Subexponential Asymptotics of a Markov-Modulated Random Walk with Queueing Applications

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Abstract

Let $\{(X_n, J_n)\}$ be a stationary Markov-modulated random walk on $\mathbb{R} \times E$ (E finite), defined by its probability transition matrix measure $F = \{F_{ij}\}, F_{ij}(B) = \mathbb{P}[X_1 \in B, J_1 = j | J_0 = i], B \in \mathcal{B}(\mathbb{R}), i, j \in E$. If $F_{ij}([x,\infty))/(1 - H(x)) \to W_{ij} \in [0,\infty)$, as $x \to \infty$, for some long-tailed distribution function H, then the ascending ladder heights matrix distribution $G_+(x)$ (right Wiener-Hopf factor) has long-tailed asymptotics. If $\mathbb{E}X_n < 0$, at least one $W_{ij} > 0$, and $H(x)$ is a subexponential distribution function, then the asymptotic behavior of the supremum of this random walk is the *same* as in the i.i.d. case, and it is given by $\mathbb{P}\left[\sup_{n \geq 0} S_n > x\right] \to (-\mathbb{E}X_n)^{-1} \int_x^\infty \mathbb{P}[X_n > u] du$ as $x \to \infty$, where $S_n = \sum_1^n X_k, S_0 = 0$. Two general queueing applications of this result are given.

First, if the same asymptotic conditions are imposed on a Markov-modulated G/G/1 queue, then the waiting time distribution has the *same* asymptotics as the waiting time distribution of a GI/GI/1 queue, i.e., it is given by the integrated tail of the service time distribution function divided by the negative drift of the queue increment process. Second, the autocorrelation function of a class of processes constructed by embedding a Markov chain into a subexponential renewal process has a subexponential tail. When a fluid flow queue is fed by these processes, the queue length distribution is asymptotically *proportional* to its autocorrelation function.

Keywords: Non-Cramér type conditions; Subexponential distributions; Markov-modulated random walk; Weiner-Hopf factorization; Supremum distribution; Subexponential dependency; Fluid flow queue

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1. Introduction

Our goal is to examine the asymptotics of the queue length distribution when the Cramér type conditions are replaced by subexponential assumptions. The main method that we use to investigate the asymptotic behavior of the queue length distribution is *random walk ladder heights technique.* This is based on the direct connection between the distribution of the supremum of a random walk and the distribution of the corresponding single server queue [FEL71], [BIL86], section 24. Our main result extends the existing result on the asymptotic behavior of the supremum of an i.i.d. random walk to the Markov-modulated setting. We apply this random walk result to investigate two canonical queueing scenaria that are of practical importance in engineering broadband network multiplexers. Besides queueing application, the obtained random walk results have a variety of other applications, e.g. in Insurance Risk Theory [AHK94].

This paper is organized as follows. In section 2 we give the definitions, a few classical results on subexponential and long-tailed distributions, and some new results on the long-tailed asymptotics of signed measures. In subsection 3.1 we define a Markov-modulated random walk and present some recent results [ASM89, ASM91] on its ladder heights analysis approach. Explicit long-tailed asymptotic behavior of the ascending ladder heights matrix distribution of the Markov-modulated random walk is examined in subsection 3.2. The same subsection contains our main result (Theorem 4) on the asymptotic behavior of the distribution of the supremum of the Markov-modulated random walk. This theorem extends on the classical result by [PAK75, VER77], and recent Markov-modulated generalizations in [AHK94]. In this result we essentially show that the asymptotics of the supremum of an i.i.d. random walk with subexponential right tail and negative drift is *invariant* under the Markov modulation.

In sections 4 and 5 we present two important queueing applications. (Practical motivation for this queueing investigation is given in subsection 4.1.) First, in section 4 we extend the classical result by Pakes on the subexponential $GI/GI/1$ queue ([PAK75]) asymptotics to the Markov-modulated G/G/1 queue. Generalization of the Pakes' result to the Markov-modulated $M/G/1$ queue was recently explored in [AHK94]. However, the asymptotic constant of proportionality in [AHK94] was left in a complex form. We show that even with the more general setting of the Markov-modulated G/G/1 queue this constant is the same as in the corresponding GI/GI/1 queue. Second, in section 5 we consider a class of processes that have a subexponential autocorrelation function. These processes are obtained by embedding a Markov chain into a stationary subexponential renewal process. When these processes are fed into a fluid flow queue, the queue length distribution is *asymptotically proportional to the autocorrelation (autocovariance) function of the arrival process.* The paper concludes in section 6 with a brief discussion on the applicability of these results to broadband network admission control.

2. Subexponential Distributions

In this section we first state the definitions on subexponential and long-tailed distributions. In section 2.1 we enumerate some of the classical results on subexponential distributions. This is followed in section 2.2 by new results on the long-tailed asymptotics of the convolution of signed measures.

Definition 1 *A distribution function F on $[0,\infty)$ is called* long-tailed *($F \in \mathcal{L}$) if*

$$\lim_{x\to\infty} \frac{1-F(x-y)}{1-F(x)} = 1, \quad y \in \mathbb{R}. \tag{2.1}$$

Definition 2 *A distribution function F on $[0,\infty)$ is called* subexponential *($F \in \mathcal{S}$) if*

$$\lim_{x\to\infty} \frac{1-F^{*2}(x)}{1-F(x)} = 2, \tag{2.2}$$

where F^{*2} denotes the 2-nd convolution of F with itself, i.e., $F^{*2}(x) = \int_{[0,\infty)} F(x-y)F(dy)$.

The class of subexponential distributions was first introduced by Chistakov [CHI64]. The definition is motivated by the simplification of the asymptotic analysis of the convolution tails. Some examples of distribution functions in $\mathcal{S}$ are:

(I) the Pareto family

$$F(x) = 1 - (x - \beta + 1)^{-\alpha},$$

$x \geq \beta \geq 0, \alpha > 0.$

(II) the lognormal distribution

$$F(x) = \Phi\left(\frac{\log x - \mu}{\sigma}\right), \quad \mu \in \mathbb{R}, \sigma > 0,$$

where Φ is the standard normal distribution.

(III) Weibull distribution

$$F(x) = 1 - e^{-x^{\beta}},$$

for $0 < \beta < 1$.

(IV)

$$F(x) = 1 - e^{-x(\log x)^{-a}},$$

for $a > 0$.

(V) Benktander Type I distribution [KLU88]

$$F(x) = 1 - cx^{-a-1}x^{-b\log x}(a + 2b\log x),$$

$a > 0, b > 0$, and c appropriately chosen.

(V) Benktander Type II distribution [KLU88]

$$F(x) = 1 - cax^{-(1-b)}\exp\{-(a/b)x^{b}\},$$

$a > 0, 0 < b < 1$, and c appropriately chosen.

2.1 Classical Results

In what follows we will state a few classical results from the literature on subexponential distributions. The general relation between $\mathcal{S}$ and $\mathcal{L}$ is the following.

Lemma 1 *(Athreya and Ney, [ATN72])* $\mathcal{S} \subset \mathcal{L}$.

Lemma 2 *If* $F \in \mathcal{L}$ *then* $(1 - F(x))e^{\alpha x} \to \infty$ *as* $x \to \infty$, *for all* $\alpha > 0$.

Note: Lemma 2 clearly shows that for long-tailed distributions Cramér type conditions are not satisfied.

The proofs of the following results can be found in Embrechts, Goldie and Veraverbeke [EGV79]. To simplify the notation, for any distribution F we denote by $\bar{F}(x) = 1 - F(x)$.

Lemma 3 *Let* $F \in \mathcal{S}$. *Then,*

(i) *If* G *is a probability distribution such that* $\bar{G}(x) = o(\bar{F}(x))$ *as* $x \to \infty$, *then* $\overline{F * G}(x) \sim \bar{F}(x)$.

(ii) *If* $\lim_{x\to\infty} \bar{G}(x)/\bar{F}(x) = c \in (0, \infty)$, *where* G *is a distribution function on* $[0, \infty)$, *then* $G \in \mathcal{S}$.

Often in renewal theory it is of interest to investigate the *integrated tail* of a distribution function. For that reason for any distribution F we define $\hat{F}(x) \stackrel{\text{def}}{=} \int_x^\infty [1 - F(t)]dt$, and $F_1(x) \stackrel{\text{def}}{=} m^{-1}(1 - \hat{F}(x))$, where $m = \hat{F}(0)$.

Definition 3 $F \in \mathcal{S}^*$ *if*

$$\int_0^x \frac{\bar{F}(x-y)}{\bar{F}(x)} \bar{F}(y)dy \to 2m_F < \infty, \quad \text{as} \quad x \to \infty,$$

where $m_F = \int_0^\infty yF(dy)$.

This class has the property that $F \in \mathcal{S}^* \Rightarrow F_1 \in \mathcal{S}$, and that $\mathcal{S}^* \subset \mathcal{S}$. Sufficient conditions for $F \in \mathcal{S}^*$ can be found in [KLU89], where it was explicitly shown that lognormal, Pareto, and certain Weibull distributions are in $\mathcal{S}^*$.

An extensive treatment of subexponential distributions (and further references) can be found in Cline [CLI86, CLI87].

2.2 Long-Tailed Asymptotics of Signed Measures

In this section we prove a few general results (which might be of independent interest) on the long-tailed asymptotics of the convolution of signed measures. A combination of these results essentially will give rise to the proofs of Theorems 2, 3, on the long-tailed asymptotics of the ascending ladder heights matrix distribution presented in section 3.2.

Let $\mathcal{B}(\mathbb{R})$ be a Borel σ-algebra on $\mathbb{R}$. The convolution of two measures $\mu_i, i = 1, 2$, is defined by ([BIL86], pp. 272) $(\mu_1 * \mu_2)(B) = \int_{-\infty,\infty} \mu_1(B - x)\mu_2(dx)$, $B \in \mathcal{B}(\mathbb{R})$, $B - x = \{y : y + x \in B\}$.

Lemma 4 *Let μ, μ_- be two* finite *(signed) measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, such that $\mu([x,\infty))/\bar{H}(x) \to c$, as $x \to \infty$, $H(x) \in \mathcal{L}$, $|c| < \infty$, and μ_- has a support on $(-\infty, 0]$. Then, $\nu \stackrel{\text{def}}{=} \mu_- * \mu$ satisfies*

$$\lim_{x\to\infty} \frac{\nu([x,\infty))}{\bar{H}(x)} = c\mu_-((-\infty, 0]).$$

Proof: Given in Appendix A. $\diamond$

Lemma 5 *Let μ, μ_-, μ_+, be* finite *(possibly signed) measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$; with μ_- having a support on $(-\infty, 0]$, $|\mu_-((-\infty, 0])| > 0$; μ_+ has a support on $[0, \infty)$ it is strictly positive on $[a, \infty)$, $a > 0$, and $\lim_{x\to\infty} \mu([x,\infty))/\bar{H}(x) = c$, $H(x) \in \mathcal{L}$, $|c| < \infty$. If $\mu = \mu_- * \mu_+$, then*

$$\lim_{x\to\infty} \frac{\mu_+([x,\infty))}{\bar{H}(x)} = \frac{c}{\mu_-((-\infty, 0])}. \tag{2.3}$$

Proof: Given in Appendix A. $\diamond$

Proposition 1 *Let $\mu = \mu_- * \mu_+$, where measures μ, μ_-, μ_+ satisfy the conditions from the* *previous lemma, and, in addition,* $\mu_-((-\infty,0])=0$, *and* $0<\left|\int_{(-\infty,0]} u\mu_-(du)\right|<\infty$. *Then,*

$$\lim_{x\to\infty}\frac{\mu_+([x,\infty))}{\hat{H}(x)}=\frac{c}{\int_{(-\infty,0]} u\mu_-(du)},$$

(recall $\hat{H}(x)=\int_{[x,\infty)}\bar{H}(u)du$*).*

Proof: Given in Appendix A. ◇

3. Markov-Modulated Random Walk

The natural probability technique for analyzing the supremum of a random walk is through the ladder heights distributions and Weiner-Hopf factorization [FEL71]. This approach has been recently generalized by Asmussen [ASM89, ASM91] to the Markov-modulated random walk. The necessary definitions and basic results are presented in the following subsection. (We borrow the notation from [ASM89].)

3.1 Markov-Modulated Random Walk and Ladder Heights

Let $\{J_t\}$ be a stationary irreducible aperiodic Markov chain with a finite state space E (say with N elements) and transition matrix P, and let $\{X_t\}$ be a sequence of real valued random variables. A stationary Markov process $\{(J_t,X_t)\}$ on $E\times\mathbb{R}$ whose transition distribution depends only on the first coordinate is called a Markov-modulated random walk (MMRW). This process is completely defined through its transition matrix measure $F_{ij}(B)=\mathbb{P}[J_1=j,X_1\in B|J_0=i]$, and $F=\{F_{ij}\}$ (note that $\|F\|=F((-\infty,\infty))=P$). Let $\{(J_t^r,X_t^r)\}$ denote the associated reversed process. This process is determined by the set of transition measures $F_{ij}^r(B)=\mathbb{P}[J_0=j,X_1\in B|J_1=i]$, with $F^r=\{F_{ij}^r\}$ being the corresponding transition matrix measure.

Further, define $S_0 = 0, S_n = \sum_{t=1}^{n} X_t$, $(\mathbb{P}_i[\cdot] \stackrel{\text{def}}{=} \mathbb{P}[\cdot|J_0 = i])$,

$$\begin{aligned}
&\tau_+ = \inf\{n > 0 : S_n > 0\}, \\
&G_+(i,j;B) = \mathbb{P}_i[J_{\tau_+} = j, S_{\tau_+} \in B, \tau_+ < \infty], \\
&\|G_+(i,j)\| = G_+(i,j;(0,\infty)), \\
&G_+(B) = \{G_+(i,j;B)\}_{i,j\in E}, \\
&\|G_+\| = \{\|G_+\|\}_{i,j\in E}.
\end{aligned}$$

The convolution of the matrix measure G_+ is naturally extended to

$$\begin{aligned}
G_+^{*2}(i,j) &= \sum_{k\in E} G_+(i,k) * G_+(k,j), \\
G_+^{*2} &= \{G_+^{*2}(i,j)\}_{i,j\in E};
\end{aligned}$$

higher convolution powers are similarly defined.

Then in [ASM91] the following extension of the Pollaczek-Khinchine identity is provided for $M = \sup_{n\geq 0} S_n$.

Theorem 1 $\mathbb{P}_i[M \in B]$ *is the ith component of the vector*

$$\sum_{n=0}^{\infty} G_+^{*n}(B)(I - \|G_+\|)e,$$

where e is the column vector of ones and I is the identity matrix.

Let $\tau_- = \inf\{n \geq 1 : S_n^r \leq 0\}$, where $S_n^r = \sum_{t=1}^{n} X_t^r$, and define

$$\begin{aligned}
&G_-(i,j;B) = \mathbb{P}_i[S_{\tau_-}^r \in B, J_{\tau_-}^r = j, \tau_- < \infty], \\
&{}^{\#}G_-(i,j) = \frac{\pi_j}{\pi_i} G_-(j,i),
\end{aligned}$$

where $\pi_i \stackrel{\text{def}}{=} \mathbb{P}[J_t = i]$. Then, the following Wiener-Hopf identity holds ([ASM89], Theorem

4.1)

$$I - F(B) = (I - {}^{\#}G_{-}) * (I - G_{+})(B), \tag{3.1}$$

where B is any real Borel set.

3.2 Subexponential Asymptotics of the Distribution of the Supremum of MMRW

This section contains our results on the subexponential asymptotics of MMRW. Theorems 2 and 3 explicitly give the long-tailed asymptotic behavior of the ascending ladder heights matrix distributions. Our main result on the subexponential asymptotic behavior of the supremum of the MMRW is stated in Theorem 4.

In order to state the results we need to introduce some additional notation. Let $A(x) = \{A_{ij}(x)\}$ be a matrix composed of distribution functions, and its Fourier transform defined as $\tilde{A}(\omega) = \{\tilde{A}_{ij}(\omega)\}$, $\tilde{A}_{ij}(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} dA_{ij}(x)$. We will use the symbol $\mathcal{F}^{-1}(\cdot)$ to denote the operation of taking the inverse Fourier transform. Note that there is a one-to-one correspondence between the distribution functions on $\mathbb{R}$ and its Fourier transforms (see [CHT88], section 8.3). Then, the Wiener-Hopf factorization equation (3.1) can be written as

$$(I - \tilde{F}(\omega)) = (I - {}^{\#}\tilde{G}_{-}(\omega))(I - \tilde{G}_{+}(\omega)). \tag{3.2}$$

Observe that $\tilde{A}_{ij}(0) = A((-\infty, \infty))$. Also, if we assume that $\int_{-\infty}^{\infty} |x|^{1+\delta} A_{ij}(dx) < \infty$, for some $\delta > 0$, then by Theorem 1, pp. 277 [CHT88], the Fourier transforms of $\tilde{A}_{ij}$ will have continuous first derivatives, and in particular $-i\tilde{A}'_{ij}(0) = \int_{-\infty}^{\infty} x dA_{ij}(x)$. For any matrix A let us define the *adjoint* matrix $adj(A)$, $adj(A)_{ij} = (-1)^{i+j} \det(A^{ij})$, where A^{ij} denotes the matrix obtained by deleting the ith row and jth column from A. If A is invertible then $A^{-1} = (\det(A))^{-1} adj(A)$. Assume $\mathbb{E}X_n < 0$ (negative drift) and $\int_{-\infty}^{\infty} |x|^{1+\delta} F_{ij}(dx) < \infty$, for some $\delta > 0$. Let π be a row vector with its ith element being π_i $(= \mathbb{P}[J_t = i])$.

Theorem 2 *Let* $\lim_{x\to\infty} \bar{F}(x)/\bar{H}(x) = W$, $W = \{W_{ij}\}$, $W_{ij} \in [0,\infty)$, $H(x) \in \mathcal{L}$. *Then,*

$$\lim_{x\to\infty} \frac{1}{\hat{H}(x)} \bar{G}_+(x) = \frac{D}{\int_{(-\infty,0]} u\mu_{G_-}(du)} e\pi W, \tag{3.3}$$

where $D \equiv D_{\|{}^{\#}G_-\|} \stackrel{\text{def}}{=} \lim_{\alpha\to 1} \frac{\det(I-\alpha\|{}^{\#}G_-\|)}{1-\alpha} > 0$, e *is a* $N \times 1$ *column vector of ones, and* $\mu_{G_-} \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\det(I - {}^{\#}\tilde{G}_-(\omega)))$ $(\int_{(-\infty,0]} u\mu_{G_-}(du) > 0)$.

Remark: This result represents an extension of Theorem 1 (C) of [VER77], where for the case of an i.i.d. random walk with $F \in \mathcal{S}$, it was shown that $G_+([x,\infty))/\hat{F}(x) \sim 1/\int_{(-\infty,0]} |u| G_-(du)$.

Proof: First, let us observe that $\det(I - \tilde{F})(\omega)$ has a zero of order one for $\omega = 0$. $\omega = 0$ is a zero since $\det(I - \tilde{F})(0) = \det(I - \|F\|) = 0$, as $\|F\|$ $(\|F\|_{ij} = p_{ji})$ is a stochastic transition matrix; that this zero is of order one follows from $\mathbb{E}X_n < 0$ and Lemma 9 of Appendix B (lemma implies that $|\det(I - \tilde{F})'(0)| > 0$). Furthermore, since $\|G_+\|$ is substochastic (see [ASM89], Proposition 4.2), we have that $|\det(I - \tilde{G}_+)(0)| > 0$, which implies (by equation (3.2)) that $\omega = 0$ is also a zero of order one for the $\det(I - {}^{\#}\tilde{G}_-)(\omega)$, implying $0 < |\det(I - {}^{\#}\tilde{G}_-)'(0)| < \infty$; finiteness follows from $\det(I - \tilde{F})'(0) = \det(I - \tilde{G}_+)(0)\det(I - {}^{\#}\tilde{G}_-)'(0)$.

Let us define the measure $\mu_{G_-} \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\det(I - {}^{\#}\tilde{G}_-(\omega)))$. Note that this measure is finite with the support $(-\infty, 0]$, and $\int_{(-\infty,0]} u\mu_{G_-}(du) = -i\det(I - {}^{\#}\tilde{G}_-)'(0) > 0$ (the positive sign follows from Lemma 9 (see Appendix B), $\mathbb{E}X_n < 0$, and $\det(I - \tilde{F})'(0) = \det(I - \tilde{G}_+)(0)\det(I - {}^{\#}\tilde{G}_-)'(0)$). Also, equation (3.2) can be written as

$$adj(I - {}^{\#}\tilde{G}_-(\omega))(\tilde{F}(\omega) - I) = (\tilde{G}_+(\omega) - I)\det(I - {}^{\#}\tilde{G}_-(\omega)),$$

or component-wise

$$(\tilde{G}_+(\omega) - I)_{ij} \det(I - {}^{\#}\tilde{G}_-(\omega)) = \sum_{k=1}^{N} adj(I - {}^{\#}\tilde{G}_-(\omega))_{ik}(\tilde{F}(\omega) - I)_{kj}.$$

If $\mu_{ij} \stackrel{\text{def}}{=} \mathcal{F}^{-1} adj(I - {}^{\#}\tilde{G}_-(\omega))_{ik}(\tilde{F}(\omega) - I)_{kj})$, then, by Lemma 4, $\mu_{ij}([x,\infty))/\bar{H}(x) \to \sum_{k=1}^{N} adj(I - {}^{\#}\tilde{G}_-(0))_{ik} W_{kj}$, as $x \to \infty$. If $\mu_{+ij} \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\tilde{G}_+(\omega) - I)_{ij}$, then $\mu_{ij} = \mu_{+ij} * \mu_{G_-}$, where $\mu_{ij}, \mu_{+ij}, \mu_{G_-}$, satisfy the conditions of Proposition 1. Thus, by the same proposition,

$$\frac{1}{\hat{H}(x)} \bar{G}_{+ij}(x) \to \frac{1}{-i \det(I - {}^{\#}\tilde{G}_-)'(0)} \sum_{k=1}^{N} adj(I - {}^{\#}\tilde{G}_-(0))_{ik} W_{kj},$$

as $x \to \infty$ (recall $\int_{(-\infty,0]} u \mu_{G_-}(du) = -i \det(I - {}^{\#}\tilde{G}_-)'(0)$), or in the matrix form

$$\frac{\bar{G}_+(x)}{\hat{H}(x)} \to \frac{1}{-i \det(I - {}^{\#}\tilde{G}_-)'(0)} adj(I - {}^{\#}\tilde{G}_-(0)) W, \tag{3.4}$$

as $x \to \infty$. Since ${}^{\#}\tilde{G}_-(0) = \|{}^{\#}G_-\|$ is a stochastic, aperiodic, irreducible matrix with stationary probability distribution π (follows easily from the definition of ${}^{\#}G_-$, $\mathbb{E}X_n < 0$, and assumptions on P) by Lemma 8 of Appendix B we get $adj(I - {}^{\#}\tilde{G}_-(0)) = De\pi$; when we replace this result in (3.4) we obtain the desired statement of the theorem. $\diamond$

Although, the case of positive drift is not of our interest (unstable queue) for the reason of completeness we state the following theorem. (The theorem is an extension of Theorem 1 (B), [VER77].)

Theorem 3 *Let* $\lim_{x\to\infty} \bar{F}(x)/\bar{H}(x) = W$, $W = \{W_{ij}\}$, $W_{ij} \in [0,\infty)$, $H(x) \in \mathcal{L}$. *If* $\mathbb{E}X_n > 0$, *then*

$$\lim_{x\to\infty} \frac{\bar{G}_+(x)}{\bar{H}(x)} = (I - \|{}^{\#}G_-\|)^{-1} W. \tag{3.5}$$

Remark: The i.i.d. version of this theorem (Theorem 1 (B), [VER77]) states that $\bar{G}_+(x)/\bar{F}(x) \sim 1/(1 - \|G_-\|)$ as $x \to \infty$.

Proof: Note that $\|{}^{\#}G_-\|$ is substochastic (see [ASM89]). Then, if we define $\mu_{ij}, \mu_{+ij}, \mu_{G_-}$ exactly as in the proof of Theorem 2, it follows (by Lemma 4) that $\mu_{ij}([x,\infty))/\bar{H}(x) \to \sum_{k=1}^{N} adj(I - {}^{\#}\tilde{G}_-(0))_{ik} W_{kj}$, as $x \to \infty$. Now, by applying Lemma 5 to the measures

$\mu_{ij}, \mu_{+ij}, \mu_{G_-}$, we obtain

$$\frac{\bar{G}_{+ij}(x)}{\bar{H}(x)} \to \frac{1}{\det(I - {}^{\#}\tilde{G}_-)(0)} \sum_{k=1}^{N} adj(I - {}^{\#}\tilde{G}_-(0))_{ik} W_{kj},$$

as $x \to \infty$, which, in matrix form, is tantamount to equation (3.5). This finishes the proof of the theorem. $\diamond$

Now we will apply Theorem 2 to obtain the asymptotics of the supremum of the MMRW, i.e., asymptotics of $\mathbb{P}_i[M > x]$ as $x \to \infty$. In order to do so we need an asymptotic estimate for the Pollaczek-Khinchine formula (Theorem 1). This will be obtained from the following lemma, an extension of the result given in [AHK94].

Lemma 6 *Let $G = \{G_{ij}\}$ be a matrix of non-negative measures on $[0, \infty)$, such that $\|G\| \stackrel{\text{def}}{=} G(0, \infty)$ is substochastic (the spectral radius is < 1). If there exists some probability distribution $H \in \mathcal{S}$ such that $\bar{G}_{ij}(x)/\bar{H}(x) \to l_{ij} \in [0, \infty)$ as $x \to \infty$ for some matrix $L = \{l_{ij}\}$, then*

$$\frac{1}{\bar{H}(x)} \sum_{n=0}^{\infty} \overline{G^{*n}}(x)(I - \|G\|) \to (I - \|G\|)^{-1} L \quad \text{as} \quad x \to \infty.$$

Remark: In [AHK94] it was assumed that $l_{ij} > 0$ for all $i, j \in E$. The fact that the same result holds in the more general case suggests that the (matrix) constants in Theorems 4.1 and 4.2, in [AHK94], have to be of the same form, and indeed this is the case. (Algebraic simplification of the matrix constant $A^{(4)}$ in Theorem 4.2, [AHK94], reduces to the same form as in Theorem 4.1, in [AHK94].)

Proof: Given in Appendix C. $\diamond$

The combination of this lemma and Theorem 2 implies the following theorem.

Theorem 4 *Let $\lim_{x\to\infty} \bar{F}(x)/\bar{H}(x) = W$, $W = \{W_{ij}\}$, $W_{ij} \in [0, \infty)$, $H(x) \in \mathcal{L}, H_1(x) \in \mathcal{S}$, with at least one $W_{ij} > 0$. If $\mathbb{E}X_n < 0$, and $\|F\|$ irreducible and aperiodic, then,*

$$\lim_{x\to\infty} \frac{1}{\hat{H}(x)} \bar{M}(x) = \frac{1}{-\mathbb{E}X_n} e\pi W e, \tag{3.6}$$

where $\bar{M}(x)$ is a column vector with its ith component equal to $\mathbb{P}_i[M > x]$. In particular

$$\mathbb{P}[M > x] \sim \frac{1}{-\mathbb{E}X_n}\pi We\hat{H}(x) \sim \frac{1}{-\mathbb{E}X_n}\int_x^\infty \mathbb{P}[X_n > u]du, \tag{3.7}$$

as $x \to \infty$.

Remarks: (i) The i.i.d. version of this theorem was given in [VER77], Theorem 2 (B). **(ii)** We have found that a weaker version of this theorem has appeared in the literature in [ARN80]. The assumptions given there are more restrictive (also our method of proof is completely different). In [ARN80] it was assumed that: a) H belongs to a class of distributions of extended regular variation [KAR30, BGT87]; this class is smaller than the subexponential class (for example from the distribution families given in section 2 it contains only the Pareto family), b) $W_{ij} > 0$ for all $i, j \in E$, c) absolute continuity assumptions are imposed on F_{ij}.
Proof: Combining Theorem 2, more precisely equation (3.4), and Lemma 6, we get

$$\frac{1}{\bar{H}(x)}\bar{M}(x) \sim \frac{1}{-i\det(I - {}^{\#}\tilde{G}_-)'(0)}(I - \tilde{G}_+(0))^{-1}adj(I - {}^{\#}\tilde{G}_-(0))We, \tag{3.8}$$

as $x \to \infty$. Now from the Wiener-Hopf identity (eq. (3.2)) we obtain

$$(I - \tilde{G}_+(\omega))^{-1}adj(I - {}^{\#}\tilde{G}_-(\omega))\det(I - \tilde{F}(\omega)) = \det(I - {}^{\#}\tilde{G}_-(\omega))adj(I - \tilde{F}(\omega)).$$

By taking the derivative in the previous equation, and noting that $\det(I - {}^{\#}\tilde{G}_-(0)) = \det(I - \tilde{F}(0)) = 0$, we get

$$\frac{1}{\det(I - {}^{\#}\tilde{G}_-)'(0)}(I - \tilde{G}_+(0))^{-1}adj(I - {}^{\#}\tilde{G}_-(0)) = \frac{1}{\det(I - \tilde{F})'(0)}adj(I - \tilde{F}(0)). \tag{3.9}$$

Replacing (3.9) into (3.8) we obtain

$$\frac{1}{\bar{H}(x)}\bar{M}(x) \sim \frac{1}{-i\det(I-\tilde{F})'(0)} adj(I-\tilde{F}(0))We, \tag{3.10}$$

as $x \to \infty$. Finally, by using Lemma 8, 9 of Appendix B we get $-i\det(I-\tilde{F})'(0) = -\mathbb{E}X_n D_{||F||}$, and $adj(I-\tilde{F}(0)) = D_{||F||}e\pi$; replacing these in (3.10) we obtain (3.6).

For (3.7) we observe that $\mathbb{P}[M > x] = \sum_{i\in E}\pi_i\mathbb{P}_i[M > x] \sim 1/(-\mathbb{E}X_n)\pi e\pi We\bar{H}(x) = 1/(-\mathbb{E}X_n)\pi We\bar{H}(x)$, as $x \to \infty$. This proves the first part of (3.7). The second asymptotic relation follows from $\int_x^\infty \mathbb{P}[X_1 > u]du = \sum_{i,j\in E}\pi_i\int_x^\infty \mathbb{P}[X_1 > u, J_1 = j|J_0 = i] \sim \sum_{i,j\in E}\pi_i W_{ij}\hat{H}(x) = \pi We\hat{H}(x)$, as $x \to \infty$. This finishes the proof of the theorem. $\diamond$

4. Markov-modulated G/G/1 Queue

In this section we will apply our main result, Theorem 4, to derive the asymptotic behavior of the waiting times of a Markov-modulated G/G/1 queue (Theorem 6). This is done through the classical connection between the extreme value theory and the queue waiting time distribution.

4.1 Motivation

Practical motivation for the queueing investigation in this and the following section originates in the problem of statistical multiplexing in modern broadband networks. The idea of multiplexing is to increase the network utilization by efficient sharing of network bandwidth and buffer resources. Main entities that are multiplexed are calls established from various sources. Each of these calls has some quality of service requirements that have to be satisfied in order for a call to operate properly. Quality of service requirements are usually expressed as bounds on performance measures associated with the multiplexer. The most basic model of statistical multiplexing is an infinite buffer discrete time single server queue. The fundamental performance measure is the queue length distribution ($\mathbb{P}[Q > x]$).

Under a variety of assumptions of Cramér type (exponentially bounded marginals and autocorrelation function of the arrival processes) many published results in the literature have shown that the queue length distribution of a network multiplexer has exponential asymptotics, i.e., $\mathbb{P}[Q > x] \sim \alpha e^{-\theta^* x}$ as $x \to \infty$. Some authors have argued that an even simpler approximation holds $\mathbb{P}[Q > x] \sim e^{-\theta^* x}$. This has led to the development of the so called effective bandwidth based admission control (for an extensive list of references on this topic see [GLW94, EHL95]).

However, recent statistical observations presented in [GAW94, JLS95b] show that the (marginal) distribution and the autocorrelation function of arrival processes that appear in communications networks may have a long (subexponential) tail. For such processes the Cramér type conditions are not satisfied. Motivated with this statistical findings, in this paper we further advance the calculus for approximating the queue length distribution of a single server queue with subexponential characteristics.

To get some intuition about the behavior of the queue when the arrival process has a subexponential (long-tailed) marginal distribution function let us examine the following example. (All the examples in this paper are calculated using the z-transform technique and Mathematica 2.2.)

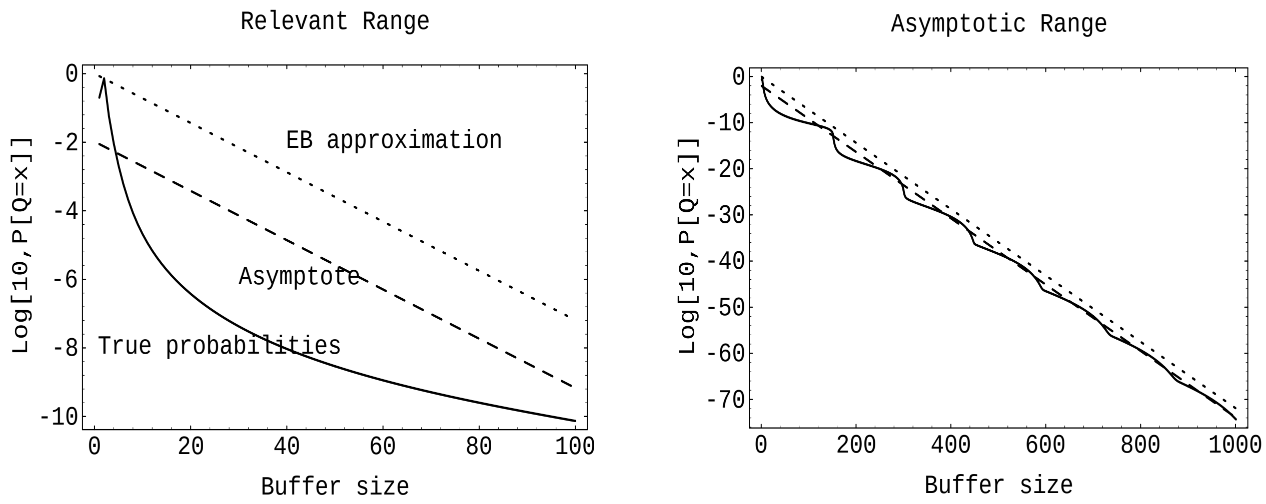

Figure 1: Illustration for Example 1.

Example 1 Consider a discrete time queue (whose evolution is given by Lindley's equation

(4.1)), with a service rate of one packet per slot ($C_t = 1$), and an arrival process characterized by a sequence A_t of i.i.d. random variables distributed as $\mathbb{P}[A_0 = 0] = 0.2, \mathbb{P}[A_0 = i] = d/i^6, 1 \leq i \leq 150$, $d = 0.77151$. Thus, this source (arrival process) has a truncated heavy tailed marginal distribution with peak rate of 150 packets. Since this process is bounded from above, its cumulant function $\mathbb{E}e^{\theta A_0}$ exists for all $\theta > 0$ and, therefore, the queue tail is asymptotically exponential. However, the range of the exponential asymptotic may be far out of the relevant range (for communication networks applications) of probabilities. On the right-hand side of Figure 1 we can see that the exponential asymptotics starts to work for very small probabilities (roughly smaller than 10^{-40}). However, in the relevant range of probabilities ($10^{-4} - 10^{-10}$) we see, on the left-hand side of Figure 1, that the exponential approximation fails. In this region, the queue length probabilities have a functional form approximately proportional to the integrated tail of A_n ($1/i^5$).

Thus, we can see that *even in the case of bounded heavy tailed arrivals*, for which the queue length asymptotics is eventually exponential, the relevant part of the queue length distribution may be subexponential. Consequently, in the rest of the paper we will examine the effect of the subexponential arrival process characteristics on the asymptotic queueing behavior.

4.2 Discrete Time Queue

Here, we define a discrete time single server queue process (or equivalently the waiting time process of a G/G/1 queue). Let $\{A_t, C_t, t \in \mathbb{N}_0\}$ be a sequence of random variables (on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$). Then, for any initial random variable Q_0 the following (Lindley's) equation

$$Q_{t+1} = (Q_t + A_t - C_t)^+ \tag{4.1}$$

defines the *discrete time queue length process* $\{Q_t, t \geq 0\}$. Throughout this paper we assume that $\{A_t, C_t, t \in \mathbb{N}_0\}$ is stationary and ergodic, and that $\mathbb{E}A_t < \mathbb{E}C_t$ (stability condition).

Then, according to the classical result by [LOY62] there exists a unique stationary solution to recursion (4.1), and for all initial conditions the queue length process converges (in finite time) to this stationary process. In this paper we assume that the queue is in its stationary regime, i.e., that $\{Q_t, t \geq 0\}$ is the stationary solution to (4.1). The dynamics of a broadband network multiplexer are modeled by the previous recursion. Q_t represents the workload at the end of the time slot t, A_t represents the amount of traffic (packets) that arrives at the multiplexer, and C_t represents the amount of traffic that is served during the slot t.

Note: Recursion (4.1) also represents the waiting time process of the $G/G/1$ queue with C_t being interpreted as the interarrival time between the customer t and $t+1$, A_t as the customer's t service requirement, and Q_t as the customer's t waiting time. For that reason terms *waiting time distribution* for the G/G/1 queue and the *queue length distribution* for the discrete time queue will be used interchangeably in this paper.

Let $X_t = A_t - C_t, t \geq 0$ be a *queue increment* process. Assume X_t is a sequence of i.i.d. random variables with distribution function F, and A_t independent of C_t. Then the following result on the waiting time distribution asymptotics of the $GI/GI/1$ queue holds (see Veraverbeke [VER77]). Let K be the distribution function of A_t.

Theorem 5 **(i)** *$F_1 \in \mathcal{S} \Longleftrightarrow K_1 \in \mathcal{S}$ and $\lim_{x\to\infty} \frac{\hat{F}(x)}{\hat{K}(x)} = 1$.*

(ii) *If $K \in \mathcal{L}$ and $K_1 \in \mathcal{S}$ (or $K \in \mathcal{S}^*$), then*

$$\mathbb{P}[Q_t > x] \sim \frac{1}{\mathbb{E}C_t - \mathbb{E}A_t} \int_x^\infty \mathbb{P}[A_t > u]du, \quad \text{as} \quad x \to \infty. \tag{4.2}$$

This theorem was first proved by [PAK75]. In [VER77] the same result was shown using a random walk technique, a technique also adopted in this paper. Some of the first applications of long-tailed distributions in queueing theory were done by Cohen [COH73], and Borovkov [BOR76] for the functions of regular variations [KAR30, BGT87]. Recent results on long-tailed and subexponential asymptotics of a $GI/GI/1$ queue are given in [ACW94, WIT92]. (Also, in [ACW94] further motivation is given for the application of long-tailed distributions to communication networks.)

In the following section we will generalize the result of Theorem 5 to the Markov-modulated case.

4.3 Subexponential Queue Asymptotics

A simple iteration of Lindley's equation (4.1) gives

$$Q_t = \sup_{n\geq 0} S_n^{r,t},$$

where $S_n^{r,t} \stackrel{\text{def}}{=} \sum_{i=0}^{n} X_{t-i}$; by stationarity it follows that $\mathbb{P}[Q_t > x | J_t = i] = \mathbb{P}[\sup_{n\geq 0} S_n^{r,0} | J_0 = i]$, $i \in E$. Therefore, investigating the stationary queue length distribution is equivalent to investigating of the associated reversed random walk $S_n^r \equiv S_n^{r,0}$. It becomes clear that by using the already obtained results on the supremum of random walk the desired queueing results can be readily derived.

More formally, let (J_t, A_t) and (J_t, C_t) be two MMRWs such that A_t and C_t are conditionally independent given J_{t-1}, J_t; $\{A_t\}$ and $\{C_t\}$ are arrival and service processes, respectively. Let K and D be the corresponding transition measures for these MMRWs, i.e., $K = \{K_{ij}\} = \{\mathbb{P}[A_1 \in B, J_1 = j | J_0 = i]\}$, and $D = \{D_{ij}\} = \{\mathbb{P}[C_1 \in B, J_1 = j | J_0 = i]\}$; the reversed transition measure for the arrival process is $K^r = \{K^r_{ij}\} = \{\mathbb{P}[A_1 \in B, J_0 = j | J_1 = i]\}$, $B \in \mathcal{B}(\mathbb{R})$. Then the following theorem holds.

Theorem 6 *Let* $\lim_{x\to\infty} \overline{K^r}(x)/\bar{H}(x) = W$, *as* $x \to \infty$, $W = \{W_{ij}\}, W_{ij} \in [0, \infty)$, $H(x) \in \mathcal{L}, H_1(x) \in \mathcal{S}$ *(or* $H \in S^*$*), with at least one* $W_{ij} > 0$. *If* $\mathbb{E}C_t > \mathbb{E}A_t$, *and* P *(*$= \|K\| = \|D\|$*) is irreducible and aperiodic, then,*

$$\frac{1}{\hat{H}(x)}\bar{Q}(x) \to \frac{1}{\mathbb{E}C_t - \mathbb{E}A_t} e\pi W e, \quad \text{as} \quad x \to \infty, \tag{4.3}$$

where $\bar{Q}(x)$ *is a column vector with its ith component equal to* $\mathbb{P}_i[Q_t > x]$. *In particular*

$$\boxed{\mathbb{P}[Q_t > x] \sim \frac{1}{\mathbb{E}C_t - \mathbb{E}A_t} \int_x^\infty \mathbb{P}[A_t > u]du, \quad \text{as} \quad x \to \infty.} \tag{4.4}$$

Note: Surprisingly enough, we see that the asymptotic behavior of the queue is not affected at all by the Markovian dependency structure, i.e., it is structurally the same as the $GI/GI/1$ queue asymptotics.

Proof: Component-wise the asymptotic proportionality of the tails of the matrix distributions $F^r(x)$ and $K^r(x)$ follows from Lemma 4. The conclusion of the theorem follows from Theorem 4. $\diamond$

An illustration of the preceding theorem is given in the following numerical example.

Example 2 Consider a constant server queue with $C_t = 1$ and two state (e.g. $\{0,1\}$) Markov-modulated arrivals (source). The transition probabilities for the modulating Markov chain are $p_{01} = 1/3, p_{10} = 3/4$. When in state 0 the source is producing zero arrivals, and when in state 1 the source is producing (independently of the previous state) arrivals according to the distribution $\mathbb{P}[A_t = 0|J_t = 1] = 0.327144, \mathbb{P}[A_t = 1|J_t = 1] = 0$, and $\mathbb{P}[A_t = i|J_t = 1] = w/i^5, w = 18.220859, 2 \leq i \leq 350$; $\rho_1 = \mathbb{E}[A_t|J_t = 1] = 3/2$. (Note that these are bounded arrivals.) Thus, according to the previous theorem, the queue length distribution is proportional to $1/i^4$, and the constant of proportionality is easily calculated to be $c = w\pi_1/(4(1-\rho_1\pi_1)) = w/7 = 2.603$. The comparison between the true probabilities and the approximation c/i^4 is shown in Figure 2.

5. Asymptotics of a Fluid Flow Queue with Subexponentially Correlated Arrivals

In this section we construct a class of processes for which we show that its autocorrelation function (acf) is subexponential. Furthermore, when these processes are fed to a fluid flow queue, we prove the asymptotic proportionality of the queue length distribution with the

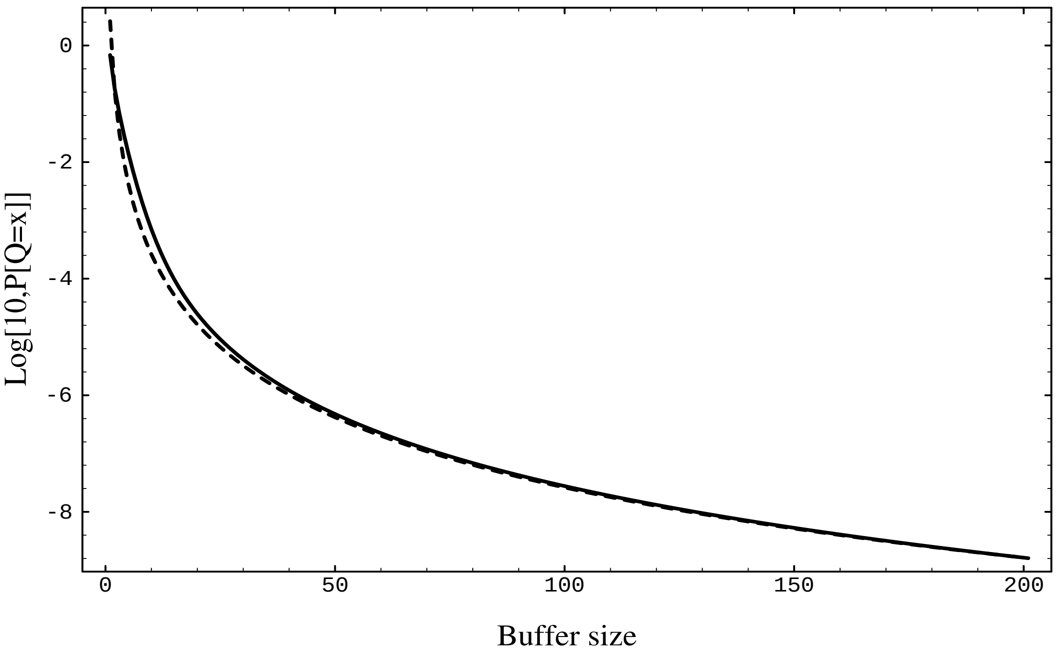

Figure 2: Graph of $\log_{10} \mathbb{P}[Q = i]$ versus buffer size i from Example 2; the solid line represents the true probabilities, and the dashed line represents the approximation $2.603/i^{-4}$.

arrival process acf. Throughout this section we assume a continuous time model (of course all the results are valid for discrete time also).

5.1 Stationary Subexponentially Correlated Arrivals

Consider a point process $T = \{T_0 \leq 0, T_n, n \geq 1\}$ such that $T_n - T_{n-1}, n \geq 1$ are i.i.d. with subexponential distribution function F. Further, let $J_n, n \geq 0$ be an irreducible aperiodic Markov Chain with finite state space $\{1, \dots, K\}$, transition matrix $\{P_{ij}\}$, and stationary probability distribution $\pi_i, 1 \leq i \leq K$. In order to make this point process stationary (see [CIN75], section 9.3), we choose the residual time at zero until the first jump to be distributed as an integrated tail of F, i.e., $F_1(t) = \mathbb{P}[T_1 \leq t] = {m_F}^{-1} \int_{0,t} \bar{F}(u)du$, $m_F = \mathbb{E}(T_n - T_{n-1})$.

Now we construct a process A_t which takes values in in $\{a_i \geq 0 \ : \ 1 \leq i \leq K\}$ and whose dynamics is described as:

$$A_t = a_{J_n} \quad \text{for} \quad T_n \leq t < T_{n+1}, \tag{5.1}$$

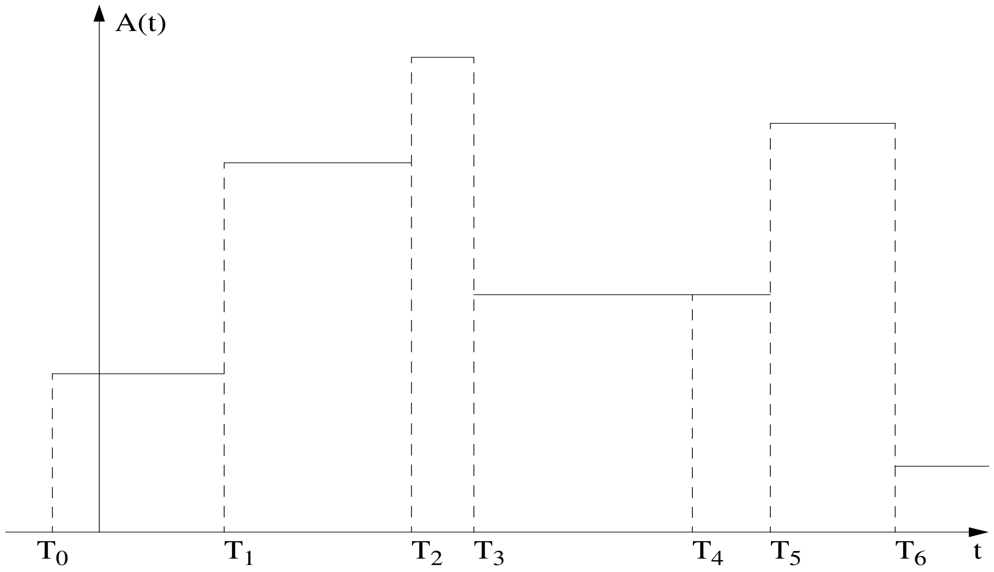

Figure 3: A possible realization of a Markov chain embedded into a renewal process.

called a Markov Chain Embedded in a Stationary Subexponential Renewal Process (MCESSR). A typical sample path of this process is given in Figure 3. It is well known that under fairly general conditions, a Markov chain converges exponentially fast to the steady state distribution. However, the process that we have constructed, because of the subexponentially distributed sojourn times, converges with subexponential speed to its steady state. This is stated in the following proposition. (Note that $\mathbb{P}[A_t = a_j] = \mathbb{P}[J_t = j] = \pi_j$.)

Proposition 2 *If $F, F_1 \in \mathcal{S}$, then*

$$(\mathbb{P}_i[A_t = a_j] - \pi_j)\bar{F}_1(t)^{-1} \to (\delta_{ij} - \pi_j),$$

as $t \to \infty$, $\delta_{ij} = 1$ if $i = j$ and zero otherwise.

In the proof of the above result we will use the following simple lemma.

Lemma 7 *If $F \in \mathcal{L}$, then $\bar{F}(t) = o(\bar{F}_1(t))$ as $t \to \infty$.*

Proof: Observe that for any $a > 0$

$$\frac{\bar{F}(t)}{\int_t^\infty \bar{F}(u)du} \leq \frac{\bar{F}(t)}{\int_t^{t+a} \bar{F}(u)du} \leq \frac{\bar{F}(t)}{a\bar{F}(t+a)}.$$

From this inequality, since $F \in \mathcal{L}$, we obtain that $\limsup_{t\to\infty} \bar{F}(t)/\bar{F}_1(t) \le m_F/a$, (recall $m_F = \int_0^\infty \bar{F}(u)du$, $F_1(t) = 1/m_F \int_t^\infty \bar{F}(u)du$); now, by letting $a \to \infty$ we finish the proof of the lemma. $\diamond$

Proof of Proposition 2: Observe that

$$\mathbb{P}_i[A_t = a_j] - \pi_j = \mathbb{P}[T_1 > t](\delta_{ij} - \pi_j) + \sum_{n=1}^{\infty}(p_{ij}^{(n)} - \pi_j)\mathbb{P}[T_n \le t < T_{n+1}]. \tag{5.2}$$

Thus, if the sum on the right-hand side of the above equation is $o(\bar{F}_1(t))$ as $t \to \infty$, the proposition will follow. For any $n > 0$, $\mathbb{P}[T_n > t] \sim \bar{F}_1(t)$ as $t \to \infty$; this follows from Lemmas 7 and 10 (a) (Appendix C). Since $\mathbb{P}[T_n \le t < T_{n+1}] = \mathbb{P}[T_{n+1} > t] - \mathbb{P}[T_n > t]$, it follows that $\mathbb{P}[T_n \le t < T_{n+1}] = o(\bar{F}_1(t))$ as $t \to \infty$. Therefore, if the limit (with respect to t) and summation can be interchanged

$$\sum_{n=1}^{\infty}(p_{ij}^{(n)} - \pi_j)\mathbb{P}[T_n \le t < T_{n+1}] = o(\bar{F}_1(t)),$$

as $t \to \infty$. The justification for doing so follows from Lemma 10 (b), the fact that $|(p_{ij}^{(n)} - \pi_j)| \le a\rho^n$, for some $a \ge 0, 0 \le \rho < 1$, and the dominated convergence theorem. This finishes the proof of proposition. $\diamond$

We will illustrate this lemma by the following example.

Example 3 Let F be a discrete distribution function with support $[1, 1000]$, $\mathbb{P}[T_2 - T_1 = 1] = 0.186532$, and $\mathbb{P}[T_2 - T_1 = i] = w/i^5, w = 22.028625$, $2 \le i \le 1000$; choose a two state Markov chain with transition probabilities $p_{01} = 1/3$ and $p_{10} = 3/4$. Then, the functions $(d_{i,1}(t) \stackrel{\text{def}}{=} (\mathbb{P}_i[A_t = a_1] - \pi_1)(\bar{F}_1(t)(\delta_{i1} - \pi_1))^{-1}$, $i = 0, 1$, converge to one as $t \to \infty$, with subexponential rate. This can be clearly seen in Figure 4.

Now it is easy to prove that the autocorrelation function $\rho(t) \stackrel{\text{def}}{=} (\mathbb{E}A_0 A_t - (\mathbb{E}A_0))^2/\text{Var}(A_0)$ of the MCESSR process satisfies the following asymptotic relation. Let $\text{Var}(A_0) > 0$ be the variance of A_0.

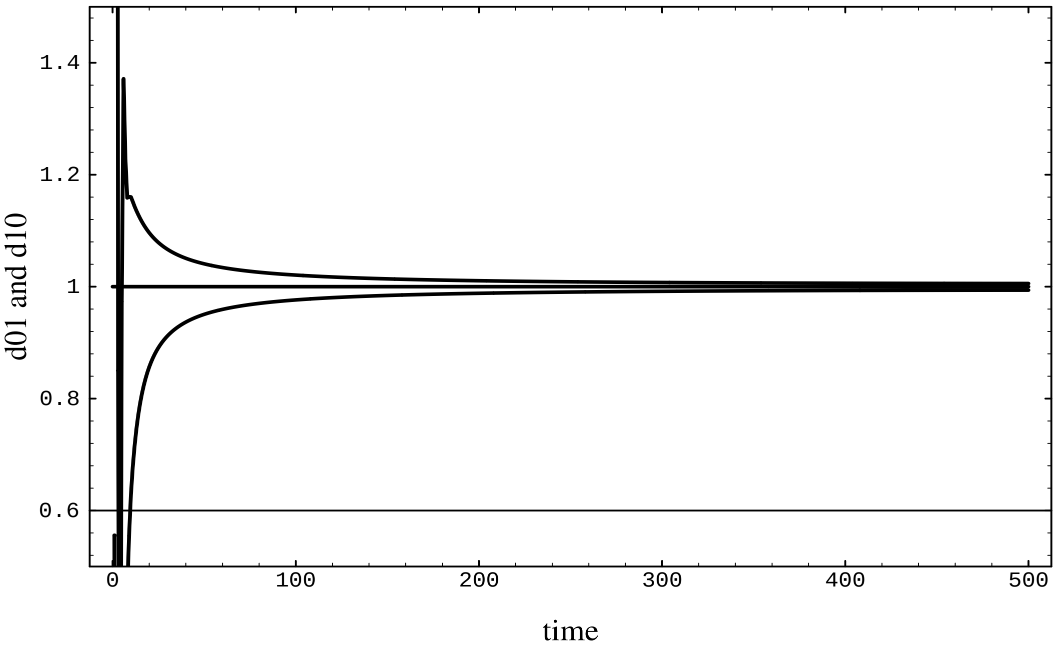

Figure 4: Functions $d_{i,1}(t) \stackrel{\text{def}}{=} (\mathbb{P}_i[A_t = a_1] - \pi_1)(\bar{F}_1(t)(\delta_{i1} - \pi_1))^{-1}$, $i = 0, 1$. The graph shows that $d_{i,1}(t) \to 1$ as $t \to 1$.

Theorem 7 *If $F, F_1 \in \mathcal{S}$, then*

$$\rho(t) \sim \bar{F}_1(t),$$

as $t \to \infty$.

Proof: By applying the previous lemma and after some simple algebraic manipulations we obtain

$$\begin{aligned}
&\left(\mathbb{E}A_0 A_t - (\mathbb{E}A_0)^2\right) \bar{F}_1(t)^{-1} \\
&= \sum_{i,j} a_i a_j (\pi_i \mathbb{P}_i[A_t = a_j] - \pi_i \pi_j) \bar{F}_1(t)^{-1} \\
&\sim \sum_{i,j} a_i a_j \pi_i (\delta_{ij} - \pi_j) \\
&= \sum_i \pi_i a_i^2 - \sum_{i,j} \pi_i \pi_j a_i a_j = \mathrm{Var}(A_0),
\end{aligned}$$

as $t \to \infty$. This completes the proof of the theorem. $\diamond$

5.2 Subexponential Asymptotics of a Fluid Flow Queue

Let us investigate the queue length distribution of a fluid queue fed with a MCESSR process. We assume that both the arrival process A_t and the server process C_t are MCESSR processes embedded into the same renewal process $\{T_n\}$, such that when the Markov chain J_n is equal to i, $A_t = a_i \geq 0$ and $C_t = c_i \geq 0$, $T_n \leq t < T_{n+1}$. Intuitively the pair A_t, C_t represents a fluid queueing model in which $A_t = a_i$ means that the flow is arriving to the queue with rate a_i, and $C_t = c_i$ means that the flow is departing from the queue with rate c_i. We will calculate the queue length distribution at the jump times (Palm probability); $Q_n \equiv Q(T_n)$ satisfies the recursion

$$Q_{n+1} = (Q_n + x_{J_n} \Delta T_n)^+,$$

where $x_i = a_i - c_i$, and $\Delta T_n = T_{n+1} - T_n$. (Again, we assume that Q_n is the stationary solution to the recursion above.)

We are now ready to state the following result on the asymptotic proportionality between the queue length distribution and its autocorrelation function. To avoid trivial cases we assume that at least for one i, $x_i > 0$; also $Var(A_0) > 0$.

Theorem 8 *Let the stability condition $\mathbb{E}x_{J_n} = \sum_i \pi_i x_i < 0$ be satisfied, and for all $x_i > 0$, $\mathbb{P}[\Delta T_n > t/x_i]/\bar{F}(t) \to w_i$, as $t \to \infty$, with at least one $w_i > 0$, and $F, F_1 \in \mathcal{S}$. Then, there exist a positive constant r such that*

$$\mathbb{P}[Q_n > t] \sim r\ \rho(t),$$

where $\rho(t)$ is the autocorrelation function of the arrival process A_t (same as in Theorem 7).

Proof: It follows by straightforward combination of Theorems 6, 7. $\diamond$

Remarks: **(i)** If the distribution function of ΔT_n belongs to the Pareto family the assumption $\mathbb{P}[\Delta T_n > t/x_i] \sim w_i \bar{F}(t), w_i > 0$ will be satisfied for all $x_i > 0$. **(ii)** To the best of our knowledge this is the first rigorous result of this kind and generality. Also, with an appropriate extension of Theorem 7 this result can be extended to the general class of *subexponential semi-Markov* arrivals.

6. Conclusion

For a Markov-modulated random walk with long-tailed right tail the ascending ladder height matrix distribution is asymptotically proportional to a long-tailed distribution. When this random walk has a negative drift and a subexponential right tail its asymptotics is the **same** as in the corresponding i.i.d. case. This result has a variety of applications, e.g. in Queueing Theory and Insurance Risk Theory.

In the queueing context the application of the random walk result showed that the queue length (waiting time) distribution of a Markov-modulated discrete time (G/G/1) queue has structurally the *same* asymptotics as the i.i.d. discrete time (GI/GI/1) queue. Furthermore, we constructed a general class of processes, for which the autocorrelation (covariance) function has a subexponential tail. When these processes are fed into a fluid flow queue, the queue length distribution is asymptotically *proportional to its autocorrelation function.*

Informally, the queueing results derived in this paper can be summarized as follows (m.d.f. denotes marginal distribution function)

- (subexp. m.d.f. + exp. acf) $\Rightarrow$ (queue distribution is determined by m.d.f.),
- (bounded (exp.) m.d.f. + subexp. acf) $\Rightarrow$ (queue distribution is determined by acf).

We expect that the above results may have an impact on the design of efficient broadband network admission control policies. When these types of conditions are met in practice, the admission controllers may decide their admission control policy based on either the marginal distributions or the autocorrelation functions of the arrival streams, depending on which conditions are satisfied.

Acknowledgments

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Appendix

A. Proof of the results from section 2.2

In this section we present the proofs of Lemmas 4, 5, and Proposition 1.

Proof of Lemma 4: Let $c > 0$; then for every $0 < \epsilon < c$ we can choose an x_0 such that for all $x \geq x_0$ we have $(c-\epsilon)\bar{H}(x) \leq \mu([x,\infty)) \leq (c+\epsilon)\bar{H}(x)$. Let's also assume that μ_- is a positive measure, then

$$\nu([x,\infty)) = \int_{(-\infty,0]} \mu([x-y,\infty))\mu_-(dy) \leq (c+\epsilon)\mu_-((-\infty,0])\bar{H}(x).$$

This implies that, $\limsup_{x\to\infty} \nu([x,\infty))/\bar{H}(x) \leq c\mu_-((-\infty,0])$. For the lower bound we have for any $z > 0$

$$\nu([x,\infty)) \geq (c-\epsilon)\int_{(-z,0]} \bar{H}(x-y)\mu_-(dy) \geq (c-\epsilon)\bar{H}(x+z)\mu_-((-z,0]).$$

From this, and $H \in \mathcal{L}$ it follows that

$$\liminf_{x\to\infty} \nu([x,\infty))/\bar{H}(x) = \liminf_{x\to\infty} \nu([x,\infty))/\bar{H}(x+z) \geq (c-\epsilon)\mu_-((-z,0]);$$

Now, by passing $z \to \infty$ and $\epsilon \to 0$ we get $\liminf_{x\to\infty} \nu([x,\infty))/\bar{H}(x) \geq c\mu_-((-\infty,0])$. This proves the case $c > 0$ and μ_- being positive measure. For $c < 0$ (μ_- positive measure) we have $(-\nu) = (-\mu) * \mu_-$, $(-\mu) \sim |c|\bar{H}(x)$ and the same arguments apply. In case μ_- is a signed measure we can represent it (by Hahn's decomposition theorem, [BIL86], pp. 441) as $\mu_- = \mu_-^+ - \mu_-^-$ where μ_-^+, μ_-^- are two positive measures. Therefore, by applying what we have proven, we obtain $\nu([x,\infty)) \sim (c\mu_-^+((-\infty,0]) - c\mu_-^-((-\infty,0]))\bar{H}(x) = c\mu_-((-\infty,0])\bar{H}(x)$.

It is left to consider the case $c = 0$. For all sufficiently large x, $-\epsilon\bar{H}(x) \leq \mu([x,\infty)) \leq \epsilon\bar{H}(x)$. From this, by using similar type of arguments, it easily follows that $\nu([x,\infty))/\bar{H}(x) \to 0$ as $x \to \infty$, and the proof of the lemma follows. $\diamond$

Proof of Lemma 5: Let again $\mu_- = \mu_-^+ - \mu_-^-$. Then, for any $x > 0$

$$\mu([x,\infty)) = \int_{(-\infty,0]} \mu_+([x-y,\infty))\mu_-^+(dy) - \int_{(-\infty,0]} \mu_+([x-y,\infty))\mu_-^-(dy).$$

From this it follows for any $z > 0$ and $x > a$

$$\begin{aligned}\mu([x,\infty)) &\leq \int_{(-\infty,0]} \mu_+([x-y,\infty))\mu_-^+(dy) - \int_{(-z,0]} \mu_+([x-y,\infty))\mu_-^-(dy) \\ &\leq \mu_+([x,\infty))\mu_-^+((-\infty,0]) - \mu_+([x+z,\infty))\mu_-^-((-z,0]).\end{aligned}$$

Let $c > 0$, then for any $0 < \epsilon < c$ there is an $x_0 > a$ such that for all $x > x_0$, $(c-\epsilon)\bar{H}(x) \leq \mu([x,\infty)) \leq (c+\epsilon)\bar{H}(x)$. Then, for $x > x_0$

$$1 \leq \frac{\mu_+([x,\infty))}{(c-\epsilon)\bar{H}(x)}\mu_-^+((-\infty,0]) - \frac{\mu_+([x+z,\infty))}{(c+\epsilon)\bar{H}(x)}\mu_-^-((-z,0]).$$

Also assume for the time being that both $\mu_-^+((-\infty,0]) > 0$, and $\mu_-^-((-\infty,0]) > 0$; then by taking the limit infimum we get

$$1 \leq \mu_-^+((-\infty,0])(c-\epsilon)^{-1}\liminf_{x\to\infty}\frac{\mu_+([x,\infty))}{\bar{H}(x)} - \mu_-^-((-z,0])(c+\epsilon)^{-1}\limsup_{x\to\infty}\frac{\mu_+([x+z,\infty))}{\bar{H}(x)}.$$

Since H is long-tailed we have $\limsup_{x\to\infty}\frac{\mu_+([x+z,\infty))}{\bar{H}(x)} = \limsup_{x\to\infty}\frac{\mu_+([x+z,\infty))}{\bar{H}(x+z)}\lim_{x\to\infty}\frac{\bar{H}(x+z)}{\bar{H}(x)} = \limsup_{x\to\infty}\frac{\mu_+([x,\infty))}{\bar{H}(x)}$; therefore,

$$1 \leq \mu_-^+((-\infty,0])(c-\epsilon)^{-1}\liminf_{x\to\infty}\frac{\mu_+([x,\infty))}{\bar{H}(x)} - \mu_-^-((-z,0])(c+\epsilon)^{-1}\limsup_{x\to\infty}\frac{\mu_+([x,\infty))}{\bar{H}(x)};$$

and finally by letting $z \to \infty$, and $\epsilon \to 0$ we obtain

$$1 \leq \mu_-^+((-\infty,0])c^{-1}\liminf_{x\to\infty}\frac{\mu_+([x,\infty))}{\bar{H}(x)} - \mu_-^-((-\infty,0])c^{-1}\limsup_{x\to\infty}\frac{\mu_+([x,\infty))}{\bar{H}(x)}. \tag{6.1}$$

Similarly, for $x > a$ we have

$$\mu([x,\infty)) \geq \mu_+([x+z,\infty))\mu_-^+([-z,0]) - \mu_+([x,\infty))\mu_-^-([-\infty,0]);$$

and, by using the same type of arguments as before, we arrive at

$$1 \geq \mu_-^+((-\infty,0])c^{-1}\limsup_{x\to\infty}\frac{\mu_+([x,\infty))}{\bar{H}(x)} - \mu_-^-((-\infty,0])c^{-1}\liminf_{x\to\infty}\frac{\mu_+([x,\infty))}{\bar{H}(x)}. \tag{6.2}$$

Finally, from inequalities (6.1), (6.2) we get

$$\liminf_{x\to\infty}\frac{\mu_+([x,\infty))}{\bar{H}(x)} = \limsup_{x\to\infty}\frac{\mu_+([x,\infty))}{\bar{H}(x)} = \lim_{x\to\infty}\frac{\mu_+([x,\infty))}{\bar{H}(x)} = \frac{c}{\mu_-((-\infty,0])}, \tag{6.3}$$

which is aymptotics claimed in the lemma.

If one of $\mu_-^+((-\infty,0]) = 0$ or $\mu_-^-((-\infty,0]) = 0$ we argue as follows. Say $\mu_-^-((-\infty,0]) = 0$, then

$$\mu([x,\infty)) \leq \mu_+([x,\infty))\mu_-^+((-\infty,0]),$$

which implies

$$1 \leq \mu_-^+((-\infty,0])(c-\epsilon)^{-1}\liminf_{x\to\infty}\frac{\mu_+([x,\infty))}{\bar{H}(x)},$$

and by passing $\epsilon \to 0$

$$1 \leq \mu_-^+((-\infty,0])c^{-1}\liminf_{x\to\infty}\frac{\mu_+([x,\infty))}{\bar{H}(x)}. \tag{6.4}$$

Similarly

$$\mu([x,\infty)) \geq \mu_+([x,\infty))\mu_-^+((-z,0]),$$

which implies

$$1 \geq \mu_-^+((-z,0])(c+\epsilon)^{-1}\limsup_{x\to\infty}\frac{\mu_+([x,\infty))}{\bar{H}(x)};$$

and by letting $z \to \infty$, and $\epsilon \to 0$

$$1 \geq \mu_-^+((-\infty, 0])c^{-1} \limsup_{x\to\infty} \frac{\mu_+([x,\infty))}{\bar{H}(x)}. \tag{6.5}$$

From inequalities (6.4) and (6.5), (6.3) follows. We argue similarly if $\mu_-^+([-\infty, 0)) = 0$. For $c < 0$ we have $(-\mu) = (-\mu_-) * \mu_+$, with $(-\mu)([x,\infty)) \sim |c|\bar{H}(x)$, as $x \to \infty$, and the result immediately follows. The case $c = 0$ can be analyzed in the similar fashion. This finishes the proof. ◇

Proof of Proposition 1: Let $\mu^1([x,\infty)) \stackrel{\text{def}}{=} \int_x^\infty \mu([u,\infty))du$, and $\mu_-^1([z,0]) \stackrel{\text{def}}{=} \int_{[z,0]} \mu_-([u,0])du$. Observe that $\mu^1([x,\infty)) \sim c\hat{Y}(x)$ as $x \to \infty$. Also, from the assumptions it follows that μ_-^1 defines a finite measure on $(-\infty, 0]$, since $\mu_-^1((-\infty,0]) = \int_{-\infty}^0 -\mu_-((-\infty,u))du = \int_{(-\infty,0]} u\mu_-(du)$. Then, by applying Fubini's theorem (see [CHT88], pp. 180), we get

$$\begin{aligned}
\mu^1([y,\infty)) &= \int_y^\infty du \int_{[0,\infty)} \mu_-([u-x,0])\mu_+(dx) \\
&= \int_{[0,\infty)} \mu_+(dx) \int_y^x \mu_-([u-x,0])du \\
&= \int_{[0,\infty)} \mu_-^1([y-x,0])\mu_+(dx).
\end{aligned}$$

Thus, $\mu_1([y,\infty))$ is obtained through the convolution of the finite measures μ_+, μ_-^1. Therefore, by applying Lemma 5, the conclusion of the proposition follows (recall that $\mu_-^1((-\infty,0]) = \int_{(-\infty,0]} u\mu_-(du)$).

◇

B.

Consider an irreducible aperiodic probability transition matrix $P = \{p_{ij}\}$ with a stationary probability distribution $\pi = (\pi_1, \ldots, \pi_n)$ (row vector). Then, the following lemma holds.

Lemma 8

$$adj(I-P) = D_P e\pi,$$

where e is a $(n\times 1)$ column vector of ones, and $D_P \stackrel{\text{def}}{=} \lim_{\alpha\uparrow 1}\det(I-\alpha P)/(1-\alpha)(>0)$.

Proof: For any $0<\alpha<1$

$$\begin{aligned} & adj(I-\alpha P) \\ &= \det(I-\alpha P)\sum_{k=0}^{\infty}\alpha^k P^k \\ &= \frac{\det(I-\alpha P)}{1-\alpha}(1-\alpha)\sum_{k=0}^{\infty}\alpha^k P^k. \end{aligned}$$

Thus, $\frac{\det(I-\alpha P)}{1-\alpha}$ converges to a positive limit as $\alpha\uparrow 1$ (we call it D_P) by Perron-Frobenius theorem, and $(1-\alpha)\sum_{k=0}^{\infty}\alpha^k P^k$ converges to $e\pi$ as $\alpha\uparrow 1$, since $P^k\to e\pi$ as $k\to\infty$ (Tauberian theorem, [CHU60], pp. 52, Theorem 2). $\diamond$

Let P be as in Lemma 8 and let us define $F=\{p_{ij}f_{ij}\}$, where f_{ij} are probability measures on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ such that $\int_{-\infty,\infty}|u|f_{ij}(du)<\infty$ for all i,j; denote with $\rho_{ij}=\int_{-\infty,\infty}uf_{ij}(du)$. If $\tilde{F}=\{p_{ij}\tilde{f}_{ij}\}$ is its Fourier transform then the following result holds.

Lemma 9

$$i\det(I-\tilde{F})'(0) = D_P\sum_{i,j}\pi_i p_{ij}\rho_{ij},$$

where D_P is as in Lemma 8.

Proof: From the identity $(I-\tilde{F}(\omega))adj(I-\tilde{F}(\omega)) = \det(I-\tilde{F}(\omega))$ it follows

$$-i\tilde{F}'(0)adj(I-P) + i(I-P)adj(I-\tilde{F})'(0) = i\det(I-\tilde{F})'(0).$$

By multiplying the previous equation from the left with π (row vector) and observing that $-i\tilde{F}'(0) = \{p_{ij}\rho_{ij}\}$ we arrive at

$$\pi\{p_{ij}\rho_{ij}\}adj(I-P) = \pi(i\det(I-\tilde{F})'(0)).$$

At last, by multiplying the above equation on the right with a column vector of ones e, and applying Lemma 8 the result follows. $\diamond$

C.

The following lemma is an improved version of Lemma 4.2, [AHK94].

Lemma 10 *Let $H, H_1, \ldots, H_m$ be probability distributions such that $\bar{H}_j(x)/\bar{H}(x) \sim c_j \in [0,\infty)$ as $u \to \infty$, $j = 1, \ldots, m$. If $H \in \mathcal{S}$ then*

a) *For all $k_1, \ldots, k_m \in \mathbb{N}$,*

$$\overline{H_1^{*k_1} * \cdots * H_m^{*k_m}}(x)/\bar{H}(x) \to \sum_{j=1}^{m} k_j c_j, \quad x \to \infty;$$

moreover, if $\max_{1\le j\le m} c_j > 0$, then

$$H_1^{*k_1} * \cdots * H_m^{*k_m} \in \mathcal{S}.$$

b) *For each $\epsilon > 0$ there exists some $K_\epsilon < \infty$ such that*

$$\overline{H_1^{*k_1} * \cdots * H_m^{*k_m}}(x)/\bar{H}(x) \le K_\epsilon(1+\epsilon)^{k_1+\cdots+k_m}\bar{H}(x)$$

for all $x \ge 0$ and $k_1, \ldots, k_m$.

Note: This lemma has been proven in [AHK94] under the conditions that all $c_j > 0$.

Proof: First part of **a)** follows from Theorem 1, [CLI86], and Lemma 3 **(i)**; second part of **a)** follows from Lemma 3 **(ii)**.

When all $c_i > 0$, **b)** follows from [AHK94], Lemma 4.2. The case when some of the $c_i = 0$ we prove using the method of stochastic dominance. Without loss of generality we may assume that the first n coefficients $c_i = 0, 1 \leq i \leq n \leq m$; and that the rest are strictly positive. Then, for any $\delta > 0$, there exists an $a > 0$ such that for all $x > a$, $\bar{H}_i(x) \leq \delta \bar{H}(x), 1 \leq i \leq n$. We can define a distribution function

$$H'(x) = \begin{cases} 0 & 0 \leq x \leq a, \\ \delta H(x) & x > a. \end{cases}$$

This distribution function dominates $H_i, 1 \leq i \leq n$, i.e., $\bar{H}'(x) \geq \bar{H}_i(x), x \geq 0, 1 \leq i \leq n$. Assume that each distribution function $H_i, 1 \leq i \leq m$ is associated with a sequence of random variables $X_i^l, l \geq 1$. Then, by Strassen's theorem on stochastic dominance [BAB94], pp. 174, we can construct a sequence of independent random variables ${X'}_i^l, 1 \leq i \leq n$, such that they all have a common distribution H' and $X_j^l \leq {X'}_j^l, l \geq 1, 1 \leq j \leq n$ almost surely. Then, we obtain the following inequality

$$\begin{aligned} & \overline{H_1^{*k_1} * \cdots * H_m^{*k_m}}(x) \\ &= \mathbb{P}\left[\sum_{j=1}^{m}\sum_{l_j=1}^{k_j} X_j^{l_j} > x\right] \\ &\leq \mathbb{P}\left[\sum_{j=1}^{n}\sum_{l_j=1}^{k_j} {X'}_j^{l_j} + \sum_{j=n+1}^{m}\sum_{l_j=1}^{k_j} X_j^{l_j} > x\right] \\ &= \overline{H'^{*(k_1+\cdots+k_n)} * \cdots * H_m^{*k_m}}(x). \end{aligned} \tag{6.6}$$

Note that the tails of all the distributions in (6.6) are asymptotically proportional to $\bar{H}(x)$, and the conclusion follows from Lemma 4.2, [AHK94]. $\diamond$

Proof of Lemma 6: The proof is basically the same as in the proof of Lemma 4.3, [AHK94], when the usage of Lemma 4.2, [AHK94], is replaced by Lemma 10. $\diamond$

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