Packing Random Intervals On-Line

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Abstract

Starting at time 0, unit-length intervals arrive and are placed on the positive real line by a unit-intensity Poisson process in two dimensions; the probability of an interval arriving in the time interval $[t, t+\Delta t]$ with its left endpoint in $[y, y+\Delta y]$ is $\Delta t \Delta y + o(\Delta t \Delta y)$. Fix $x \geq 0$. An arriving interval is *accepted* if and only if it is contained in $[0, x]$ and overlaps no interval already accepted.

We study the number $N_x(t)$ of intervals accepted during $[0, t]$. By Laplace-transform methods, we derive large-x estimates of $\mathrm{E}N_x(t)$ and $\mathrm{Var}N_x(t)$ with error terms exponentially small in x uniformly in $t \in (0, T)$, where T is any fixed positive constant. We prove that, as $x \to \infty$, $\mathrm{E}N_x(t) \sim \alpha(t)x$, $\mathrm{Var}N_x(t) \sim \mu(t)x$, uniformly in $t \in (0, T)$, where $\alpha(t)$ and $\mu(t)$ are given by explicit, albeit complicated formulas. Using these asymptotic estimates we show that $N_x(t)$ satisfies a central limit theorem, i.e., for any fixed t

$$\frac{N_x(t) - \mathrm{E}N_x(t)}{\sqrt{\mathrm{Var}(N_x(t))}} \xrightarrow{d} \mathcal{N}(0,1) \quad \text{as} \quad x \to \infty,$$

where $\mathcal{N}(0,1)$ is a standard normal random variable, and $\xrightarrow{d}$ denotes convergence in distribution. This stochastic, on-line interval packing problem generalizes the classical parking problem, the latter corresponding only to the absorbing states of the interval packing process, where successive packed intervals are separated by gaps less than 1 in length. We verify that, as $t \to \infty$, $\alpha(t)$ and $\mu(t)$ converge to $\alpha_* = .748\ldots$ and $\mu_* = .03815\ldots$, the constants of Renyi and Mackenzie for the parking problem. Thus, by comparison with the parking analysis in a single space variable, ours is a transient analysis involving both a time and space variable.

Our interval packing problem has applications similar to those of the parking problem in the physical sciences, but the primary source of our interest is the modeling of reservation systems, especially those designed for multi-media communication systems to handle high-bandwidth, real-time demands.

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1 Introduction

Let $(a_1, b_1), \ldots, (a_n, b_n)$ be a sequence of n independent random intervals in $\mathbf{R}_+$ to be packed on-line in a given interval $[0, x]$; an interval $I_i = (a_i, b_i)$ is packed (marked as occupied in $[0, x]$) if and only if $I_i \subseteq [0, x]$ and I_i does not overlap any interval in $I_1, \ldots, I_{i-1}$ that has already been packed. Suppose the I_i are drawn independently from a given interval distribution. To avoid trivialities, we assume that the distribution is such that $I_i \subseteq [0, x]$ with probability 1. The *on-line interval packing problem* asks for the expectation of the number $N_x(n)$ of intervals packed as a function of n and the parameters of the interval distribution.

The asymptotic behavior of the expected number packed on-line was found in [4] for the case where, for each i, a_i and b_i are taken as the smaller and larger of two independent random draws from $[0, x]$. In this case, x is only a scale factor and does not figure in the result. The asymptotic expected number packed is given by

$$\mathrm{E}N_x(n) \sim cn^{(\sqrt{17}-3)/4}$$

as $n \to \infty$, where $c \approx 0.98$ approximates the value of an explicit formula given in [4].

In this paper, by a completely different analysis, we obtain the corresponding result for the case where the a_i are independent uniform random draws from $[0, x-1]$ and all intervals have unit lengths ($b_i = a_i + 1$). Now, the result clearly depends on x; indeed, we prove in Section 2 the following limit law for the expected number packed: If for some constant $\lambda > 0$, we put $n = \lfloor \lambda x \rfloor$, then

$$\lim_{x \to \infty} \frac{\mathrm{E}N_x(\lfloor \lambda x \rfloor)}{x} = \alpha(\lambda), \tag{1}$$

where

$$\alpha(\lambda) = \int_0^\lambda \exp\left(-2\int_0^v \frac{1-e^{-x}}{x}dx\right) dv. \tag{2}$$

We note in passing that the probabilistic analysis of optimal *off-line* interval packing has been worked out in [10], where the number packed is to be maximized, and in [5], where the unoccupied space of the packing is to be minimized. Also, for the worst-case analysis of combinatorial models of on-line interval packing, see [8, 11].

Our interest in on-line interval packing originated in a stochastic version, where the arrival times and left endpoints of the intervals to be packed form a two-dimensional Poisson process with a mean normalized to 1 per unit time per unit distance. For this version, which we will call the *Poisson* model, we use the same notation $N_x(t)$ for the number packed during $[0, t]$ in $[0, x]$; except where noted otherwise, a continuous argument signals that it refers to the Poisson model. We let $K(t, x)$ denote the expected number packed during $[0, t]$ into $[0, x]$, and we prove in Section 2 that, as might be expected, $K(t, x)$ has a property similar to that in (1),

$$K(t, x) \sim \alpha(t)x \tag{3}$$

as $x \to \infty$ with t fixed, where $\alpha(.)$ is given by (2). As we will verify, the Poissonization of the input length makes the analysis easier; we then obtain (1) from (3) via a simple argument based on the concentration of the Poisson distribution around its mean.

The classical parking problem of Renyi [17] has an intimate relationship with on-line interval packing. In the former problem, unit-length cars are parked sequentially along a curb (interval) $[0,x]$, $x>1$. Each car chooses a parking place independently and uniformly at random from those available, i.e., from those where it will not overlap cars already parked or the curb boundaries. Thus, the left endpoint of the first car is a uniform random draw from $[0,x-1]$, and if the first car is parked in $[y,y+1]$, then the left endpoint of the second car parked is drawn uniformly at random from $[0,y-1]$, $[0,y-1]\cup[y+1,x-1]$, or $[y+1,x-1]$ according as $y\geq 1$ and $y>x-2$, $1\leq y\leq x-2$, or $y<1$ and $y\leq x-2$, respectively. This uniform parking of cars continues until every unoccupied gap is less than 1 in length, i.e., no further cars can be parked. Renyi [17] showed that the mean of $\tilde{N}_x$, the number parked at the conclusion of the process, satisfies $\mathrm{E}\tilde{N}_x \sim \alpha_* x,\ x\rightarrow\infty$, with the *Renyi constant* $\alpha_* = \lim_{\lambda\rightarrow\infty}\alpha(\lambda)$ determined by (2). This result actually follows from a more refined estimate of $\mathrm{E}\tilde{N}_x$ which was later improved by Dvoretzky and Robbins [6]. (The more precise estimate is given in Section 3 for comparison with our new results.) For many other improvements and extensions of the results on the parking problem, see [6, 7, 12, 13, 14] and the references therein.

Although the parking process differs from the on-line (unit) interval packing process, it is easy to verify that, given n cars (unit intervals) already parked (packed) in $[0,x]$, the conditional joint distribution of the $n+1$ gaps is the same under both processes. Moreover, (1) proves

$$\lim_{\lambda\rightarrow\infty}\lim_{x\rightarrow\infty}\frac{\mathrm{E}N_x(\lfloor\lambda x\rfloor)}{x} = \lim_{x\rightarrow\infty}\frac{\mathrm{E}\tilde{N}_x}{x} = \alpha_*. \tag{4}$$

Thus, in the limit, the fraction of the positive real line that is occupied is the same under both processes.

The interval packing process is said to have converged to an absorbing state if it has reached a state in which all gaps have lengths less than 1. Although one expects some strong form of convergence to an absorbing state, i.e., of $N_x(t)$ to $\tilde{N}_x$ as $t\rightarrow\infty$, it is surprising at first to find that, for all $x>2$, the expected time to absorption of the interval packing process is infinite. We will show in Section 2 that, if T_x denotes the time-to-absorption of the interval packing process, then for any fixed x, $\mathbf{P}(T_x>t)$ tends to 0 like $1/t$, and hence, T_x is finite almost surely, but $\mathrm{E}T_x=\infty$.

Mackenzie [12] showed that the asymptotic variance, as $x\rightarrow\infty$, is given by

$$\mathrm{Var}(\tilde{N}_x) \sim \mu_* x, \tag{5}$$

where, in terms of $\tilde{\alpha}(y) := \alpha_* - \alpha(y)$, and

$$\beta(y) := \exp\left(-2\int_0^y \frac{1-e^{-z}}{z}dz\right), \tag{6}$$

the constant is given by

$$\begin{aligned}\mu_* &= 4\int_0^\infty \left[e^{-y}(1-e^{-y})\frac{\tilde{\alpha}(y)}{y} - \frac{e^{-2y}(e^{-y}-1+y)\tilde{\alpha}^2(y)}{\beta(y)y^2}\right]dy - \alpha_*, \\ &= 0.03815\ldots.\end{aligned} \tag{7}$$

For the Poisson model of on-line interval packing, the variance of the number packed in $[0,x]$ during $[0,t]$ has a similar form. As shown in Section 4,

$$\mathrm{Var}(N_x(t)) \sim \mu(t)x, \tag{8}$$

as $x \to \infty$, where $\mu(t)$ depends only on t and is strictly positive for all $t > 0$; a complicated, explicit formula is given later and shown to have the property that $\mu(t) \to \mu_*$, as $t \to \infty$, which is not surprising in view of the convergence of the on-line interval packing problem to the parking problem.

Dvoretzky and Robbins [6] gave a central limit theorem for the parking problem, basing the second of their two proofs on the bound $\tilde{N}_x \leq x$ and the fact that the variance of $\tilde{N}_x$ was asymptotically linear in x. Since $N_x(t)$ has the same properties, one might hope to adapt the technique to our problem so as to obtain a central limit theorem for any fixed t. This is indeed possible, as we will see in Section 4; the difficulties arising from our more general model are easily handled.

We conclude this section with a brief discussion of applications. There have been many applications of the parking problem in the physical sciences, including space-filling problems, molecular adsorption on surfaces, order-disorder theory, and problems in the theory of liquids (see [12, 15, 17]). We expect that the on-line interval packing problem is also worth considering in these settings, especially as a model of time-dependent behavior, but our interest stems from the scheduling problems of existing and proposed multimedia communication systems. The application of this type cited in [4] arises in a one-dimensional loss network (modeled by the interval $[0,1]$) where the intervals to be packed represent calls between pairs of communicating stations (points in $[0,1]$).

Modeling reservation protocols in communication systems was the source of our stochastic version of on-line interval packing (see e.g., [9] and the references therein). In a baseline reservation model, there is a single resource and there are randomly arriving requests, each specifying a future time interval during which it wants to use the resource. A request arriving at time t identifies the desired interval $[t_1, t_2]$ by giving the advance notice $t_1 - t$ and the duration $t_2 - t_1$. Scheduling decisions are made on-line: a requested reservation is approved/accepted if and only if the specified interval does not overlap an interval already reserved for some earlier request. Now consider the following stochastic set-up: requests are Poisson arrivals at rate λ, advance notices are independently and uniformly distributed over $[0,a]$ for some given a, and intervals have unit durations. Suppose that, at some time t in equilibrium, we look at the pattern of unit intervals that were reserved during $[t-x, t]$ for some large x. If a is large relative to x, one expects that, except for negligible edge effects, this pattern is approximately the same stochastically as the pattern of intervals packed in $[0,x]$ by the Poisson model of on-line interval packing during the time interval $[0,\lambda]$. This statement is made rigorous and a corresponding limit law proved in [3]; certain generalizations of the above model are also accommodated.

2 Absorption Time

Consider the Poisson model on $[0,x]$, and recall that unit intervals in this model continue to be packed until all of the gaps between packed intervals are smaller than one, i.e., the process absorbs. Let the time to absorption be denoted by T_x.

It is clear that, for $0 \le x < 1$, $T_x \equiv 0$; for $x = 1$, $T_x = \infty$; and for $1 < x \le 2$, T_x is exponentially distributed

$$\mathbf{P}[T_x > t] = e^{-t(x-1)}. \tag{9}$$

Theorem 1 *For $x > 2$, there exist two positive functions $\tau_1(x)$ and $\tau_2(x)$ such that*

$$\frac{\tau_1(x)}{t} \le \mathbf{P}(T_x > t) \le \frac{\tau_2(x)}{t}, \tag{10}$$

the right inequality holding for $t > 0$ and the left for $t > 1$.

Remark: A simple consequence of this theorem is that

$$\mathrm{E}T_x = \infty, \quad T_x < \infty \quad \text{a.s.,} \quad \text{and} \quad N_x(T_x) \stackrel{d}{=} \tilde{N}_x, \tag{11}$$

where $\stackrel{d}{=}$ denotes equality in distribution.

Proof:
Upper bound. We prove the upper bound for $x \in (n, n+1]$ by induction on $n = 2, 3, \ldots$. Let v be the arrival time of the first interval to be packed in $[0, x]$, and let y, $0 \le y \le x-1$, be the position of its left endpoint. Then

$$\begin{aligned}\mathbf{P}(T_x > t) &= \int_0^t e^{-(x-1)v} dv \int_0^{x-1} \mathbf{P}(\max(T_y, T_{x-y-1}) > t - v) dy + e^{-t(x-1)} \\ &\le \int_0^t e^{-(x-1)v} dv \int_1^{x-1} 2\mathbf{P}(T_y > t - v) dy + e^{-t(x-1)}.\end{aligned} \tag{12}$$

We break down the integral over $[0, t]$ into integrals over $[0, t/2]$ and $[t/2, t]$, and use the trivial bounds $\mathbf{P}(T_y > t - v) \le 1$,

$$\begin{aligned}\int_0^{t/2} e^{-(x-1)v} dv &< \int_0^\infty e^{-(x-1)v} dv = \frac{1}{x-1} \\ \int_{t/2}^t e^{-(x-1)v} dv &< \int_{t/2}^\infty e^{-(x-1)v} dv = \frac{e^{-t(x-1)/2}}{x-1},\end{aligned}$$

to get from (12)

$$\begin{aligned}\mathbf{P}(T_x > t) &\le 2\int_0^{t/2} e^{-(x-1)v} dv \int_1^{x-1} \mathbf{P}(T_y > t/2) dy + 2(x-2)\int_{t/2}^t e^{-(x-1)v} dv + e^{-t(x-1)} \\ &\le \frac{2}{x-1} \int_1^{x-1} \mathbf{P}(T_y > t/2) dy + 2\frac{x-2}{x-1} e^{-t(x-1)/2} + e^{-t(x-1)}.\end{aligned} \tag{13}$$

For the basis of the induction, we now upper bound $\int_1^{x-1} \mathbf{P}(T_y > t/2) dy$ with $2 < x \le 3$. From (9), we obtain

$$\int_1^{x-1} \mathbf{P}(T_y > t/2) dy \le \int_1^\infty e^{-(y-1)t/2} dy \le \frac{2}{t},$$

and so from (13) and $e^{-z} < 1/z,\ z > 0$, we obtain the upper bound in (10) for $2 < x \le 3$ with the choice

$$\tau_2(x) = \frac{5}{x-1} + 4\frac{x-2}{(x-1)^2}.$$

Now let $n < x \le n+1,\ n > 2$, and assume that the upper bound in (10) holds for $2 < x \le n$. By (9), we get

$$\begin{aligned}\int_1^{x-1} \mathbf{P}(T_y > t/2)dy &= \int_1^2 \mathbf{P}(T_y > t/2)dy + \int_2^{x-1} \mathbf{P}(T_y > t/2)dy \\ &\le \int_1^\infty e^{-(y-1)t/2}dy + \frac{2(x-3)}{t} \max_{2<y\le x-1} \tau_2(y) \\ &= \frac{2}{t} + \frac{2(x-3)}{t} \max_{2<y\le x-1} \tau_2(y),\end{aligned}$$

so from (13) we obtain the upper bound in (10) with the choice

$$\tau_2(x) = \frac{5}{x-1} + 4\frac{x-2}{(x-1)^2} + \frac{4(x-3)}{x-1} \max_{2<y\le x-1} \tau_2(y),\ \ x > 3.$$

Lower bound. For the easier lower bound proof an inductive argument is not needed. Let $T_y^0, y > 1$, be the first time a unit interval is packed in $[0, y]$; then, the conditional probability of $T_x > t$, given that the first unit interval in $[0, x]$ arrived at time $v \in (0, t)$, with its left end-point at $y \in (1, x-1)$ is bounded by

$$\mathbf{P}(T_x > t|v, y) \ge \mathbf{P}(T_y^0 > t - v) = e^{-(y-1)(t-v)},$$

which implies

$$\begin{aligned}\mathbf{P}(T_x > t) &\ge \int_0^t e^{-(x-1)v}dv \int_1^{x-1} e^{-(y-1)(t-v)}dy \\ &= \int_0^t e^{-(x-1)v}\frac{1-e^{-(x-2)(t-v)}}{t-v}dv. \qquad (14)\end{aligned}$$

By calculus, one checks that

$$\varphi(v) := \frac{1-e^{-(x-2)(t-v)}}{t-v}$$

increases over $[0, t]$. Hence, we conclude from (14) that

$$\mathbf{P}(T_x > t) \ge \varphi(0) \int_0^t e^{-(x-1)v}dv = \frac{1-e^{-(x-2)t}}{t}\frac{1-e^{-(x-1)t}}{x-1}. \qquad (15)$$

Since $1 - e^{-t}$ is increasing in t, (15) yields the lower bound in (10) with the choice

$$\tau_1(x) = \frac{(1-e^{-(x-2)})(1-e^{-(x-1)})}{x-1}.$$

∎

3 Expected Number Packed

In Section 3.1, we begin with the Poisson model and derive a first-order estimate of the expected number of intervals packed in $[0, x]$ during $[0, t]$, which we have denoted by $K(t, x)$. We then show how this result can be used to prove (1), the corresponding result for the model with fixed-length input. In Section 3.2, we prove a much more precise estimate for $K(t, x)$.

3.1 Proof of (1)

Consider the Poisson model and note that $K(t, x) = 0$ if $0 \leq x < 1$, since no interval of length 1 can fit into $[0, x]$ in that case. As in Renyi's analysis in one dimension, we will compute a Laplace transform in the space dimension, eventually to obtain a first-order partial differential equation for

$$\mathcal{K}(t, u) := \int_0^\infty K(t, x) e^{-ux} dx, \tag{16}$$

which can be solved by standard methods. Note that, since $K(t, x) \leq x,\ t, x \geq 0$, we know that $\mathcal{K}(t, u)$ converges for $t \geq 0,\ \Re u > 0$. Furthermore, for fixed $t \geq 0$, $\mathcal{K}(t, u)$ is analytic for $\Re u > 0$.

First, we relate the expected number of intervals packed in time $t + \Delta t$ to the expected number packed in time t based on the events in the first Δt time units. Arriving intervals with left endpoints in $(x-1, x]$ are rejected, so if no interval with its left endpoint in $[0, x-1]$ arrives at some time in $[0, \Delta t]$, then nothing happens and $K(t + \Delta t, x) = K(t, x)$. On the other hand, if an interval $[y, y+1]$ arrives and $y \in [0, x-1]$, then $K(t + \Delta t, x)$ will be the sum of $K(t, y)$ and $K(t, x - y - 1)$ plus 1 for the interval packed. Thus, we have

$$K(t + \Delta t, x) = [1 - (x-1)\Delta t] K(t, x) + \Delta t \left[\int_0^{x-1} \left(K(t, y) + K(t, x - y - 1) + 1 \right) dy \right], \tag{17}$$

ignoring terms of order $o(\Delta t)$ (which correspond to the probabilities of events in which multiple arrivals occur). By symmetry, the integral simplifies and the above becomes

$$K(t + \Delta t, x) = [1 - (x-1)\Delta t] K(t, x) + \Delta t \left[2 \int_0^{x-1} K(t, y) dy + x - 1 \right], \tag{18}$$

which yields

$$\frac{\partial K}{\partial t}(t, x) = (x-1)[1 - K(t, x)] + 2 \int_0^{x-1} K(t, z) dz, \tag{19}$$

for $t \geq 0$ and $x \geq 1$. Now multiply both sides by e^{-ux}, where $u > 0$, and integrate with respect to x over $[1, \infty)$. Exploiting the fact that $K(t, x) = 0,\ 0 \leq x < 1$, we get for the transform of (19)

$$\frac{\partial \mathcal{K}}{\partial t} = \frac{e^{-u}}{u^2} + \mathcal{K} + \frac{\partial \mathcal{K}}{\partial u} + 2 \int_1^\infty e^{-ux} dx \int_0^{x-1} K(t, z) dz. \tag{20}$$

Next, reverse the order of integration in (20) so that the integral becomes $\int_{z=0}^{\infty}\int_{x=z+1}^{\infty}(.)dzdx$. Carrying out the integration gives $2(e^{-u}/u)\mathcal{K}$ for the last term in (20), so (20) can be written

$$\frac{\partial\mathcal{K}}{\partial t} = \frac{e^{-u}}{u^2} + (1+\frac{2e^{-u}}{u})\mathcal{K} + \frac{\partial\mathcal{K}}{\partial u} \tag{21}$$

For the boundary conditions, observe that $K(0,x) = 0$ implies that $\mathcal{K}(0,u) = 0$. The solution to the first-order partial differential equation satisfying this initial condition can be determined by standard methods. In our case, we simply input (20) into *Mathematica*©, and found that

$$\mathcal{K}(t,u) = e^{-2\mathrm{Ei}(-u)}\int_0^t e^{-u+2\mathrm{Ei}(-u-v)}\frac{dv}{(u+v)^2}, \tag{22}$$

where $\mathrm{Ei}(x) = \int_{-\infty}^{x}\frac{e^y}{y}dy,\ x \neq 0$, is the exponential integral. However, it is also easy to obtain (22) directly, as (21) can be reduced to an ordinary differential equation with the change of variables $r = u+t,\ s = t$. To put (22) into a form that we can check against Renyi's result in the limit $t \to \infty$, we rewrite (22) as

$$\begin{aligned}
\mathcal{K}(t,u) &= \int_0^t e^{-u+2(\mathrm{Ei}(-u-v)-\mathrm{Ei}(-u))}\frac{dv}{(u+v)^2} \\
&= e^{-u}\int_0^t \exp\left(2\int_u^{u+v}\frac{e^{-x}}{x}dx\right)\frac{dv}{(u+v)^2} &(23)\\
&= \frac{e^{-u}}{u^2}\int_0^t \exp\left(-2\int_u^{u+v}\frac{1-e^{-x}}{x}dx\right)dv. &(24)
\end{aligned}$$

At this point, we can easily prove

Theorem 2 *The fraction of* $\mathbf{R}_+$ *that is occupied after time* t *is given by*

$$\lim_{x\to\infty}\frac{K(t,x)}{x} = \alpha(t) = \int_0^t \exp\left(-2\int_0^v \frac{1-e^{-x}}{x}dx\right)dv. \tag{25}$$

Remark. In the limit $t \to \infty$, no remaining gap in the final, absorbing state of $\mathbf{R}_+$ is large enough to accommodate an interval, so we obtain (4) for the Poisson model.

Proof: From (24) it easily follows that

$$\mathcal{K}(t,u) \sim \frac{\alpha(t)}{u^2} \quad \text{as} \quad u \searrow 0. \tag{26}$$

Now, since $\int_0^x K(t,z)dz$ is monotonically increasing in x, application of Karamata's Tauberian theorem [2] p. 37, yields

$$\int_0^x K(t,z)dz \sim \alpha(t)\frac{x^2}{2} \quad \text{as} \quad x \to \infty, \tag{27}$$

where $\alpha(t)$ is the same as in (25). Next, since x is the expected number of arrivals per unit time in $[0,x]$, we have $\frac{\partial}{\partial t}K(t,x) \leq x$, which implies in particular that

$$\frac{\frac{\partial}{\partial t}K(t,x)}{x^2} \to 0, \quad \text{as} \quad x \to \infty. \tag{28}$$

Now divide (19) by x^2, let $x \to \infty$, and substitute (27) and (28) to obtain

$$K(t,x) \sim \alpha(t)x \quad \text{as} \quad x \to \infty. \tag{29}$$

∎

Let us now return to the problem with fixed-length input.

Theorem 3 *The limit in (1) holds for the on-line interval packing problem when intervals are unit-length and have left endpoints independently and uniformly drawn from* $[0, x-1]$.

Proof: We may express $K(\lambda, x+1)$ as an average of $N_x(X)$ where X is Poisson distributed with parameter λx

$$K(\lambda, x+1) = \sum_{n=0}^{\infty} e^{-\lambda x} \frac{(\lambda x)^n}{n!} \mathrm{E} N_x(n). \tag{30}$$

Next, choose a small $\epsilon > 0$; break up the sum in (30) into three partial sums, the middle one straddling the mean; and apply elementary bounds to obtain

$$\begin{aligned} K(\lambda, x+1) \quad &\leq \quad \sum_{n=0}^{\lfloor(\lambda-\epsilon)x\rfloor} e^{-\lambda x}\frac{(\lambda x)^n}{n!} \cdot n + \sum_{n=\lfloor(\lambda-\epsilon)x\rfloor+1}^{\lfloor(\lambda+\epsilon)x\rfloor} e^{-\lambda x}\frac{(\lambda x)^n}{n!} \mathrm{E}N_x(\lfloor(\lambda+\epsilon)x\rfloor) \\ &\quad + \sum_{n=\lfloor(\lambda+\epsilon)x\rfloor+1}^{\infty} e^{-\lambda x}\frac{(\lambda x)^n}{n!} \cdot n \end{aligned} \tag{31}$$

It is an elementary exercise in calculus to show that for any $\epsilon > 0$, there exists a constant $\theta \equiv \theta(\epsilon) > 0$, such that

$$1 - \sum_{n=\lfloor(\lambda-\epsilon)x\rfloor+1}^{\lfloor(\lambda+\epsilon)x\rfloor} e^{-\lambda x}\frac{(\lambda x)^n}{n!} = O\left(e^{-\theta x}\right). \tag{32}$$

By replacing (32) in (31) we obtain

$$K(\lambda, x+1) \leq \mathrm{E}N_x(\lfloor(\lambda+\epsilon)x\rfloor)(1+o(1)) + o(1),$$

as $x \to \infty$. Let $0 < \epsilon < \lambda$, and replace λ by $\lambda - \epsilon$ in the above to arrive at

$$\underline{\lim}_{x\to\infty} \frac{\mathrm{E}N_x(\lfloor\lambda x\rfloor)}{x} \geq \lim_{x\to\infty} \frac{K(\lambda-\epsilon, x+1)}{x} = \alpha(\lambda-\epsilon) \nearrow \alpha(\lambda)$$

as $\epsilon \to 0$.

Similarly, for a lower bound, we find

$$K(\lambda, x+1) \quad \geq \quad \mathrm{E}N_x(\lfloor(\lambda-\epsilon)x\rfloor)(1+o(1)),$$

as $x \to \infty$, and so

$$\overline{\lim}_{x\to\infty} \frac{\mathrm{E}N_x(\lfloor\lambda x\rfloor)}{x} \leq \alpha(\lambda+\epsilon) \searrow \alpha(\lambda),$$

as $\epsilon \to 0$, which completes the proof of the theorem. ∎

It is interesting to note that we can in fact prove (1) without direct reference to the stochastic version, although this seems much easier to see in hindsight. We start with a recurrence

$$\mathrm{E}N_{x+1}(n) = \frac{2}{x}\sum_{j=0}^{n-1}\binom{n-1}{j}\int_0^x \left(\frac{y-1}{x}\right)^j \left(\frac{x-y+1}{x}\right)^{n-j-1} \mathrm{E}N_y(j)dy + 1, \tag{33}$$

which expresses the fact that $\mathrm{E}N_{x+1}(n)$ is one plus the sum of the expected numbers packed to the left and right of the first interval packed; the factor of 2 comes from symmetry and the binomial law describes the number ($\leq n-1$) packed to the left of the first-packed interval $[y, y+1]$. At this point, we view the expected value

$$K(\lambda, x+1) = \sum_{n=0}^{\infty} e^{-\lambda x}\frac{(\lambda x)^n}{n!}\mathrm{E}N_x(n)$$

as a transform, apply it to (33), and after a fair amount of algebra, we obtain (19). The remainder of the analysis is as before.

3.2 A Refined Estimate

In principle, we can compute exact formulas for $K(t,x)$, $0 \leq x \leq j$, inductively as we increase $j = 1, 2, \ldots$. To derive an equation for this purpose, we can view (19), for fixed x, as an ordinary differential equation in t with the initial condition $K(0,x) = 0$. The solution is

$$\begin{aligned} K(t,x) &= e^{-t(x-1)}\int_0^t \left[x - 1 + 2\int_0^{x-1} K(v,z)dz\right] e^{v(x-1)} \\ &= 1 - e^{-t(x-1)} + 2e^{-t(x-1)}\int_0^t e^{v(x-1)}dv \int_0^{x-1} K(v,z)dz. \end{aligned} \tag{34}$$

Remark: We can also derive (34) by probabilistic reasoning. Our first recurrence in (17) and (18) was based on events in the first Δt time units. But we can also write a recurrence based on the time and place of the first arrival. Thus, using $e^{-v(x-1)}\Delta v \Delta y$ as the probability that the first arrival occurs during $[v, v+\Delta v]$ with left endpoint at $[y, y+\Delta y]$, we obtain

$$K(t,x) = \int_0^t \int_0^{x-1} e^{-v(x-1)}[K(t-v,y) + K(t-v, x-y-1) + 1]dvdy,$$

which is easily put into the form of (34). ∎

Let us now calculate exact formulas for $0 \leq x \leq 3$. By (34) and the trivial fact that $K(t,x) = 0$, $0 \leq x < 1$, we have, for $1 \leq x \leq 2$,

$$K(t,x) = 1 - e^{-t(x-1)}, \tag{35}$$

noting that the limit $t \to \infty$ is 1, as it should be, and for $2 \leq x \leq 3$,

$$\begin{aligned}
K(t,x) &= 1 - e^{-t(x-1)} + 2e^{-t(x-1)} \int_0^t dv \int_1^{x-1} e^{v(x-1)}(1 - e^{-v(z-1)})dz \\
&= 1 - e^{-t(x-1)} + 2e^{-t(x-1)} \int_0^t \left[\frac{e^v - e^{v(x-1)}}{v} + (x-2)e^{v(x-1)} \right] dv \\
&= \left(3 - \frac{2}{x-1}\right)\left(1 - e^{-t(x-1)}\right) - 2e^{-t(x-1)} \left[\frac{x-2}{x-1} - \int_0^t \frac{e^v - e^{v(x-1)}}{v} dv \right], \quad (36)
\end{aligned}$$

the limiting value $3 - 2/(x-1)$ as $t \to \infty$ agreeing, as we expect, with the corresponding parking result of Dvoretzky and Robbins [6, Equations (2.8) and (2.9)].

With the growth in complexity shown for the values $j = 1, 2, 3$, one is well advised to develop useful asymptotics. For the parking problem, the following refined estimate of $\mathrm{E}\tilde{N}_x$ is well known. There exists a constant $\xi > 0$ such that, as $x \to \infty$

$$\mathrm{E}\tilde{N}_x = \alpha_* x + \alpha_* - 1 + O(e^{-\xi x \log x}) \tag{37}$$

Renyi [17] proved this result with a larger error term (viz., $O(1/x^m)$ for every m); Dvoretzky and Robbins [6] supplied the tighter error term. The estimate (37) is the limit $t \to \infty$ of the estimate below for the Poisson model of on-line (unit) interval packing, the main result of this section.

Theorem 4 *For any fixed ξ, $0 < \xi < 1$, and $T > 0$, we have*

$$\sup_{0 \leq t \leq T} |K(t,x) - (\alpha(t)x + \alpha(t) + \beta(t) - 1)| = O\left(e^{-\xi x \log x}\right). \tag{38}$$

Before giving the proof, we briefly discuss our approach and prove two lemmas. To this point, u has been treated as a positive real, but for the remainder of the section it is to be taken as complex. For simplicity, we will in many occasions suppress t and write $\mathcal{K}(u) = \mathcal{K}(t,u)$, t then being any fixed non-negative constant. A similar convention will apply to $\alpha = \alpha(t)$ and $\beta = \beta(t)$.

The inversion formula for Laplace transforms gives, for $x > 0$,

$$K(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \mathcal{K}(u) e^{xu} du, \tag{39}$$

the integration path being the vertical line $\Re u = \sigma$ directed upward, where σ is any positive real. The main idea in the proof of Theorem 4 is to shift this integration path as far left as possible and to use the Cauchy residue theorem to deal with singularities encountered in this shift. To do this, we need to study the analytic continuation of $\mathcal{K}(u)$ and to obtain a growth estimate for $|\mathcal{K}|$.

Lemma 5 *The function $\mathcal{K}(u)$ is analytic for all $u \neq 0$. At $u = 0$, $\mathcal{K}(u)$ has a second order pole and the expansion*

$$\mathcal{K}(u) = \frac{\alpha}{u^2} + \frac{\alpha + \beta - 1}{u} + \cdots, \tag{40}$$

where α and β are as given in (25) and (6).

Proof: Note that $\alpha(u)$ and $\beta(u)$ are entire functions of the complex variable u, so if we rewrite (24) as

$$\mathcal{K}(u) = \frac{e^{-u}}{u^2} \cdot \frac{1}{\beta(u)} \int_0^t \beta(u+v)dv = \frac{e^{-u}}{u^2\beta(u)}[\alpha(u+t) - \alpha(u)], \tag{41}$$

we see that $\mathcal{K}(u)$ is analytic for $u \neq 0$. The expansions of $e^{-u}/\beta(u)$ and $\alpha(u+t) - \alpha(u)$ about $u = 0$ are given by

$$\begin{aligned} \frac{e^{-u}}{\beta(u)} &= e^{-u+2\int_0^u (1-y/2+\cdots)dy} = e^{u-u^2/2+\cdots} \\ &= 1 + u + 0 \cdot u^2 + \cdots \end{aligned}$$

and

$$\begin{aligned} \alpha(u+t) - \alpha(u) &= [\alpha(t) - \alpha(0)] + [\beta(t) - \beta(0)]u + \cdots \\ &= \alpha(t) + [\beta(t) - 1]u + \cdots . \end{aligned}$$

Multiplying these gives the expansion in (40). ∎

Lemma 6 *Let $t > 0$, and σ_o be real numbers. Then*

$$|\mathcal{K}(t,u)| \leq \frac{c(t,\sigma_o)}{|u|^2}, \text{ for } |u| \geq 2t, \ \Re u \geq \sigma_o, \tag{42}$$

where $c(t,\sigma_o)$ denotes a positive constant depending on t and σ_o.

Proof: Use the elementary estimates

$$\begin{aligned} |e^{-u}| &\leq e^{-\sigma_o} \text{ for } \Re u \geq \sigma_o, \\ |u+v| &\geq |u|/2 \geq t \text{ for } |u| \geq 2t, \ 0 \leq v \leq t, \end{aligned}$$

to obtain

$$\left|\exp\left(2\int_u^{u+v}(e^{-y}/y)dy\right)\right| \leq \exp\left(\left|2\int_u^{u+v}(e^{-y}/y)dy\right|\right) \leq e^{2e^{-\sigma_o}}, \ |u| \geq 2t, \ \Re u \geq \sigma_o, \ 0 \leq v \leq t,$$

so that, from (23),

$$|\mathcal{K}(t,u)| \leq \frac{4t\exp(-\sigma_o + 2e^{-\sigma_o})}{|u|^2}, \ |u| \geq 2t, \ \Re u \geq \sigma_o;$$

which gives (42) with the choice

$$c(t,\sigma_o) = 4t\exp(-\sigma_o + 2e^{-\sigma_o}). \tag{43}$$

∎

We observe that $c(t,\sigma_o)$ increases in t, a fact to be used in the proof of Theorem 4.

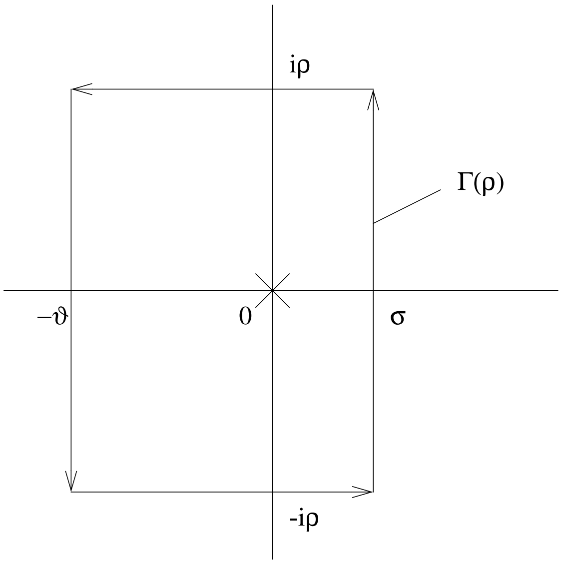

Figure 1: Rectangular contour of integration.

Proof of Theorem 4: We begin by proving the theorem with the weaker error estimate: $O(e^{-\vartheta x})$ for all $\vartheta > 0$. We shift the vertical integration path in (39) to the left of 0 and apply the residue theorem. First observe that, by Lemma 5, $\mathcal{K}(u)e^{xu}$ is analytic for all u, except for a pole at $u = 0$. From (40) and the power series expansion $e^{xu} = 1 + xu + \cdots$, we conclude that

$$\mathcal{K}(u)e^{xu} = \frac{\alpha}{u^2} + \frac{\alpha x + \alpha + \beta - 1}{u} + \cdots, \quad u \neq 0. \tag{44}$$

Apply the residue theorem to the rectangular contour $\Gamma(\rho)$ sketched in Fig. 1, where $\rho > 0$ and $\sigma, \vartheta \geq 2T$. As $\mathcal{K}(u)$ is analytic on and inside $\Gamma(\rho)$, except for a pole at $u = 0$, we get from (44)

$$\frac{1}{2\pi i} \int_{\Gamma(\rho)} \mathcal{K}(u)e^{xu} du = \alpha x + \alpha + \beta - 1. \tag{45}$$

By Lemma 6, the total contribution of the integrals along the horizontal sides of $\Gamma(\rho)$ tends to 0 as $\rho \to \infty$, so (45) becomes

$$\frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \mathcal{K}(u)e^{xu} du = \alpha x + \alpha + \beta - 1 + \frac{1}{2\pi i} \int_{-\vartheta - i\infty}^{-\vartheta + i\infty} \mathcal{K}(u)e^{xu} du. \tag{46}$$

But if we write $u = -\vartheta + iy,\ -\infty < y < \infty,\ \vartheta > 0$, then by Lemma 6,

$$\sup_{0 \leq t \leq T} \left| \frac{1}{2\pi i} \int_{-\vartheta - i\infty}^{-\vartheta + i\infty} \mathcal{K}(u)e^{xu} du \right| \leq \frac{c(T, -\vartheta)e^{-\vartheta x}}{2\pi} \int_{-\infty}^{\infty} \frac{dy}{\vartheta^2 + y^2} = \frac{c(T, -\vartheta)}{2|\vartheta|} e^{-\vartheta x}, \tag{47}$$

and so $\sup_{0 \leq t \leq T} |K(t, x) - (\alpha(t)x + \alpha(t) + \beta(t) - 1)| = O(e^{-\vartheta x})$ for all $\vartheta > 0$ follows from (46) and (47). Finally, the $O(e^{-\vartheta x})$ error term can be improved to $O(e^{-\xi x \log x})$ for any $0 < \xi < 1$ by using in (47) the explicit value for $c(T, -\vartheta)$ provided by(43) and maximizing over $\vartheta > 0$ for fixed x. This is a routine calculus problem, so we omit the details. ∎

4 Second Moment

The main result of this section is the following estimate for the second moment $M(t,x) := \mathrm{E}N_x^2(t)$.

Theorem 7 *For any $T > 0$, we have*

$$\sup_{0\le t\le T}\left|M(t,x) - (\alpha^2(t)x^2 + m(t)x + m_1(t))\right| = O(e^{-\xi x \log x}), \tag{48}$$

for all $\xi \in (0,1)$, where $m(t)$ and $m_1(t)$ are explicitly computable constants (the computations are given below in Lemma 10).

As in the previous section, before proving Theorem 7, we compute the transform in the space dimension

$$\mathcal{M}(t,u) := \int_0^\infty M(t,x)e^{-xu}dx. \tag{49}$$

Since $M(t,x) \le x^2,\ t,x \ge 0$, $\mathcal{M}(t,u)$ converges for $t \ge 0$ and $\Re u > 0$. Furthermore, for fixed $t \ge 0$, $\mathcal{M}(t,u)$ is analytic for $\Re u > 0$. After computing the transform in Section 3.1, we prove analyticity properties of $\mathcal{M}(t,u)$ in Section 3.2, and then complete the proof of Theorem 7.

4.1 The Transform $\mathcal{M}(t,u)$

The derivation of the formula for $\mathcal{M}(t,u)$ duplicates that for $\mathcal{K}(t,u)$, so we will be brief. In analogy with (17), we have

$$M(t+\Delta t) = \mathrm{E}N_x^2(t+\Delta t) = [1-(x-1)\Delta t]M(t,x) +$$

$$\Delta t\int_0^{x-1}\mathrm{E}[1 + N_y^2(t) + N_{x-y-1}^2(t) + 2N_y(t) + 2N_{x-y-1}(t) + 2N_y(t)N_{x-y-1}(t)]dy + o(t),$$

and hence

$$\begin{aligned}\frac{\partial M}{\partial t}(t,x) &= (x-1)[1-M(t,x)] + \\ & \quad 2\int_0^{x-1} M(t,y)dy + 4\int_0^{x-1} K(t,y)dy + 2\int_0^{x-1} K(t,y)K(t,x-y-1)dy,\end{aligned} \tag{50}$$

the analogue of (19). Now take the transform of both sides of (50) (see (49)) and obtain

$$\frac{\partial \mathcal{M}}{\partial t}(t,u) = \frac{\partial \mathcal{M}}{\partial u}(t,u) + \mathcal{M}(t,u)\left(1 + \frac{2e^{-u}}{u}\right) + \frac{e^{-u}}{u^2}\mathcal{A}(t,u),\ t \ge 0,\ u > 0, \tag{51}$$

where

$$\mathcal{A}(t,u) = 1 + 4u\mathcal{K}(t,u) + 2(u\mathcal{K}(t,u))^2, \tag{52}$$

with the boundary condition $\mathcal{M}(0,u) = 0$, as before. Solving (51) as we did (21), we find

$$\mathcal{M}(t,u) = e^{-u}\int_0^t \frac{\mathcal{A}(t-v,u+v)}{(u+v)^2}\exp\left(2\int_u^{u+v}\frac{e^{-y}}{y}dy\right)dv \tag{53}$$

which is equivalent to

$$\begin{aligned}\mathcal{M}(t,u) &= \frac{e^{-u}}{u^2}\int_0^t \mathcal{A}(t-v,u+v)\exp\left(-2\int_u^{u+v}\frac{1-e^{-y}}{y}dy\right)dv \\ &= \frac{e^{-u}}{u^2\beta(u)}\int_u^{t+u}\mathcal{A}(t+u-v,v)\beta(v)dv. \end{aligned} \tag{54}$$

4.2 Analytic Properties of $\mathcal{M}(t,u)$

We first establish the analytic properties of $\mathcal{K}(t-v,v)$, $\mathcal{A}(t-v,v)$ and $\mathcal{B}(t,v) := \mathcal{A}(t-v,v)\beta(v)$ as functions of v for fixed t; in the following two lemmas both t and v are assumed to be complex variables.

Lemma 8 *The function $\mathcal{K}(t-v,v)$ is analytic for all t and for all $v \neq 0$. At $v=0$, $\mathcal{K}(t-v,v)$ has a second-order pole and the expansion*

$$\mathcal{K}(t-v,v) = \frac{\alpha(t)}{v^2} + \frac{\alpha(t)-1}{v} + O(1), \tag{55}$$

as $v \to 0$.

Proof: From (41) we obtain

$$\mathcal{K}(t-v,v) = \frac{e^{-v}}{v^2\beta(v)}[\alpha(t)-\alpha(v)]. \tag{56}$$

Then the analytic properties of $\mathcal{K}(t-v,v)$ for $v \neq 0$ are evident from (41). At $v=0$, the right-hand side of (56) can be expanded to

$$\left(\frac{1}{v^2}+\frac{1}{v}+O(v)\right)\left(\alpha(t)-v+O(v^2)\right),$$

which, after simple algebra, becomes equal to the right-hand side of (55). ∎

Lemma 9
(i) $\mathcal{A}(t-v,v)$ is analytic for all t and for all $v \neq 0$ and has a second-order pole at $v=0$ along with the expansion

$$\mathcal{A}(t-v,v) = \frac{2\alpha^2(t)}{v^2} + \frac{4\alpha^2(t)}{v} + O(1), \tag{57}$$

as $v \to 0$.
(ii) $\mathcal{B}(t,v)$ is analytic for all t and $v \neq 0$ and has a second-order pole at $v=0$ along with the expansion

$$\mathcal{B}(t,v) = \frac{2\alpha^2(t)}{v^2} + O(1), \tag{58}$$

as $v \to 0$.

Proof: For $v \neq 0$, analyticity properties of $\mathcal{A}(t-v,v)$ and $\mathcal{B}(t,v)$ follow from Lemma 8 and the definition of $\mathcal{A}$ in (52). Expansion (57) follows from (52) and (55).

For part (ii), note that $\beta(u)$ is entire and that its expansion at $u=0$ is given by

$$\begin{aligned}\beta(v) &= \exp\left(-2\int_0^v \frac{1-e^{-y}}{y}dy\right) = e^{-2\int_0^v (1+y/2+\cdots)dy} \\ &= e^{-2v+v^2+\cdots} = 1-2v+\cdots. \qquad (59)\end{aligned}$$

Now multiply (57) by (59) to obtain (58). ∎

Define

$$\kappa(t,v) := \mathcal{B}(t,v) - 2\alpha^2(t)/v^2. \qquad (60)$$

Lemma 10 *The transform $\mathcal{M}(t,u)$ has an analytic continuation for all $u \neq 0$, and at $u=0$ it has the expansion*

$$\mathcal{M}(t,u) = \frac{2\alpha^2(t)}{u^3} + \frac{m(t)}{u^2} + \frac{m_1(t)}{u} + O(1), \qquad (61)$$

as $u \to 0$, where

$$m(t) = 4\alpha(t)\beta(t) + 2\alpha^2(t)\left(1-t^{-1}\right) + \int_0^t \kappa(t,v)dv, \qquad (62)$$

and

$$\begin{aligned}m_1(t) &= 2[[\beta^2(t) + \alpha(t)\beta'(t)] + 4\alpha(t)\beta(t)(1-1/t) + 2\alpha^2(t)(1/t^2 - 1/t) \\ &\quad + \kappa(t,t) - \kappa(t,0) + \int_0^t \frac{\partial\kappa(t,v)}{\partial t}dv + \int_0^t \kappa(t,v)dv. \qquad (63)\end{aligned}$$

Proof: The analytic continuation of $\mathcal{M}(t,u)$ follows from the properties of $\mathcal{A}(t-v,v)$, $\beta(v)$, and (54). Next, for the expansion at $u=0$, note that part (ii) of Lemma 9 implies that $\kappa(t,v)$ is an entire function in both complex variables t,v. Here, (54) can be rewritten as

$$\begin{aligned}\mathcal{M}(t,u) &= \frac{e^{-u}}{u^2\beta(u)}\left(2\alpha^2(t+u)\int_u^{t+u}\frac{dv}{v^2} + \int_u^{t+u}\kappa(t+u,v)dv\right) \\ &= \frac{e^{-u}}{u^2\beta(u)}\left(\frac{2\alpha^2(t+u)}{u} - \frac{2\alpha^2(t+u)}{t+u} + \int_u^{t+u}\kappa(t+u,v)dv\right). \qquad (64)\end{aligned}$$

To expand (64) at $u=0$, observe that

$$\frac{e^{-u}}{u^2\beta(u)} = \frac{1}{u^2} + \frac{1}{u} + O(u), \qquad (65)$$

as $u \to 0$, and that

$$\frac{2\alpha^2(t+u)}{u} = \frac{2\alpha^2(t)}{u} + 4\alpha(t)\beta(t) + 2(\beta^2(t) + \alpha(t)\beta'(t))u + O(u^2), \qquad (66)$$

as $u \to 0$. Similarly,

$$\begin{aligned}\frac{2\alpha^2(t+u)}{t+u} &= \frac{2}{t}(\alpha^2(t)+2\alpha(t)\beta(t)u+O(u^2))\left(1-\frac{u}{t}+O(u^2)\right)\\ &= \frac{2\alpha^2(t)}{t}+u\left(\frac{4\alpha(t)\beta(t)}{t}-\frac{2\alpha^2(t)}{t^2}\right)+O(u^2),\end{aligned} \tag{67}$$

and

$$\int_u^{t+u}\kappa(t+u,v)dv = \int_0^t \kappa(t,v)dv + u\left(\kappa(t,t)-\kappa(t,0)+\int_0^t \frac{\partial\kappa(t,v)}{\partial t}dv\right)+O(|u|^2), \tag{68}$$

as $u \to 0$. Substitution of the estimates (65)-(68) into (64) proves the lemma. ∎

Next, we need a bound on the growth of $|\mathcal{M}(u)|$.

Lemma 11 *Let $t > 0$ and $\sigma_o := \Re u$ be real numbers. Then for $|u| > 3t, \Re u \geq \sigma_o$,*

$$|\mathcal{M}(t,u)| \leq \frac{c_1(t,\sigma_o)}{|u|^2}, \tag{69}$$

where $c_1(t,\sigma_o)$ denotes a positive constant depending on t and σ_o.

Proof: Let v satisfy $0 < v \leq t$, so that, by the conditions of the lemma, $t-v>0$, $|u+v| \geq 2t$, and $\Re(u+v) \geq \sigma_o$. We conclude from Lemma 6 that

$$|(u+v)\mathcal{K}(t-v,u+v)| \leq \frac{c(t-v,\sigma_o)}{|u+v|} \leq \frac{c(t,\sigma_o)}{2t}, \tag{70}$$

where $c(t,\sigma_o)$ is defined by (43). Now (52) and (70) give for the above ranges of t,u,v

$$|\mathcal{A}(t-v,u+v)| \leq 1+2\frac{c(t,\sigma_o)}{t}+\frac{1}{2}\left[\frac{c(t,\sigma_o)}{t}\right]^2. \tag{71}$$

Using (53) and (71), the remainder of the proof mimics that of Lemma 6, so we omit the details. We obtain the bound (69) with the choice

$$c_1(t,\sigma_o) = \left(1+2\frac{c(t,\sigma_o)}{t}+\frac{1}{2}\left[\frac{c(t,\sigma_o)}{t}\right]^2\right)c(t,\sigma+o).$$

∎

We observe that $c_1(t,\sigma_o)$ increases in t, a fact to be used in the proof below of Theorem 7.

Proof of Theorem 7: Proceeding as in the proof of Theorem 4, we have by Lemma 10 that $\mathcal{M}(u)e^{xu}$ is analytic for all u, except for a pole at $u=0$. From Lemma 10 and the power series expansion $e^{xu} = 1+xu+(xu)^2/2\cdots$, we conclude that the residue of $\mathcal{M}(t,u)e^{xu}$ at $u=0$ is $\alpha^2x^2+mx+m_1$. Apply the residue theorem to the rectangular contour $\Gamma(\rho)$

sketched in Fig. 1, where $\rho > 0$ and $\sigma, \xi \geq 3t$. As $\mathcal{M}(u)$ is analytic on and inside $\Gamma(\rho)$, except for a pole at $u = 0$, we get

$$\frac{1}{2\pi i} \int_{\Gamma(\rho)} \mathcal{M}(u) e^{xu} du = \alpha^2 x^2 + mx + m_1. \tag{72}$$

The remainder of the proof uses the bound in Lemma 11 and follows exactly the proof of Theorem 4. Again, we omit the details. ∎

5 Central Limit Theorem

In this section, our first result is a precise variance estimate in Theorem 13 for the Poisson model. Then, in Theorem 14, we show that $N_x(t)$ satisfies a central limit theorem.

Before turning to the Poisson model, we give, again for contrast, a more precise variance estimate for the parking problem than was given in the introduction.

Theorem 12 (Mackenzie [12] and Dvoretzky and Robbins [6]) *With μ_* given by (7),and with any constant ξ, $0 < \xi < 1$, the variance of the number parked in the parking problem satisfies*

$$\mathrm{Var}(\tilde{N}_x) = \mu_* x + \mu_* + O(e^{-\xi x \log x}), \quad \text{as } x \to \infty. \tag{73}$$

Remark: Dvoretzky and Robbins [6] proved (73) without giving an explicit formula for μ_*. Earlier, by computing the continuous limit of a result derived in a discretized version of the parking problem, Mackenzie [12] obtained formula (7) for μ_*. ∎

The combined analysis of Mackenzie and of Dvoretzky and Robbins gives a roundabout proof of the variance estimate which is unnecessary. A direct proof can be based on a careful analysis of an inverse Laplace transform, as in the previous section. A sketch of our new proof of (73) is given in the appendix.

Theorem 13 *For any $T > 0$, we have*

$$\sup_{0 \leq t \leq T} |\mathrm{Var}(N_x(t)) - (\mu(t)x + \mu_1(t))| = O(e^{-\xi x \log x}), \tag{74}$$

for all $\xi \in (0,1)$,

$$\mu(t) = m(t) - 2\alpha(t)(\alpha(t) + \beta(t) - 1), \tag{75}$$

$$\mu_1(t) = m_1(t) - (\alpha(t) + \beta(t) - 1)^2, \tag{76}$$

with $m(t)$ and $m_1(t)$ given by (62) and (63). In addition, $\mu(t) > 0$ for all $t > 0$.

Remarks: From (75), (76), (62), and (63), we find that $\lim_{t\to\infty} \mu_1(t) = \lim_{t\to\infty} \mu(t) = \mu_*$ consistent with (73) and the fact that the on-line interval packing problem becomes the parking problem in the limit $t \to \infty$. Numerical values of $\mu(t), 0 < t \leq 10$, are illustrated in Figure 2. We prove below that $\mu(t) > 0$ for $t > 0$; but a proof that $\mu(t)$ is unimodal as indicated in the figure seems difficult. We note that $\mu(t) > 0$ is needed for the existence of a central limit theorem. ∎

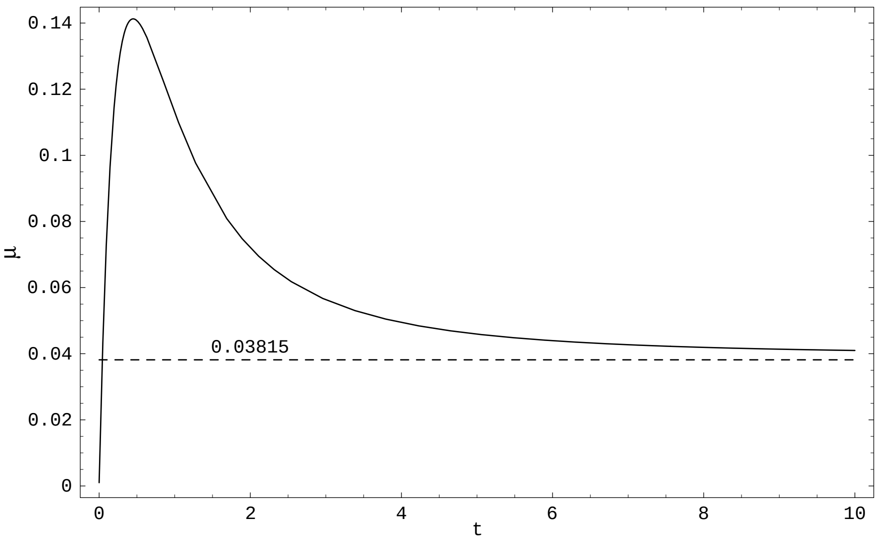

Figure 2: Numerical values of $\mu(t), 0 < t \leq 10$.

Proof: Estimate (74) is an immediate consequence of Theorems 4 and 7.

It remains to prove that $\mu(t) > 0$ for all $t > 0$; we proceed as follows. (The proof will not attempt to analyze directly the complicated expression for $\mu(t)$.) We first write formulas for $\mathrm{Var}(N_x(t)),\ x \geq 0$, that will facilitate the argument. (In the sequel, $t \geq 0$ is always tacitly assumed.) For $1 \leq x \leq 2$, we have

$$M(t,x) = 1 - e^{-t(x-1)}, \tag{77}$$

which when combined with (35) yields

$$\mathrm{Var}(N_x(t)) = e^{-t(x-1)} - e^{-2t(x-1)},\ 1 \leq x \leq 2. \tag{78}$$

Now condition on the event that the first interval arrives at time $[t-v, t-v+dv]$, and is packed with its left endpoint in $[y, y+dy]$. The expectation of the conditional variance is smaller than the variance, so we arrive at

$$\mathrm{Var}(N_x(t)) \geq 2\int_0^t e^{-(t-v)(x-1)}dv \int_0^{x-1} \mathrm{Var}(N_y(v))dy,\ x \geq 1. \tag{79}$$

Next, we claim that the function

$$\begin{aligned} f(t,x) &= \mathrm{Var}(N_x(t)),\ 0 \leq x \leq 2 & (80)\\ &= 2\int_0^t e^{-(t-v)(x-1)}dv \int_0^{x-1} f(v,y)dy,\ x \geq 2, & (81) \end{aligned}$$

has the property that

$$\mathrm{Var}(N_x(t)) \geq f(t,x), \quad x \geq 0. \tag{82}$$

This claim is easily established by proving by induction on $m = 1, 2, \ldots$ that $\mathrm{Var}(N_x(t)) \geq f(t,x),\ m \leq x \leq m+1$. We omit the details.

We are now reduced to an analysis of the function $f(t,x)$. First, note that (81) is equivalent to

$$\frac{\partial f(t,x)}{\partial t} = -(x-1)f(t,x) + 2\int_0^{x-1} f(t,y)dy, \;\; x \geq 2, \tag{83}$$

with $f(0,x) = 0, \;\; x \geq 2$. To solve (83), define $\bar{f}(t,u) := \int_2^\infty f(t,x)e^{-ux}dx$ for $u > 0$, multiply (83) by e^{-ux}, and integrate with respect to x over $[2,\infty)$. Exploiting the fact that $f(t,x) = 0, \;\; 0 \leq x \leq 1$, we get

$$\frac{\partial \bar{f}(t,u)}{\partial t} = \frac{\partial \bar{f}(t,u)}{\partial u} + \bar{f}(t,u) + \frac{2e^{-u}}{u}\left(\int_1^2 f(t,x)e^{-ux}dx + \bar{f}(t,u)\right), \tag{84}$$

with $\bar{f}(0,u) = 0, \;\; u > 0$. Rewrite (84) as

$$\frac{\partial \bar{f}(t,u)}{\partial t} = \frac{\partial \bar{f}(t,u)}{\partial u} + \left(1 + \frac{2e^{-u}}{u}\right)\bar{f}(t,u) + \frac{e^{-u}}{u^2}\mathcal{A}_f(t,u), \tag{85}$$

where

$$\mathcal{A}_f(t,u) = 2u\int_1^2 f(t,x)e^{-ux}dx. \tag{86}$$

Solving (85) as we did (51), we get

$$\bar{f}(t,u) = e^{-u}\int_0^t \frac{\mathcal{A}_f(t-v,u+v)}{(u+v)^2}\exp\left(2\int_u^{u+v}\frac{e^{-y}}{y}dy\right)dv \tag{87}$$

which reduces to

$$\bar{f}(t,u) = \frac{e^{-u}}{u^2\beta(u)}\int_u^{t+u}\mathcal{A}_f(t+u-v,v)\beta(v)dv. \tag{88}$$

Karamata's Tauberian theorem then implies

$$\int_0^x f(t,y)dy \sim \nu(t)\frac{x^2}{2}, \text{as} \quad x \to \infty, \tag{89}$$

where

$$\nu(t) = \int_0^t \mathcal{A}_f(t-v,v)dv. \tag{90}$$

From (78), (80), and (86) we conclude that $\mathcal{A}_f(t,u) > 0, \;\; u > 0$, and hence that $\nu(t) > 0$ by (90). This observation together with (74) and (82) then gives the desired inequality

$$\begin{aligned} \mu(t) &= \lim_{x\to\infty}\frac{2}{x^2}\int_0^x \mathrm{Var}(N_y(t))dy \\ &\geq \lim_{x\to\infty}\frac{2}{x^2}\int_0^x f(t,y)dy = \nu(t) > 0. \end{aligned}$$

∎

We are now ready to prove a central limit theorem for $N_x(t)$ as x becomes large, holding t fixed.

Theorem 14 *For any fixed t, we have $Z_x(t) \xrightarrow{d} \mathcal{N}(0,1)$, as $x \to \infty$, where*

$$Z_x(t) = \frac{N_x(t) - \mathrm{E}N_x(t)}{\sqrt{\mathrm{Var}(N_x(t))}},$$

$\mathcal{N}(0,1)$ is a standard normal random variable, and $\xrightarrow{d}$ denotes convergence in distribution.

Proof: We apply an approach of Dvoretzky and Robbins [6]; it is the second of their two proofs of a central limit theorem for the parking problem. The new features to be dealt with here are minor, so we only sketch the basic technique.

The idea is to observe the state of the packing process after a small number $n_x = o(x)$ of intervals have been packed, leaving a vector of successive gaps $\mathbf{y} = (y_1, \ldots, y_{n_x+1})$. The key fact is that the continuation of the packing process consists of $n_x + 1$ *independent* packing subprocesses, one taking place in each gap y_i. If τ_x is the earliest time by which the initial n_x intervals are packed, then at any time $t > \tau_x$, the $n_x + 1$ subprocesses define a triangular array of independent random variables $N_{y_i}(t - \tau_x)$ indexed by i, $1 \le i \le n_x + 1$, and $x = n_x + \sum y_i > 1$. After a suitable normalization, we apply a version of Liapunov's theorem for triangular arrays (see [6, Lemma 6]) and obtain a conditional central limit theorem for $Z_x(t)$, the normalized version of $N_x(t) = n_x + \sum N_{y_i}(t - \tau_x)$, given $(\tau_x, \mathbf{y})$. Finally, an extension of the Dvoretzky-Robbins argument shows that the central limit theorem holds uniformly over a set of $(\tau_x, \mathbf{y})$ whose probability tends to 1 as $x \to \infty$. This extension is straightforward once it is observed that $\tau_x \xrightarrow{d} 0$ as $x \to \infty$, and that the estimate of the variance of $N_x(t)$ is uniform in t (see Theorem 13). It follows that the central limit theorem also holds for the unconditional packing process $Z_x(t)$. ∎

6 Final Remarks

We have focused on asymptotics in x for fixed t. It would also be of interest to obtain good large-t estimates for fixed x, and more information on the convergence to the parking problem. Figure 3 was produced from exact formulas for small x (see (34)), and is instructive in connection with time dependence. The figure illustrates that convergence of $K(t,x)$ to $\alpha(t)x + \alpha(t) + \beta(t) - 1$ is very fast *uniformly* in t.

A more technical problem of interest is the extension of our results on the Poisson model to the model with fixed-length input. Here, we have shown only that the leading-term asymptotics for the expected number packed carry over without change to the latter model.

The authors have made some headway in the generalization of the Poisson model to the case of a general interval-length distribution (see [15, 13] for results of this type for the parking problem). We also have partial results on the discrete version of our problem with the interval $[0, x]$ replaced by the integers $(1, 2, \ldots, m)$ and with the intervals to be packed all having integer lengths $a \le m$ (see [16, 12] for the analysis of this model of the parking problem). A paper on these results is in preparation.

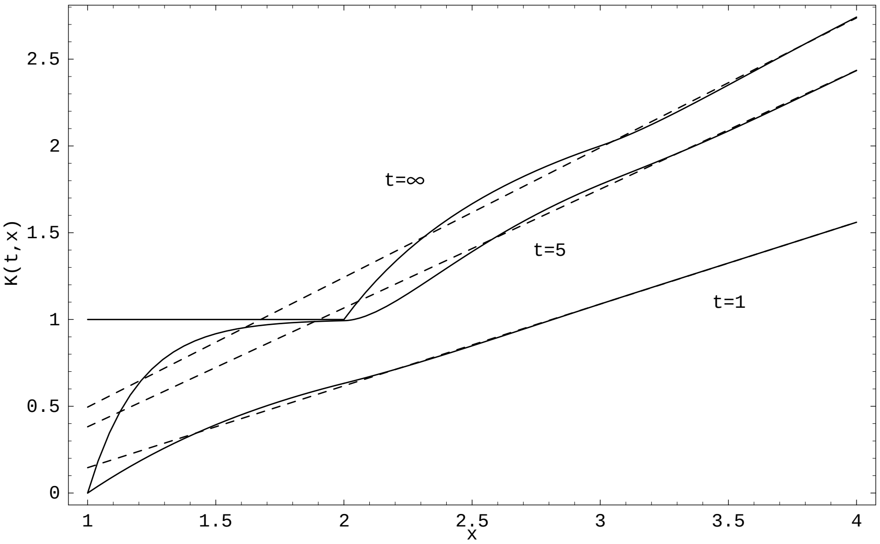

Figure 3: Solid lines represent exact values of $K(t, x)$ for $t = 1, 5, \infty$, and $1 \leq x \leq 4$; dashed lines are drawn using linear approximation $\alpha(t)x + \alpha(t) + \beta(t) - 1$.

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Appendix

Proof of (73): Using transform methods, we will derive an estimate for the second moment $\mathrm{E}\tilde{N}_x^2$ that is equal to the variance in (73) plus the square of the mean given by (37). The second-moment transform $\mathcal{G}(u) := \int_0^\infty \mathrm{E}\tilde{N}_x^2 e^{-ux}dx$ is naturally expressed in terms of the first-moment transform $\mathcal{F}(u) := \int_0^\infty \mathrm{E}\tilde{N}_x e^{-ux}dx$, which is given by (see [17]),

$$\mathcal{F}(u) = \frac{e^{-u}\tilde{\alpha}(u)}{u^2\beta(u)}, \tag{91}$$

where $\tilde{\alpha}(u) = \alpha_* - \alpha(u)$ and, as before,

$$\beta(u) = e^{-2\int_0^u \frac{1-e^y}{y}dy}, \quad \alpha(u) = \int_0^u \beta(v)dv.$$

Since $\mathrm{E}\tilde{N}_x \le x$ and $\mathrm{E}\tilde{N}_x^2 \le x^2$, we know that $\mathcal{F}(u)$ and $\mathcal{G}(u)$ converge and are analytic for $\Re u > 0$. We show that $\mathcal{F}(u)$ and $\mathcal{G}(u)$ have analytic continuations for $u \ne 0$, and at $u = 0$, they have poles of order 2 and 3, respectively. We then bound the growth of $|\mathcal{F}(u)|$ and $|\mathcal{G}(u)|$. Our estimate of the second moment will be derived from the inversion formula for Laplace transforms, the residue theorem, and the growth bounds.

The formulas derived for the parking problem (such as those for $\mathcal{F}(u)$ and $\mathcal{G}(u)$) are analogous to the previous ones obtained for on-line interval packing. Indeed, they are obtained formally from the previous ones by letting $t \to \infty$. To emphasize this analogy, we denote corresponding functions by the same letter (such as $\mathcal{A}(u)$ and $\mathcal{B}(u)$).

The main difference between the parking and on-line interval packing results is that both $|\mathcal{F}(u)|$ and $|\mathcal{G}(u)|$ are $O(\frac{1}{|u|})$ in any half-plane $\Re u \ge 0$, whereas the corresponding transforms $\mathcal{K}(t,u)$ and $\mathcal{M}(t,u)$ were $O(\frac{1}{|u|^2})$ in modulus. By adopting a trick of Renyi [17], we will avoid any difficulty caused by the larger bound.

We begin the analysis by developing the bound on $|\mathcal{F}(u)|$. Observe that $\mathcal{F}(u)$ has an analytic continuation to all complex $u \ne 0$ given by (91). From (91) we obtain that $\mathcal{F}(u)$ has a double pole at the origin and the expansion

$$\mathcal{F}(u) = \frac{\alpha_*}{u^2} + \frac{\alpha_* - 1}{u} + O(u), \tag{92}$$

as $u \to 0$ (the expansion has no constant term).

Let $\eta(u) := \int_0^u \frac{1-e^{-y}}{y}dy$.

Lemma 15 *For every real σ there exists a constant $c(\sigma) > 0$ such that*

$$|\eta(u) - \log|u|| \le c(\sigma), \quad \Re u \ge \sigma, \ |u| \ge 1.$$

Proof: Let $u := rv(\theta)$, where $r \ge 1$ and $v = v(\theta) := e^{i\theta}$. We have

$$\begin{aligned}\eta(u) &= \eta(v) + \int_v^u \frac{1-e^{-\zeta}}{\zeta}d\zeta \\ &= \eta(v) + \log r - \int_v^u \frac{e^{-\zeta}}{\zeta}d\zeta \end{aligned}\tag{93}$$

To estimate the integral, integrate by parts and get

$$\int_v^u \frac{e^{-\zeta}}{\zeta} d\zeta = -\frac{e^{-u}}{u} + \frac{e^{-v}}{v} - \int_v^u \frac{e^{-\zeta}}{\zeta^2} d\zeta,$$

then fix the path of integration of this last integral to be the straight line $\zeta = vt,\ 1 \leq t \leq r$, and obtain

$$\int_v^u \frac{e^{-\zeta}}{\zeta^2} d\zeta = \frac{1}{v}\int_1^r \frac{e^{-vt}}{t^2} dt,$$

so that

$$\left|\int_v^u \frac{e^{-\zeta}}{\zeta} d\zeta\right| \leq e^{-\sigma} + e + e^{-\sigma}\int_1^\infty \frac{dt}{t^2} = e + 2e^{-\sigma},$$

Together with (93), this bound and the choice $c(\sigma) = e + 2e^{-\sigma} + \max_{|v|=1} \eta(v)$ proves the lemma. ∎

We are now ready for the growth estimate.

Lemma 16 *For every real σ, there exists a constant $c_1(\sigma) > 0$ such that*

$$|\mathcal{F}(u)| \leq \frac{c_1(\sigma)}{|u|},\ \ |u| \geq 1,\ \ \Re u \geq \sigma.$$

Proof: Let $|u| = r$ and write

$$\alpha(u) = \int_0^r \beta(\tau) d\tau + \int_{\Gamma_0} \beta(\zeta) d\zeta,$$

where Γ_0 is the circular arc, centered at the origin, which joins r to u. Then by (91), we have

$$\mathcal{F}(u) = \frac{e^{-u}}{u^2\beta(u)}\left[\int_r^\infty \beta(\tau)d\tau - \int_{\Gamma_0}\beta(\zeta)d\zeta\right]. \tag{94}$$

Let $g(u) := \eta(u) - \log r$ so that $\beta(u) = e^{-2\eta(u)} = e^{-2g(u)}/r^2$ and (94) becomes

$$\mathcal{F}(u) = \frac{e^{-u}}{u^2} e^{2g(u)} r^2 \left[\int_r^\infty \frac{e^{-2g(\tau)}}{\tau^2} d\tau - \int_{\Gamma_0} \frac{e^{-2g(u)}}{r^2} du\right].$$

Since Γ_0 has length at most πr, we conclude from the above equation that

$$|\mathcal{F}(u)| \leq e^{-\sigma} e^{2|g(u)|}\left[\int_r^\infty \frac{e^{2g(\tau)}}{\tau^2} d\tau + \frac{\pi}{r}\max_{|u|=r} e^{2|g(u)|}\right]. \tag{95}$$

The lemma then follows from Lemma 15 and (95) with the choice $c_1(\sigma) = (1+\pi)e^{4c(\sigma)-\sigma}$. ∎

In our next step, we compute $\mathcal{G}(u)$ and prove properties similar to those for $\mathcal{F}(u)$. We begin with a recurrence for the second moment. We use the notation $\mathrm{E}[\tilde{N}_{x+1}^2|y]$ to denote the expectation of $\tilde{N}_{x+1}^2$ given that the first car parks at location $[y, y+1]$.

$$\begin{aligned}
\mathrm{E}\tilde{N}_{x+1}^2 &= \int_0^x \mathrm{E}[\tilde{N}_x^2|y]\frac{dy}{x} \\
&= \int_0^x \mathrm{E}[1 + \tilde{N}_y^2 + \tilde{N}_{x-y}^2 + 2\tilde{N}_y + 2\tilde{N}_{x-y} + \tilde{N}_y\tilde{N}_{x-y}]\frac{dy}{x} \\
&= 1 + \frac{2}{x}\int_0^x \mathrm{E}\tilde{N}_y^2 dy + \frac{4}{x}\int_0^x \mathrm{E}\tilde{N}_y dy + \frac{2}{x}\int_0^x \mathrm{E}\tilde{N}_y\tilde{N}_{x-y} dy
\end{aligned} \tag{96}$$

Take the transform of both sides of (96) and obtain after routine manipulations

$$\frac{d\mathcal{G}}{du} = -\left(1 + \frac{2e^{-u}}{u}\right)\mathcal{G} - \frac{e^{-u}}{u^2}\mathcal{A}(u), \ u > 0, \tag{97}$$

where

$$\mathcal{A}(u) = 1 + 4u\mathcal{F}(u) + 2(u\mathcal{F}(u))^2. \tag{98}$$

We have $\tilde{N}_0 = 0$ and $\tilde{N}_x \le x$, so the desired solution for $\mathcal{G}(u)$ is

$$\mathcal{G}(u) = \frac{e^{-u}}{u^2\beta(u)}\int_u^\infty \mathcal{A}(\tau)\beta(\tau)d\tau, \ u > 0. \tag{99}$$

Observe that by Lemma 16 and (98), we get $\lim_{u\to\infty}\mathcal{A}(u) = 1$, which implies that the integral appearing in (99) is finite. We know from the properties of $\mathcal{F}(u)$ that $\mathcal{A}(u)$ and $\mathcal{B}(u) := \mathcal{A}(u)\beta(u)$ are analytic for $u \ne 0$. It is routine to verify from (92) and (98) that, at $u = 0$, we have

$$\mathcal{A}(u) = \frac{2\alpha_*^2}{u^2} + \frac{4\alpha_*^2}{u} + (2\alpha_*^2 - 1) + \cdots \tag{100}$$

$$\mathcal{B}(u) = \frac{2\alpha_*^2}{u^2} - (\alpha_*^2 + 1) + \cdots. \tag{101}$$

Define

$$\kappa(u) := \mathcal{B}(u) - \frac{2\alpha_*^2}{u^2}$$

and note that, by (101), $\kappa(u)$ is entire. By Lemma 15 and the fact that $\lim_{u\to\infty}\mathcal{A}(u) = 1$, we conclude that, for $\Re u \ge \sigma$, $\mathcal{B}(u)$ and hence $\kappa(u)$ are $O(\frac{1}{|u|^2})$ as $s \to \infty$. In particular, $\kappa(u)$ is integrable on $[0,\infty)$.

As noted earlier, $\mathcal{G}(u)$ is analytic for $u \ne 0$. If we let $\kappa_* := \int_0^\infty \kappa(\tau)d\tau$, then we can rewrite (99) as

$$\begin{aligned}\mathcal{G}(u) &= \frac{e^{-u}}{u^2\beta(u)}\int_u^\infty \left[\kappa(\tau) + \frac{2\alpha_*^2}{\tau^2}\right]d\tau \\ &= \frac{e^{-u}}{u^2\beta(u)}\left[\frac{2\alpha_*^2}{u} + \kappa_* - \int_0^u \kappa(\tau)d\tau\right],\end{aligned} \tag{102}$$

which shows that $\mathcal{G}(u)$ is analytic for $u \ne 0$. The function $e^{-u}/(u^2\beta(u))$ can be expanded as in the derivation of (92) from (91), and by (101) we have $\int_0^u \kappa(\tau)d\tau = -(\alpha_*^2+1)u + \cdots$. Thus, (102) yields

$$\mathcal{G}(u) = \frac{2\alpha_*^2}{u^3} + \frac{\kappa_* + 2\alpha_*^2}{u^2} + \frac{\kappa_* + \alpha_*^2 + 1}{u} + \cdots. \tag{103}$$

The bound on the growth of $\mathcal{G}(u)$ is as follows.

Lemma 17 *For every real σ, there exists a constant $c_2(\sigma) > 0$ such that*

$$|\mathcal{G}(u)| \le \frac{c_2(\sigma)}{|u|}, \ \Re u \ge \sigma, \ |u| \ge 1.$$

Proof: Use (102) and mimic the proof of Lemma 16. We omit the details. ∎

Finally, we estimate $\mathrm{E}\tilde{N}_x^2$ from the inversion formula

$$\mathrm{E}\tilde{N}_x^2 = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{xu}\mathcal{G}(u)du, \tag{104}$$

for any $\sigma > 0$. As is readily verified, $e^{xu}\mathcal{G}(u)$ is analytic for $u \neq 0$ and has a third order pole at the origin. Multiplying (103) by $e^{xu} = 1 + xu + (xu)^2/2 + \cdots$ the residue of $e^{xu}\mathcal{G}(u)$ is found to be $\alpha_*^2x^2 + (\kappa_* + 2\alpha_*^2)x + \kappa_* + \alpha_*^2 + 1$. Thus, by the residue theorem (see the argument in the proof of Theorem 3), we get

$$\mathrm{E}\tilde{N}_x^2 = \alpha_*^2x^2 + (\kappa_* + 2\alpha_*^2)x + \kappa_* + \alpha_*^2 + 1 + \frac{1}{2\pi i}\int_{-\sigma-i\infty}^{-\sigma+i\infty} e^{xu}\mathcal{G}(u)du, \tag{105}$$

for any $\sigma > 0$. It remains to estimate the integral.

The estimate in (17) is not sufficient, as it fails to guarantee the absolute convergence of the integral. To get around this difficulty, we adopt a trick of Renyi [17]. Rewrite (97) as

$$\frac{d}{du}(e^u\mathcal{G}(u)) = -\frac{2}{u}\mathcal{G}(u) - \frac{\mathcal{A}(u)}{u^2}$$

and deduce from Lemma 17 that

$$\left|\frac{d}{du}(e^u\mathcal{G}(u))\right| \le \frac{2c_2(\sigma)+1}{|u|^2},\ \Re u \ge \sigma,\ |u| \ge 1. \tag{106}$$

An integration by parts and substitution of (106) gives

$$\begin{aligned}
\int_{-\sigma-i\infty}^{-\sigma+i\infty} e^{xu}\mathcal{G}(u)du &= \int_{-\sigma-i\infty}^{-\sigma+i\infty} e^{(x-1)u}(e^u\mathcal{G}(u))du,\ \sigma > 0,\\
&= -\frac{1}{x-1}\int_{-\sigma-i\infty}^{-\sigma+i\infty} e^{(x-1)u}d(e^u\mathcal{G}(u)),\ \sigma > 0\\
&= O(e^{-\sigma x}) \quad \text{for all} \quad \sigma > 0.
\end{aligned}$$

As at the end of the proof of Theorem 3, we remark that an optimization of parameters improves the bound to $O(e^{-\xi x\log x})$ for all $\xi \in (0,1)$; as before, the details are routine and left to the interested reader. This bound together with (105) gives an estimate for the second moment which, after subtracting the square of the mean in (37), yields (73) with $\mu_* = \kappa_* + 2\alpha_*$, where

$$\begin{aligned}
\kappa_* &= \int_0^\infty \kappa(u)du\\
&= \int_0^\infty \left\{\beta(u)[1 + 4u\mathcal{F}(u) + 2u^2\mathcal{F}^2(u)] - \frac{2\alpha_*^2}{u^2}\right\}du
\end{aligned} \tag{107}$$

The calculations of the discrete model leading to (7) were omitted in [12], and it is not obvious that $\kappa_* + 2\alpha_*$ and (7) give the same constant. To prove that they do, we start by integrating the first term in (107), getting $\int_0^\infty \beta(u)du = \alpha_*$, and then writing $\kappa_* + 2\alpha_*$ in the form

$$\kappa_* + 2\alpha_* = 4\alpha_* + \int_0^\infty \frac{\mathcal{Q}(u)}{u^2}du - \alpha_*, \tag{108}$$

where

$$\mathcal{Q}(u) := u^2\beta(u)[4\mathcal{F}_1(u) + 2\mathcal{F}_1^2(u)] - 2\alpha_*^2$$

with

$$\mathcal{F}_1(u) := u\mathcal{F}(u) = \frac{e^{-u}\tilde{\alpha}(u)}{u\beta(u)}. \tag{109}$$

Comparing (108) with (7) written in terms of $\mathcal{F}_1$, we see that $\kappa_* + 2\alpha_*$ and (7) yield the same constant if

$$4\alpha_* + \int_0^\infty \frac{\mathcal{Q}(u)}{u^2}du = 4\int_0^\infty \beta(u)\mathcal{F}_1(u)(1 - e^{-u})du - 4\int_0^\infty \beta(u)\mathcal{F}_1^2(u)(e^{-u} - 1 + u)du. \tag{110}$$

We work with the left-hand side and begin by noting that, since $\kappa(u) = \beta(u) + \mathcal{Q}(u)/u^2$ is entire (by design), then so is $\mathcal{Q}(u)/u^2$. Thus, $\mathcal{Q}(u)/u$ vanishes at 0, and since $\mathcal{F}_1(u)$ is exponentially small in u, we have $\mathcal{Q}(u)/u = O(\frac{1}{u})$ as $u \to \infty$ and so $\mathcal{Q}(u)/u$ also vanishes at infinity. Therefore, an integration by parts and use of

$$\beta'(u) = -2\frac{1 - e^{-u}}{u}\beta(u) \tag{111}$$

gives, after some simplification,

$$\begin{aligned} 4\alpha_* + \int_0^\infty \mathcal{Q}(u)\frac{du}{u^2} &= \int_0^\infty \mathcal{Q}'(u)\frac{du}{u} \\ &= 4\alpha_* + 4\int_0^\infty \beta(u)e^{-u}[2\mathcal{F}_1(u) + \mathcal{F}_1^2(u)]du \qquad (112) \\ &\quad +4\int_0^\infty u\beta(u)\mathcal{F}_1'(u)du + 4\int_0^\infty u\beta(u)\mathcal{F}_1'(u)\mathcal{F}_1(u)du. \qquad (113) \end{aligned}$$

Now isolate the second to the last integral, integrate by parts, again using (111), and get

$$\begin{aligned} 4\int_0^\infty u\beta(u)\mathcal{F}_1'(u)du &= 4u\beta(u)\mathcal{F}_1(u)|_0^\infty - 4\int_0^\infty \mathcal{F}(u)[\beta(u) + u\beta'(u)]du \\ &= -4\alpha_* + 4\int_0^\infty \beta(u)(1 - 2e^{-u})\mathcal{F}_1(u)du. \end{aligned}$$

Substituting back into (113) gives, after a little algebra,

$$4\alpha_* + \int_0^\infty \frac{\mathcal{Q}(u)}{u^2}du = 4\int_0^\infty \beta(u)(\mathcal{F}_1(u) + e^{-u}\mathcal{F}_1^2(u) + u\mathcal{F}_1(u)\mathcal{F}_1'(u))du. \tag{114}$$

A straightforward calculation shows that

$$\mathcal{F}_1'(u) = \mathcal{F}_1(u)\frac{1 - u - 2e^{-u}}{u} - \frac{e^{-u}}{u}, \tag{115}$$

which is also easily deduced from the differential equation for $\mathcal{F}$ given in [6]. This gives

$$u\mathcal{F}_1(u)\mathcal{F}_1'(u) = (1 - u - 2e^{-u})\mathcal{F}_1^2(u) - e^{-u}\mathcal{F}_1(u)$$

which we substitute back into (114). In the resulting expression, we collect separately the coefficients of $\beta(u)\mathcal{F}_1(u)$ and $\beta(u)\mathcal{F}_1^2(u)$ and obtain the right-hand side of (110), as desired. ■