{-1} \frac{dy_m}{dx}$$

up to a $O(X^0)$ term.

2. Single channel case

We consider here $k = 1$, in which case the partition function $Z(x, y; \overline{q})$ can be found easily.

Proposition 2.1. *One has*

$$Z(x, y; \overline{q}) = \frac{1}{1 - yx - \sum_\ell q_\ell y^\ell}. \tag{10}$$

Proof. A standard exercise, as follows. For notational simplicity, let $q_\ell > 0$ for $\ell = 1, \ldots, L$ and $q_\ell = 0$ otherwise. From (3) and (4)

$$\begin{aligned} Z_X(x, y; \overline{q}) &= \sum_{v=0}^{\infty} \sum_{\overline{n}} N(n_1, n_2, \ldots n_L, v) x^v \prod_{\ell: q_\ell > 0} q_\ell^{n_\ell} y^{v + n_1 + 2n_2 + \cdots L n_L}, \\ &= \sum_{v=0}^{\infty} \sum_{\overline{n}} \binom{v + n_1 + \cdots + n_L}{v \;\; n_1 \;\; \cdots \;\; n_L} (xy)^v \prod_{\ell: q_\ell > 0} q_\ell y^\ell. \end{aligned}$$

Reorganize the sums, changing the double sum to $\sum_{r=0}^{\infty} \sum_{(v,\overline{n}): v + n_1 + \cdots + n_L = r}$, to obtain

$$Z_X(x, y; \overline{q}) = \sum_{r=0}^{\infty} \left(xy + \sum_{\ell=1}^{L} q_\ell y^\ell \right)^r$$

and hence (10). □

By (10),

$$Z_X|_{x=1} = \mathrm{Res}_0 \frac{y^{-X-1}}{1 - y - \sum_\ell q_\ell y^\ell} \tag{11}$$

and

$$\left.\frac{\partial Z_X}{\partial x}\right|_{x=1} = \mathrm{Res}_0 \frac{y^{-X}}{(1-y-\sum_\ell q_\ell y^\ell)^2}. \tag{12}$$

Denote the denominator of (11) or (12) by $Q(y)$ and let $y_1, \dots, y_n$ be the roots of the polynomial Q. For $X \geq 1$ the contour integral of (11) or (12) over circles of large radius tends to zero, whence

$$Z_X|_{x=1} = -\sum_m \mathrm{Res}_{y_m} \frac{y^{-X-1}}{1-y-\sum_\ell q_\ell y^\ell} \tag{13}$$

and

$$\left.\frac{\partial Z_X}{\partial x}\right|_{x=1} = -\sum_m \mathrm{Res}_{y_m} \frac{y^{-X}}{(1-y-\sum_\ell q_\ell y^\ell)^2}. \tag{14}$$

We are interested in the asymptotics of $\langle v\rangle_X/X$ as $X \to \infty$. In this situation, it is readily seen that (13) and (14) are dominated by the residues at the roots closest to the origin. One checks immediately that the (single) real root (say, y_1) is the root of Q closest to the origin and that this root is simple, i.e., the derivative of Q at y_1 does not vanish.

If we assume that A does not vanish at the point y_1, then the residue of A/Q is $A(y_1)/Q'(y_1)$ and the residue of A/Q^2 is $A'(y_1)/(Q'(y_1))^2 - A(y_1)/(Q'(y_1))^3$. Hence,

$$Z_X|_{x=1} = \frac{y_1^{-X-1}}{1+\sum_\ell \ell q_\ell y_1^{\ell-1}}(1+O(\gamma^{-X})), \tag{15}$$

where $\gamma = \max_{m>1} |y_m/y_1|$ is the spectral gap, and

$$\left.\frac{\partial Z_X}{\partial x}\right|_{x=1} = \frac{X y_1^{-X-1}}{(1+\sum_\ell \ell q_\ell y_1^{\ell-1})^2} + O(1), \tag{16}$$

where the error term comes from the second summand in the formula for residues of A/Q^2.

Summarizing, we get

$$\langle v\rangle_X/X = \frac{1}{1+\sum_\ell \ell q_\ell y_1^{\ell-1}} + O(1/X). \tag{17}$$

In fact, the last formula can be rewritten in a slightly more general fashion. If the generating function in question is meromorphic, that is the ratio P/Q of two analytic functions, then the limiting asymptotic value of $\langle v\rangle_X/X$ is equal to

$$y_1^{-1}(dy_1/dx), \tag{18}$$

where $y_1(x)$ is a local branch of the curve $Q(x,y)=0$, corresponding to the root of $Q(x,\cdot)$ with the least absolute value, provided that this root is unique and that P does not vanish at this root. We will use this observation in what follows.

Formula (17) gives the asymptotic value of the average wasted space per unit length for long intervals X. Of interest for us is the behavior of this average wasted space as the rates $q_i \to \infty$. One cannot expect any reasonable limiting behavior without further assumptions. What we need is the following:

Assumption. *Let ρ be the unique real root of the polynomial*

$$\sum_\ell q_\ell y^\ell - 1.$$

Then we assume that $\rho \to 0$ and that the (nonzero) rescaled coefficients q_ℓ/ρ^ℓ converge to nonnegative coefficients $c_\ell = \lim q_\ell \rho^{-\ell}$

We notice that $\sum_\ell c_\ell = 1$ and therefore 1 is the unique real root of the polynomial $\tilde{Q} = 1 - \sum c_\ell y^\ell$. It follows in particular that $y_1/\rho \to 1$. Further, easy calculations yield the following asymptotic result.

Proposition 2.2. *Under the assumption stated above, the average wasted space per unit length scales as ρ. More precisely, as $\rho \to 0$,*

$$\rho^{-1}\frac{\langle v\rangle_X}{X} \to \frac{1}{\sum_\ell \ell c_\ell}$$

One might interpret the denominator on the right-hand side as the average conditional length in the rescaled interval flow.

3. "Domino packing", k-channels

Now we consider the k-channel situation. In general, the analysis seems to be quite involved, so we restrict ourselves to the case of "domino" packing, where all intervals have length 2, and an essentially complete analysis is rather easy, once one discovers the inductive structure of packings. As above, we denote by $Z(x, y; q)$ the generating (or partition) function. In our current situation we have only one parameter q, the retention rate for the incoming intervals. We start with a general analysis, and then center on the asymptotics for $q \to \infty$.

Generating function. Let $f_X^j[n]$ be the number of elements in the set $\mathcal{C}_X^j[n]$ of admissible configurations on $[0, X+1]$ with exactly j dominos straddling X and n units of wasted space in $[0, X]$. Observe that, for $X > 1$, each configuration in $\mathcal{C}_X^j[n]$ is, for some m, $0 \le m \le k-j$, a configuration in $\mathcal{C}_{X-1}^m[n-(k-m-j)]$ to which dominos have been inserted into the interval $[X-1, X+1]$ of exactly j channels. Then $f_X^j[n] = \sum_{m=0}^{k-j} f_{X-1}^m[n-(k-m-n)]$, and the generating polynomial $f_X^j(x) = \sum_n f_X^j[n]x^n$ for wasted space satisfies

$$f_X^j = \sum_{m=0}^{k-j} x^{k-m-j} f_{X-1}^m, \ X > 1.$$

Now define $\overline{f}_X(x) = (f_X^0(x), \dots, f_X^k(x))$, form the generating (vector) function $\overline{f}(x, y) = \sum_X \overline{f}_X y^X$, and find immediately that it solves

$$(E - yA(x))\overline{f}(x, y) = f_1(x), \tag{19}$$

where E is the $(k+1)\times(k+1)$ identity matrix and

$$A(x) = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & x \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \dots & x^{k-2} & x^{k-1} \\ 1 & x & \dots & x^{k-1} & x^k \end{pmatrix}. \tag{20}$$

Notice that Z is just the component f^0 of $\overline{f}$.

Formula (20) implies that

$$Z = \frac{P(x,y)}{\det\left(E - yA(x)\right)},$$

in which the polynomial

$$Q(x,y) = \det(E - yA(x))$$

is central to our analysis. Note that for fixed (real) x, the zeros of Q are the reciprocal eigenvalues of the symmetric matrix $A(x)$, which is also real.

The following facts can be established by a direct if tedious analysis.

Lemma 3.1. *a)* $Q(0,y) = (y-1)^i(y+1)^i$ *for odd* k, $k = 2i-1$, *and* $Q(0,y) = (y-1)^{i+1}(y+1)^i$ *for even* k, $k = 2i$.

b) In the local coordinates (ξ,η) *near either of the points* $(0,1)$ *or* $(0,-1)$ *in the* (x,y) *plane, the polynomial* Q *has a nondegenerate homogeneous component* $q_\pm(\xi,\eta)$ *(of degree equal to the order of the zero of its restriction to the* y*-axis).*

c) The dehomogenizations of the polynomials $q_\pm$ *(defined above) given by*

$$\tilde{q}_\pm(c) = \xi^{-\deg q_\pm} q_\pm(\xi, c\xi)$$

can be expressed in terms of Chebyshev polynomials with all roots simple and real.

Proof. Multiplying $E - yA(x)$ from the right by the matrix

$$P(x) = \begin{pmatrix} 1 & -x & 0 & \dots & 0 & 0 \\ 0 & 1 & -x & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -x \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \tag{21}$$

antidiagonalizes A, so that

$$\det(E - yA(x)) = \det(P(x) - y\Delta),$$

where Δ is the matrix with 1's on the side diagonal and 0's elsewhere (here we used $\det(P) = 1$).

Rearranging the rows and columns of $P(x) - y\Delta$ we obtain that $\det(P(x) - y\Delta)$ is equal (up to a sign which is immaterial as we are concerned with the zeros of the determinant only) to the determinant of the matrix

$$S_k(x,y) = \begin{pmatrix} -y & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & -y & -x & 0 & \dots & 0 & 0 & 0 \\ 0 & -x & -y & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & -y & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & -y & -x \\ 0 & 0 & 0 & 0 & \dots & 0 & -x & 1-y \end{pmatrix} \tag{22}$$

for the matrix size $(k+1)$ even and

$$S_k(x,y) = \begin{pmatrix} -y & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & -y & -x & 0 & \dots & 0 & 0 & 0 \\ 0 & -x & -y & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & -y & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -x & -y & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & -x-y \end{pmatrix} \tag{23}$$

for $(k+1)$ odd.

For a matrix M denote by M' the matrix obtained from M by deleting the first column and the first row. (Notice that $S''_n = S_{n-2}$). One finds immediately that

$$\det(S_n) = -y\det(S'_n) - \det(S_{n-2}).$$

Further,

$$\det(S'_n) = -y\det(S_{n-2}) - x^2\det(S'_{n-2}).$$

Iterating and combining all equalities together one arrives at the recursion

$$\det(S_n)_n = (y^2 - x^2 - 1)\det(S_{n-2}) - x^2\det(S_{n-4}) \tag{24}$$

for $n \geq 4$.

One sees that the polynomials $s_n(x,y) = \det(S_n)$ split naturally into 2 series, for even and for odd matrix sizes. One more splitting: we need to consider the polynomials s near two points $(0, \pm 1)$.

Consider the polynomials $\tilde{q}_{n,\pm}(\xi,\eta) = s_n(\xi, \pm 1 + \eta)$. For small n the polynomials can be found manually:

$$\begin{aligned} \tilde{q}_{0,\pm} &= 1; \\ \tilde{q}_{1,+} &= -\eta, \quad \tilde{q}_{1,-} = 2 - \eta; \\ \tilde{q}_{2,+} &= \xi + 2\eta + \xi\eta + \eta^2, \quad \tilde{q}_{2,-} = -\xi - 2\eta + \xi\eta + \eta^2; \\ \tilde{q}_{3,+} &= -2\eta^2 + \xi^2 + \eta\xi^2 - \eta^3, \quad \tilde{q}_{3,-} = -4\eta - \xi^2 + 4\eta^2 + \xi^2\eta - \eta^3. \end{aligned}$$

From these initial data and from (24) it follows that all monomials in $\tilde{q}_{2m,\pm}$ have degree m at least. Similarly, all monomials in $\tilde{q}_{2m+1,+}$ have degree $\geq (m+1)$, and in $\tilde{q}_{2m+1,-}$ degree $\geq m$. Denote the homogeneous components of the corresponding (minimal) degree as $q_{n,\pm}$. Introduce the polarizations

$$\begin{aligned} Q^e_{m,\pm}(c) =& \frac{q_{2m,\pm}(\xi, c\xi)}{c^m}; \\ Q^o_{m,+} =& \frac{q_{2m+1,+}(\xi, c\xi)}{c^{m+1}}; \\ Q^o_{m,1} =& \frac{q_{2m+1,-}(\xi, c\xi)}{c^m}. \end{aligned}$$

Then (24) again implies that the polynomial Q_m satisfy the recursions

$$Q^{\cdot}_{m,\pm} = \pm 2cQ^{\cdot}_{m-1} - Q^{\cdot}_{m-2}.$$

Together with the inital data, these recursions allow one to give explicit formulas for the Q's:

$$\begin{aligned} Q^e_{m,\pm} =& (\pm 1)^m U^{(2)}_m + U^{(2)}_{m-1}, \\ Q^o_{m,+} =& - U^{(1)}_{m+1}, \\ Q^o_{m,-} =& (-1)^m 2U^{(2)}_m. \end{aligned}$$

Here $U^{(1)}_m(c) = \cos(m \arccos(c))$ and $U^{(2)}_m(c) = \sin((m+1)\arccos(c)/\sin(\arccos(c)))$ are Chebyshev polynomials of the 1-st and of the 2-nd kind, respectively.

It is easy to check that all the polynomials Q have simple roots. Relevant for us are the maximal roots of the polynomials $Q^{\cdot}_{\cdot,-}$ and minimal roots of the polynomials $Q^{\cdot}_{\cdot,+}$ (they correspond to the branches with least absolute value of $y(x)$). They can be immediately found to be

$$\begin{aligned} &\cos\left(\frac{2\pi}{2m+1}\right) \quad \text{for} \quad Q^e_{m,-}; \\ &\cos\left(\frac{2m\pi}{2m+1}\right) \quad \text{for} \quad Q^e_{m,+}; \\ &\cos\left(\frac{\pi}{2m}\right) \quad \text{for} \quad Q^0_{m,-}, m > 0; \\ &-\cos\left(\frac{\pi}{2(m+1)}\right) \quad \text{for} \quad Q^0_{m,+}. \end{aligned}$$

□

The lemma implies that locally at each of the points $(0,1)$ and $(0,-1)$ the curve $Q = 0$ splits into some number of smooth branches *with different slopes.* This implies that the derivative dy_1/dx (one parametrizes the branch by the x coordinate) for the branch with the smallest absolute value of y does not vanish. In its turn, this allows us to determine the asymptotics of $\langle v \rangle_X / X$ for $q \to \infty$.

Proposition 3.2. *As $q \to \infty$, the average wasted space per unit length scales as $q^{-k/2}$:*

$$\frac{\langle v \rangle_X}{X} q^{k/2} \to c(k)$$

for some $c(k) > 0$.

Proof. To study the equilibrium distribution we have to weight by q^j every configuration with j dominos placed. The number of dominos placed on the k channels over the interval $[0, X]$ with n units of wasted space is $(kX - n)/2$. Hence, each term $y^X x^n$ in the formal series Z should be weighted by $q^{(kX-n)/2}$, or, equivalently, x should be weighted by $q^{-1/2}$ and y should be weighted by $q^{k/2}$. The relevant roots of $Q = 0$ are on the line $\{x = 1\}$. As $q \to \infty$, the roots of Q in the rescaled variables sit on $\{x = q^{-1/2}\}$, and therefore converge to $(0, \pm 1)$.

Using formula (13) we conclude that

$$(\langle v \rangle_X / X) q^{k/2} \to dy_1/dx|_{x=q^{-1/2}},$$

as $X \to \infty$, and by the previous lemma, dy_1/dx tends to a nonvanishing constant as $q \to \infty$. This proves the proposition. □

The constants $c(k)$ can be found explicitly. In particular, $1 - c(k)$ scales as k^{-2}.

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