J. Appl. Prob. **41,** 691–702 (2004)
Printed in Israel

SPACE FILLING AND DEPLETION

YULIY BARYSHNIKOV,* *Bell Laboratories*
E. G. COFFMAN, JR ** *** AND
PREDRAG JELENKOVIĆ,** *Columbia University*

Abstract

For a given $k \geq 1$, subintervals of a given interval $[0, X]$ arrive at random and are accepted (allocated) so long as they overlap fewer than k subintervals already accepted. Subintervals not accepted are cleared, while accepted subintervals remain allocated for random retention times before they are released and made available to subsequent arrivals. Thus, the system operates as a generalized many-server queue under a loss protocol. We study a discretized version of this model that appears in reference theories for a number of applications, including communication networks, surface adsorption–desorption processes, and reservation systems. Our primary interest is in steady-state estimates of the vacant space, i.e. the total length of available subintervals $kX - \sum \ell_i$, where the ℓ_i are the lengths of the subintervals currently allocated. We obtain explicit results for $k = 1$ and for general k with all subinterval lengths equal to 2, the classical *dimer* case of chemical applications. Our focus is on the asymptotic regime of large retention times.

Keywords: Markov chain; many-server queue; dimer packing

2000 Mathematics Subject Classification: Primary 60J27; 60K35
Secondary 82B41

1. Introduction

The two subsections to follow define in abstract terms the Markov chain modeling space filling and depletion. The final subsection then maps this model into several important applications in the engineering and physical sciences.

1.1. Configuration space

Consider the interval $[0, X]$, X an integer, subdivided by the integers into *slots* of length 1. Call this interval the *span*, and fix an integer k, to be interpreted as a number of resource units or *channels*. An *interval* in this work is always composed of consecutive slots. A configuration $\boldsymbol{C}$ of intervals is simply a finite set of intervals in the span. For given k, a *packing* of the configuration $\boldsymbol{C}$ is a function on $\boldsymbol{C}$ with values in the set of k channels such that the intervals mapped into the same channel do not overlap; they can have common endpoints but no common interior points. We say that a configuration is *admissible* if there exists a packing of the configuration, i.e. if the intervals of the configuration can be placed over k channels with no overlapping.

Received 14 May 2001; revision received 19 January 2004.
* Postal address: Bell Laboratories, 2C-261, 600 Mountain Avenue, Murray Hill, NJ 07974, USA.
Supported by DIMACS.
** Postal address: Department of Electrical Engineering, Columbia University, 1312 Seeley W. Mudd, Mail Code 4712, 500 West 120th Street, New York, NY 10027, USA.
*** Email address: egc@ee.columbia.edu

The *counting function* of configuration $\boldsymbol{C}$ counts the number of intervals containing a given noninteger point in the span:

$$n_{\boldsymbol{C}}(x) = \#\{I \in \boldsymbol{C} : x \in I\}, \qquad x \in [0, X] \setminus \mathbb{N};$$

the counting function is not defined at integer points. Clearly, an admissible configuration $\boldsymbol{C}$ satisfies

$$n_{\boldsymbol{C}} \leq k \quad \text{on } [0, X] \setminus \mathbb{N}. \tag{1}$$

Analysis of a simple greedy algorithm, e.g. one that processes intervals in order of increasing left endpoint, shows that this condition is also sufficient.

Proposition 1. *If the condition (1) is satisfied, the configuration* $\boldsymbol{C}$ *is admissible.*

Denote the set of admissible configurations on the interval $[0, X]$ by $\mathcal{C}_X$. The asymmetry in parameters reflects the point of view here; specifically, for fixed k, the interest is in large X asymptotics.

1.2. Dynamics

Consider now the following Markov chain on the configuration space $\mathcal{C}_X$. Assume that, for any integer point i, intervals of length ℓ with left endpoint i arrive at rate λ_ℓ; the arrivals of intervals at different points and of different lengths are independent.

A newly arrived interval is included in the configuration if the resulting configuration is admissible; otherwise the interval is rejected. As there is (almost surely) at most one candidate, this definition of the filling process is unambiguous.

The depletion process, i.e. the departure of intervals from configurations, is independent of the arrival process, and has a similar description: the flow of 'killing' signals for intervals of length ℓ arrive at each integer i at rate μ_ℓ and form independent Poisson point processes. If at the time such a signal arrives there is at least one interval of length ℓ with its left endpoint at i in the configuration, then one of them leaves.

It is convenient to assume that the arrival rates λ_ℓ vanish for all but a finite number of lengths ℓ, say $\lambda_\ell > 0$ when $1 \leq \ell \leq L$ and $\lambda_\ell = 0$ otherwise.

1.3. Background

The above Markov chain finds diverse applications, although most of the known results are restricted to the pure filling model with $\mu_i = 0$ for all i. In Rényi's classical pure filling model [12], known variously as a model of the *car-parking problem* or *random sequential adsorption*, all subintervals have unit length and their left endpoints are drawn independently from the continuous uniform distribution on $[0, X - 1]$. Mackenzie [8] studied a discretized version, which is just our model with unit length subintervals and $\mu_1 = 0$.

In the physics and chemistry literature, we find applications to surface adsorption–desorption, granular densification, and polymer chain processes [4], [14]. Evans [4] gave further applications and several hundred references. For example, in a recent application Talbot *et al.* [13] investigated the kinetics of a microscopic model of hard rods on a linear substrate which are thought to be similar to certain processes of compacting granular materials. One of their more intriguing results is discussed at the end of Section 3 .

We note the special importance of the dimer case that we study in Section 4, a case that has received much attention over the years [4]. Indeed, the origin of the vast literature on sequential adsorption and related problems is commonly traced to the work of Flory [6] on a dimer problem in the 1930s. In prior work on models with desorption effects, Privman and Nielaba [11] analyzed a random dimer model in which adsorption is coupled with diffusional relaxation. See the review of Evans [4, p. 1320] for further details and additional references. Note that our model with general k provides, in a limited sense, the flexibility of a second dimension.

In his study of one-dimensional communication networks, Kelly [7] pointed out that the Markov chain models bus-connected local area networks. In this application, intervals correspond to the circuits connecting communicating parties and $[0, X]$ represents the bus. Kelly also mentioned similar applications in computer interconnection networks and database structures.

In operations research and engineering, applications to reservation systems have motivated considerable research, especially for the pure filling case [2], [3]. In such models, intervals are the times during which copies of a resource (e.g. hotel rooms, communication channels, etc.) are reserved. The span is then the reservation book, and departures model the cancellation of reservations.

To date, the best analytical results are those of Kelly, who studied a model very similar to the one defined here, and of Ziedins [16], who studied the continuous relaxation of Kelly's model. For refinements and related problems, see also [5] and [15]. Kelly's main results apply to the case $k = 1$ and to the case of general k with interval lengths governed by a geometric law. For $k = 1$, Kelly defined an alternating renewal process $\{\xi_j\}$ on the integers with values in $\{0, 1\}$; sequences of 1s between consecutive 0s represent accepted intervals. Explicit formulae were derived for computing the parameters of $\{\xi_j\}$. The probability distributions describing the finite process on $[0, X]$ were proved to be just those of $\{\xi_j,\ 0 \le j \le X\}$ conditioned on $\xi_{-1} = \xi_{X+1} = 0$. The vacant intervals of $\{\xi_j\}$, i.e. the strings of 0s separating consecutive 1s, were shown to have a geometric distribution.

The focus here is on space utilization, so our results add to the earlier theory in two principal ways. First, detailed calculations are worked out for expected vacant space when $k = 1$. Special consideration is given to small-μ asymptotics, for behavior in this limit is quite different from that seen in the 'jamming' limit (absorbing state) of Rényi's pure filling model, where $\mu = 0$; we briefly discuss this phenomenon at the end of Section 3. Second, the important dimer case of chemical applications [4], where all intervals have length 2, is worked out in detail.

As a final remark, we note that the approach of the analysis itself can be viewed in the context of a new trend that uses multivariate generating functions in probabilistic asymptotic problems; see e.g. [1], [10].

2. Analysis

Expected vacant space is calculated in general terms in this section. First, a generating function Z is calculated. The expected vacant space is then expressed in terms of a logarithmic derivative of Z and then studied by means of a singularity analysis.

2.1. Stationary distribution

Let $q_\ell = \lambda_\ell/\mu_\ell$, and note immediately that the Markov chain defined on $\mathcal{C}_X$ is reversible. Its stationary probabilities are given by

$$\pi(\boldsymbol{C}) = Z_X^{-1} \prod_{\ell : q_\ell > 0} q_\ell^{n_\ell(\boldsymbol{C})},$$

where $n_\ell(\boldsymbol{C})$ is the number of intervals of length ℓ in the configuration $\boldsymbol{C}$, and Z_X is the partition function (normalizing constant)

$$Z_X(\boldsymbol{q}) = \sum_{\boldsymbol{C}\in\mathcal{C}_X} \prod_{\ell:q_\ell>0} q_\ell^{n_\ell(\boldsymbol{C})} = \sum_{\boldsymbol{n}} N(\boldsymbol{n})\boldsymbol{q}^{\boldsymbol{n}}, \tag{2}$$

where $\boldsymbol{q} = (q_1, q_2, \ldots, q_L)$, $\boldsymbol{n}$ ranges over the L-tuples $(n_1, n_2, \ldots, n_L)$, $\boldsymbol{q}^{\boldsymbol{n}} = \prod_{\ell:q_\ell>0} q_\ell^{n_\ell(\boldsymbol{C})}$, and $N(\boldsymbol{n})$ is the number of configurations having n_1 intervals of length 1, n_2 intervals of length 2, $\ldots$, and n_L intervals of length L.

Let $u(\boldsymbol{C}) = \sum_\ell \ell n_\ell(\boldsymbol{C})$ be the total used space and $v(\boldsymbol{C}) = kX - u(\boldsymbol{C})$ the total vacant space in configuration $\boldsymbol{C}$, and extend the partition function to the following polynomial in a formal variable x:

$$Z_X(x; \boldsymbol{q}) = \sum_{\boldsymbol{C}\in\mathcal{C}_X} x^{v(\boldsymbol{C})} \prod_{\ell:q_\ell>0} q_\ell^{n_\ell(\boldsymbol{C})}. \tag{3}$$

In words, $Z_X(x, \boldsymbol{q})$ is the generating polynomial for the vacant space in admissible configurations on $[0, X]$.

To wrap up all the data uniformly, form the generating function for all spans X,

$$Z(x, y; \boldsymbol{q}) = \sum_X Z_X(x; \boldsymbol{q}) y^X.$$

Hereafter, we suppress the dependence on $\boldsymbol{q}$ in the generating function notation when there is no need for it.

2.2. Vacant space

We find that the average vacant space over admissible configurations in a span $[0, X]$ is given by

$$\langle v\rangle_X = Z_X^{-1} \left.\frac{\partial Z_X}{\partial x}\right|_{x=1},$$

where $Z_X = Z_X(x)$ is given by (3), and where the subscript in $\langle v\rangle_X$ denotes averaging over the stationary probabilities of the Markov chain on $\mathcal{C}_X$. To find the two terms on the right-hand side, we use the residue method. That is, we compute

$$Z_X|_{x=1} = \mathrm{Res}_0 y^{-X-1} Z(x, y; \boldsymbol{q})|_{x=1} \tag{4}$$

and

$$\left.\frac{\partial Z_X}{\partial x}\right|_{x=1} = \mathrm{Res}_0 y^{-X-1} \frac{\partial}{\partial x} Z(x, y; \boldsymbol{q})\Big|_{x=1}. \tag{5}$$

We are interested chiefly in the asymptotic analysis of the expected vacant space for large X. If the partition function is rational, $Z = P/Q$, then, in a standard way, we can use the residue theorem to pinpoint the leading terms of the asymptotics. Indeed, fix x and let $y_1(x), \ldots, y_n(x)$ be the roots of the polynomial Q (some of them can, of course, collide or escape to infinity as x varies). For $X \gg 1$, the contour integral of (4) or (5) over circles of large radius tends to zero, whence

$$Z_X|_{x=1} = -\sum_m \mathrm{Res}_{y_m} \frac{y^{-X-1}P}{Q} \tag{6}$$

and

$$\left.\frac{\partial Z_X}{\partial x}\right|_{x=1} = -\sum_m \mathrm{Res}_{y_m} y^{-X-1} \left.\frac{\partial}{\partial x}\frac{P}{Q}\right|_{x=1}. \tag{7}$$

It is readily seen that (6) and (7) are dominated by the residues at the roots closest to the origin.

Further, if $y_m(x)$ is a root of Q of multiplicity α, then the contribution of the corresponding residue to (6) is a polynomial in X of degree $m-1$ with the leading coefficient being

$$y_m^{-X-\alpha}\frac{P(y_m)}{\tilde{Q}(y_m)},$$

where we just write y_m for $y_m(1)$ and $\tilde{Q} = (1/m!)Q^{(m)}$.

Similarly, the leading coefficient for (7) is

$$y_m^{-X-\alpha-1}\left.\frac{\mathrm{d}}{\mathrm{d}x}y_m(x)\right|_{x=1}\frac{P(y_m)}{\tilde{Q}(y_m)}.$$

Therefore, we can deduce the following useful expression for $(\partial Z_X/\partial x)/Z_X$.

Proposition 2. *Assume that there is a single root y_m of Q with the least absolute value. Then the ratio $(\partial Z_X/\partial x)/Z_X$ is given by*

$$X y_m^{-1}\left.\frac{\mathrm{d}y_m}{\mathrm{d}x}\right|_{x=1}$$

up to an $O(X^0)$ term.

3. Single channel case

We consider here $k = 1$, in which case the partition function $Z(x, y; \boldsymbol{q})$ can be found easily.

3.1. Generating function

Proposition 3. *We have*

$$Z(x, y; \boldsymbol{q}) = \frac{1}{1 - yx - \sum_\ell q_\ell y^\ell}. \tag{8}$$

Proof. The proof is a standard exercise, as follows. For notational simplicity, let $q_\ell > 0$ for $\ell = 1, \ldots, L$ and $q_\ell = 0$ otherwise. From (2) and (3),

$$\begin{aligned} Z(x, y; \boldsymbol{q}) &= \sum_{v=0}^{\infty}\sum_{\boldsymbol{n}} N(n_1, n_2, \ldots, n_L, v) x^v y^{v+n_1+2n_2+\cdots+Ln_L} \prod_{\ell:q_\ell>0} q_\ell^{n_\ell} \\ &= \sum_{v=0}^{\infty}\sum_{\boldsymbol{n}} \binom{v + n_1 + \cdots + n_L}{v \ n_1 \ \ldots \ n_L} (xy)^v \prod_{\ell:q_\ell>0} q_\ell y^\ell. \end{aligned}$$

Reorganizing the sums, changing the double sum to $\sum_{r=0}^{\infty}\sum_{(v,\boldsymbol{n}):v+n_1+\cdots+n_L=r}$, we obtain

$$Z(x, y; \boldsymbol{q}) = \sum_{r=0}^{\infty}\left(xy + \sum_{\ell=1}^{L} q_\ell y^\ell\right)^r$$

and hence (8).

3.2. Residues

By Proposition 3,

$$Z_X|_{x=1} = \mathrm{Res}_0 \frac{y^{-X-1}}{1 - y - \sum_\ell q_\ell y^\ell} \tag{9}$$

and

$$\left.\frac{\partial Z_X}{\partial x}\right|_{x=1} = \mathrm{Res}_0 \frac{y^{-X}}{(1 - y - \sum_\ell q_\ell y^\ell)^2}. \tag{10}$$

Denote the denominator of (9) by $Q(y)$ and let $y_1, \dots, y_n$ be the roots of the polynomial Q. For $X \geq 1$, the contour integral of (9) or (10) over circles of large radius tends to zero, whence

$$Z_X|_{x=1} = -\sum_m \mathrm{Res}_{y_m} \frac{y^{-X-1}}{1 - y - \sum_\ell q_\ell y^\ell} \tag{11}$$

and

$$\left.\frac{\partial Z_X}{\partial x}\right|_{x=1} = -\sum_m \mathrm{Res}_{y_m} \frac{y^{-X}}{(1 - y - \sum_\ell q_\ell y^\ell)^2}. \tag{12}$$

We are interested in the asymptotics of $\langle v \rangle_X / X$ as $X \to \infty$. In this situation, it is readily seen that (11) and (12) are dominated by the residues at the roots closest to the origin. We can immediately check that the (single) real root (say y_1) is the root of Q closest to the origin and that this root is simple (i.e. the derivative of Q at y_1 does not vanish), so the desired residues are easy to compute. The results are

$$Z_X|_{x=1} = \frac{y_1^{-X-1}}{1 + \sum_\ell \ell q_\ell y_1^{\ell-1}} (1 + O(\gamma^{-X})),$$

where $\gamma = \max_{m>1} |y_m/y_1|$ is the spectral gap, and

$$\left.\frac{\partial Z_X}{\partial x}\right|_{x=1} = \frac{X y_1^{-X-1}}{(1 + \sum_\ell \ell q_\ell y_1^{\ell-1})^2} + O(1).$$

Summarizing, we get

$$\frac{\langle v \rangle_X}{X} = \frac{1}{1 + \sum_\ell \ell q_\ell y_1^{\ell-1}} + O(1/X). \tag{13}$$

This is, of course, in agreement with Proposition 2, which says that, if the generating function in question is the ratio P/Q of two analytic functions, then the limiting asymptotic value of $\langle v \rangle_X / X$ is equal to

$$y_1^{-1} \left(\frac{\mathrm{d} y_1}{\mathrm{d} x} \right),$$

where $y_1(x)$ is a local branch of the curve $Q(x, y) = 0$ corresponding to the root of $Q(x, \cdot)$ with the least absolute value, provided that this root is unique and that P does not vanish at this root.

3.3. Large q_i asymptotics

The formula (13) gives the asymptotic value of the average vacant space per unit length for long intervals X. Of interest to us is the behavior of this average vacant space as the rates q_i tend to ∞. We cannot expect any reasonable (nondegenerate) limiting behavior without further assumptions. What we need is the following.

Assumption 1. *Let ρ be the unique real root of the polynomial*

$$\sum_\ell q_\ell y^\ell - 1.$$

Then we assume that $\rho \to 0$ and that the (nonzero) rescaled coefficients q_ℓ/ρ^ℓ converge to nonnegative coefficients $c_\ell = \lim q_\ell \rho^{-\ell}$.

We notice that $\sum_\ell c_\ell = 1$ and therefore 1 is the unique real root of the polynomial $\tilde{Q} = 1 - \sum \ell c_\ell y^\ell$. It follows in particular that $y_1/\rho \to 1$. Further, easy calculations yield the following asymptotic result.

Proposition 4. *Under Assumption 1, the vacant-space rate scales as ρ. More precisely, as $\rho \to 0$,*

$$\rho^{-1} \frac{\langle v \rangle_X}{X} \to \frac{1}{\sum_\ell \ell c_\ell}. \tag{14}$$

We might interpret the denominator on the right-hand side of (14) as the average conditional length in the rescaled interval flow.

Transient behavior for a version of our model has been studied by Talbot *et al.* in the article [13] cited in Section 1.3. Their model is a direct generalization of Rényi's space-filling model (see Section 1.3). Through simulations, they describe convergence to statistical equilibrium starting with an empty span as the departure rate μ tends to 0^+. The process begins with an initial, essentially pure filling phase in which vacant space reduces at an $O(1/t)$ rate until the span is filled to a fraction that is approximately equal to Rényi's constant $\alpha = 0.748\ldots$. Thereafter, equilibrium behavior is approached in a very slow densification phase with vacant space decreasing at an $O(1/\log t)$ rate; as a typical event in this process, awkwardly placed subintervals straddled by gaps summing to more than 1 eventually depart and are replaced by two subintervals. Note particularly the singular perturbation point at $\mu = 0$: the occupancy of the span approaches 100% as $\mu \to 0^+$, but in the model where $\mu = 0$ the average occupancy in the jamming state is α. A rigorous proof of the details of transient behavior appears to be a challenging open problem.

4. Dimer packing, k channels

Now we consider the k-channel situation. In general, the analysis seems to be quite involved, but if we restrict ourselves to the case of dimer packing, where all intervals have length 2, an essentially complete analysis is possible once the inductive structure of packings is discovered. As above, we denote by $Z(x, y; q)$ the generating (or partition) function. In our current situation we have only one parameter q, the retention rate for the incoming intervals. We start with a general analysis, and then center on the asymptotics as $q \to \infty$. As a background note, we mention that Page [9] analyzed random dimer filling for $k = 1$.

4.1. Generating function

Let $f^j_X[n]$ be the number of elements in the set $\mathcal{C}^j_X[n]$ of admissible configurations on $[0, X+1]$ with exactly j dimers straddling X and n units of vacant space in $[0, X]$. Observe that, for $X > 1$, each configuration in $\mathcal{C}^j_X[n]$ is, for some m with $0 \le m \le k-j$, a configuration in $\mathcal{C}^m_{X-1}[n-(k-m-j)]$ to which dimers have been inserted into the interval $[X-1, X+1]$ of exactly j channels. Then $f^j_X[n] = \sum_{m=0}^{k-j} f^m_{X-1}[n-(k-m-n)]$, and the generating polynomial $f^j_X(x) = \sum_n f^j_X[n]x^n$ for vacant space satisfies

$$f^j_X = \sum_{m=0}^{k-j} x^{k-m-j} f^m_{X-1}, \qquad X > 1.$$

Now define $\boldsymbol{f}_X(x) = (f^k_X(x), \dots, f^0_X(x))^\top$, form the generating (vector) function $\boldsymbol{f}(x, y) = \sum_X \boldsymbol{f}_X y^X$, and find immediately that it solves

$$(E - yA(x))\boldsymbol{f}(x, y) = \boldsymbol{f}_1(x),$$

where E is the $(k+1)\times(k+1)$ identity matrix, $\boldsymbol{f}_1(x) = (1, x, x^2, \dots, x^k)^\top$, and

$$A(x) = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & & & 0 & 1 & x \\ \vdots & & \iddots & \iddots & x & x^2 \\ 0 & 0 & \iddots & \iddots & \vdots & \vdots \\ 0 & 1 & x & \dots & x^{k-2} & x^{k-1} \\ 1 & x & x^2 & \dots & x^{k-1} & x^k \end{pmatrix}. \tag{15}$$

Notice that Z is just the component f^0 of $\boldsymbol{f}$.

From (15) it follows that

$$Z = \frac{P(x, y)}{\det(E - yA(x))},$$

in which the polynomial

$$Q(x, y) = \det(E - yA(x))$$

is central to our analysis. Note that, for fixed (real) x, the zeros of Q are the reciprocal eigenvalues of the symmetric matrix $A(x)$, which is also real.

The following facts can be established by a direct if tedious analysis.

Lemma 1. (a) *For odd k, $k = 2i-1$, $Q(0, y) = (y-1)^i(y+1)^i$ and for even k, $k = 2i$, $Q(0, y) = (y-1)^{i+1}(y+1)^i$.*

(b) *In the local coordinates (ξ, η) near the points* $(0, 1)$ *and* $(0, -1)$ *in the (x, y) plane, the polynomial Q has nondegenerate homogeneous components $q_+(\xi, \eta)$ and $q_-(\xi, \eta)$ respectively of degree equal to the order of the zero of its restriction to the y-axis.*

(c) *The dehomogenizations of the polynomials q_+ and q_- (defined in (b)) given by*

$$\tilde{q}_+(c) = \xi^{-\deg q_+} q_+(\xi, c\xi), \qquad \tilde{q}_-(c) = \xi^{-\deg q_-} q_-(\xi, c\xi)$$

can be expressed in terms of Chebyshev polynomials with all roots simple and real.

Proof. Multiplying $E - yA(x)$ from the right by the matrix

$$P(x) = \begin{pmatrix} 1 & -x & 0 & \dots & 0 & 0 \\ 0 & 1 & -x & \ddots & \vdots & \vdots \\ 0 & 0 & 1 & \ddots & 0 & \vdots \\ \vdots & & 0 & \ddots & -x & 0 \\ 0 & & & \ddots & 1 & -x \\ 0 & 0 & \dots & \dots & 0 & 1 \end{pmatrix}$$

antidiagonalizes A, so that

$$\det(E - yA(x)) = \det(P(x) - y\Delta),$$

where $\Delta = (\Delta_{i,j})$ is the matrix with entries $\Delta_{i,i-1} = 1$ and 0s elsewhere (here we used the fact that $\det(P) = 1$).

Rearranging the rows and columns of $P(x) - y\Delta$ we obtain that $\det(P(x) - y\Delta)$ is equal (up to a sign which is immaterial as we are concerned with the zeros of the determinant only) to the determinant of the matrix

$$S_k(x, y) = \begin{pmatrix} -y & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & -y & -x & 0 & \dots & 0 & 0 & 0 \\ 0 & -x & -y & 1 & & 0 & 0 & 0 \\ 0 & 0 & 1 & -y & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & -y & 1 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 1 & -y & -x \\ 0 & 0 & 0 & 0 & \dots & 0 & -x & 1-y \end{pmatrix}$$

when the matrix size $(k + 1)$ is odd and

$$S_k(x, y) = \begin{pmatrix} -y & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & -y & -x & 0 & \dots & 0 & 0 & 0 \\ 0 & -x & -y & 1 & & 0 & 0 & 0 \\ 0 & 0 & 1 & -y & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & -y & -x & 0 \\ 0 & 0 & 0 & 0 & \ddots & -x & -y & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & -x-y \end{pmatrix}$$

when $k + 1$ is even.

For a matrix M, denote by M' the matrix obtained from M by deleting the first column and the first row. (Notice that $(S'_n)' = S_{n-2}$). We find immediately that

$$\det(S_n) = -y \det(S'_n) - \det(S_{n-2}).$$

Further,

$$\det(S'_n) = -y \det(S_{n-2}) - x^2 \det(S'_{n-2}).$$

Iterating and combining all equalities together, we arrive at the recursion

$$\det(S_n) = (y^2 - x^2 - 1)\det(S_{n-2}) - x^2 \det(S_{n-4}) \tag{16}$$

for $n \geq 4$.

We see that the polynomials $s_n(x, y) = \det(S_n)$ split naturally into two series, for even and for odd matrix sizes. One more splitting: we need to consider the polynomials s near two points, $(0, 1)$ and $(0, -1)$.

Consider the polynomials $\tilde{q}_{n,+}(\xi, \eta) = s_n(\xi, 1 + \eta)$ and $\tilde{q}_{n,-}(\xi, \eta) = s_n(\xi, -1 + \eta)$. For small n, the polynomials can be found manually:

$$\begin{aligned}
\tilde{q}_{0,+} &= 1, & \tilde{q}_{0,-} &= 1,\\
\tilde{q}_{1,+} &= -\eta, & \tilde{q}_{1,-} &= 2 - \eta,\\
\tilde{q}_{2,+} &= \xi + 2\eta + \xi\eta + \eta^2, & \tilde{q}_{2,-} &= -\xi - 2\eta + \xi\eta + \eta^2,\\
\tilde{q}_{3,+} &= -2\eta^2 + \xi^2 + \eta\xi^2 - \eta^3, & \tilde{q}_{3,-} &= -4\eta - \xi^2 + 4\eta^2 + \xi^2\eta - \eta^3.
\end{aligned}$$

From these initial data and from (16) it follows that all monomials in $\tilde{q}_{2m,+}$ and $\tilde{q}_{2m,-}$ have degree m at least. Similarly, all monomials in $\tilde{q}_{2m+1,+}$ have degree greater than m, and those in $\tilde{q}_{2m+1,-}$ degree greater than or equal to m. Denote the homogeneous components of the corresponding (minimal) degree as $q_{n,+}$ and $q_{n,-}$. Introduce the polarizations

$$\begin{aligned}
Q^{\mathrm{e}}_{m,+}(c) &= \frac{q_{2m,+}(\xi, c\xi)}{c^m},\\
Q^{\mathrm{e}}_{m,-}(c) &= \frac{q_{2m,-}(\xi, c\xi)}{c^m},\\
Q^{\mathrm{o}}_{m,+}(c) &= \frac{q_{2m+1,+}(\xi, c\xi)}{c^{m+1}},\\
Q^{\mathrm{o}}_{m,-}(c) &= \frac{q_{2m+1,-}(\xi, c\xi)}{c^m}.
\end{aligned}$$

Then (16) again implies that these polynomials satisfy the recursions

$$Q^p_{m,\pm} = \pm 2c Q^p_{m-1} - Q^p_{m-2},$$

where $p \in \{\mathrm{o}, \mathrm{e}\}$. Together with the initial data, these recursions allow us to give explicit formulae for the polarizations:

$$\begin{aligned}
Q^{\mathrm{e}}_{m,+} &= U^{(2)}_m + U^{(2)}_{m-1},\\
Q^{\mathrm{e}}_{m,-} &= (-1)^m U^{(2)}_m + U^{(2)}_{m-1},\\
Q^{\mathrm{o}}_{m,+} &= -U^{(1)}_{m+1},\\
Q^{\mathrm{o}}_{m,-} &= (-1)^m 2U^{(2)}_m.
\end{aligned}$$

Here $U^{(1)}_m(c) = \cos(m \arccos(c))$ and $U^{(2)}_m(c) = \sin((m + 1)\arccos(c)/\sin(\arccos(c)))$ are Chebyshev polynomials of the first and of the second kind respectively.

It is easy to check that all the above polynomials have simple roots. Relevant for us are the maximal roots of the polynomials $Q^p_{m,-}$ and minimal roots of the polynomials $Q^p_{m,+}$ (they correspond to the branches with least absolute value of $y(x)$). They can be immediately found to be

$$\begin{aligned}
&\cos\left(\frac{2\pi}{2m+1}\right) && \text{for } Q^{\mathrm{e}}_{m,-},\\
-&\cos\left(\frac{\pi}{2m+1}\right) && \text{for } Q^{\mathrm{e}}_{m,+},\\
&\cos\left(\frac{\pi}{2m}\right) && \text{for } Q^{\mathrm{o}}_{m,-},\ m>0,\\
-&\cos\left(\frac{\pi}{2(m+1)}\right) && \text{for } Q^{\mathrm{o}}_{m,+}.
\end{aligned}$$

4.2. Asymptotics of vacant space

These calculations allow us to determine the asymptotics of $\langle v\rangle_X/X$ as $q\to\infty$.

Theorem 1. *As $q\to\infty$, the average vacant space per unit length scales as $q^{-1/2}$:*

$$\frac{\langle v\rangle_X}{X}q^{1/2}\to\cos\left(\frac{\pi}{k+2}\right).$$

Proof. According to the scaling argument, to find the average fraction of vacant space for general q, we just have to multiply x by $q^{-1/2}$, y by $q^{k/2}$, and apply the logarithmic derivative formula of Proposition 2. As the logarithmic derivative is homogeneous of degree 0 in y and of degree -1 in x, it implies that the limit of the fraction of vacant space, as $X\to\infty$, multiplied by $q^{1/2}$ converges to the slope of the branch of $\{Q=0\}$ passing through the y-axis and closest to the x-axis, for small x. Those have been found above and result in the claimed asymptotics. This proves the theorem.

An interesting observation is that the scaling of the fraction of vacant space for large q is independent of k, that is, the holes sitting near the surface do not 'feel' the depth of the k-channel substrate below. This can be understood as an indication that the vacant slots do not form typically deep wells and are just sparsely scattered isolated singletons which travel along the surface until they meet pairwise, to form a vacant space of size 2, which can be filled immediately by a dimer. The existing dependence on k is very weak.

References

[1] Bertozzi, A. and McKenna, J. (1993). Multidimensional residues, generating functions and their applications to queueing networks. *SIAM Rev.* **35,** 239–268.

[2] Coffman, E. G., Jr., Jelenković, P. and Poonen, B. (1999). Reservation probabilities. *Adv. Perf. Anal.* **2,** 129–158.

[3] Coffman, E. G., Jr., Flatto, L., Jelenković, P. and Poonen, B. (1998). Packing random intervals on-line. *Algorithmica* **22,** 448–476.

[4] Evans, J. W. (1993). Random and cooperative sequential adsorption. *Rev. Mod. Phys.* **65,** 1281–1329.

[5] Ferrari, P. and Garcia, N. L. (1998). One-dimensional loss networks and conditioned M/G/∞ queues. *J. Appl. Prob.* **35,** 963–975.

[6] Flory, P. J. (1939). Intramolecular reaction between neighboring substituents of vinyl polymers. *J. Amer. Chem. Soc.* **61,** 1518–1521.

[7] Kelly, F. P. (1987). One-dimensional circuit-switched networks. *Ann. Prob.* **15,** 1166–1179.

[8] Mackenzie, J. K. (1962). Sequential filling of a line by intervals placed at random and its application to linear adsorption. *J. Chem. Phys.* **37,** 723–728.

[9] Page, E. S. (1959). The distribution of vacancies on a line. *J. R. Statist. Soc. B* **21,** 364–374.

[10] PEMANTLE, R. AND WILSON, M. C. (2002). Asymptotics of multivariate sequences. I. Smooth points of the singular variety. *Combin. Theory Ser. A* **97,** 129–161.

[11] PRIVMAN, V. AND NIELABA, P. (1992). Diffusional relaxation in dimer deposition. *Europhys. Lett.* **18,** 673.

[12] RÉNYI, A. (1958). On a one-dimensional problem concerning random space filling. *Magyar Tud. Akak. Mat. Kutató Int. Közl.* **3,** 109–127 (in Hungarian).

[13] TALBOT, J., TARJUS, G. AND VIOT, P. (2000). Adsorption–desorption model and its application to vibrated granular materials. *Phys. Rev. E* **61,** 5429–5438.

[14] TALBOT, J., TARJUS, G., VAN TASSEL, P. R. AND VIOT, P. (2000). From car parking to protein adsorption: an overview of sequential adsorption processes. *Colloids Surfaces A* **165,** 287–324.

[15] ZACHARY, S. AND ZIEDINS, I. (1999). Loss networks and Markov random fields. *J. Appl. Prob.* **36,** 403–414.

[16] ZIEDINS, I. (1987). Quasi-stationary distributions and one-dimensional circuit-switched networks. *J. Appl. Prob.* **24,** 965–977.