Convergence Rates in the Implicit Renewal Theorem on Trees

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Abstract: Consider distributional fixed point equations of the form

$$R \stackrel{\mathcal{D}}{=} f(C_i, R_i, 1 \leq i \leq N),$$

where $f(\cdot)$ is a possibly random real valued function, $N \in \{0, 1, 2, 3, \dots\} \cup \{\infty\}$, $\{C_i\}_{i\in\mathbb{N}}$ are real valued random weights and $\{R_i\}_{i\in\mathbb{N}}$ are iid copies of R, independent of $(N, C_1, C_2, \dots)$; $\stackrel{\mathcal{D}}{=}$ represents equality in distribution. In the recent paper [10], an Implicit Renewal Theorem was developed that enables the characterization of the power tail asymptotics of the solutions R to many equations that fall into this category. In this paper we complement the analysis in [10] to provide the corresponding rate of convergence.

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1. Introduction

Distributional fixed point equations of the form

$$R \stackrel{\mathcal{D}}{=} f(C_i, R_i, 1 \leq i \leq N), \tag{1.1}$$

where $f(\cdot)$ is a possibly random real-valued function, $N \in \mathbb{N} \cup \{\infty\}$, $\mathbb{N} = \{0, 1, 2, 3, \dots\}$, $\{C_i\}_{i\in\mathbb{N}}$ are real-valued random weights and $\{R_i\}_{i\in\mathbb{N}}$ are iid copies of R, independent of $(N, C_1, C_2, \dots)$, appear in many applications in applied probability, e.g., analysis of algorithms and statistical physics; see [1, 6, 8–11] for more details.

As previously stated in the abstract, the work in [10] provides an Implicit Renewal Theorem (Theorem 3.4) that enables the characterization of the power tail behavior of the solution R to (1.1). The results in [10] fully generalize the Implicit Renewal Theorem of Goldie (1991) [7], which was derived for equations of the form $R \stackrel{\mathcal{D}}{=} f(C, R)$ (equivalently $N \equiv 1$ in our case), to recursions (fixed point equations) on trees. The work in [7], for the $N \equiv 1$ case, also includes the rate of convergence in the Implicit Renewal Theorem. Similarly, in this paper we complement the main theorem in [10] by deriving the corresponding convergence rate.

We provide here a matrix form derivation of Corollary 3.4 in [7] that seamlessly extends to trees and that treats both the nonnegative and real-valued weights simultaneously. Our main theorem, Theorem 3.4 can be applied to various multiplicative max-plus recursions, as it was done in [9, 10]. The most important application is the multiplicative branching recursion

$$R \stackrel{\mathcal{D}}{=} \sum_{i=1}^{N} C_i R_i + Q,$$

where $N \in \mathbb{N} \cup \{\infty\}$, $\{C_i\}_{i\in\mathbb{N}}$ are real-valued random weights, Q is a real-valued random variable, and $\{R_i\}_{i\in\mathbb{N}}$ are iid copies of R, independent of $(Q, N, C_1, C_2, \dots)$, which has been studied extensively in the prior literature, e.g., see [2–4] and the references therein.

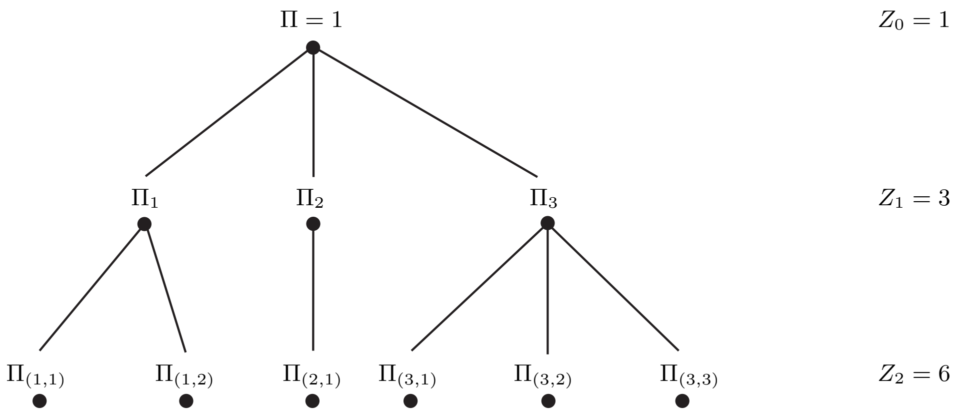

Fig 1. *Weighted branching tree*

2. Weighted Branching Tree

We use the model from [10] for defing a weighted branching tree. First we construct a random tree $\mathcal{T}$. We use the notation $\emptyset$ to denote the root node of $\mathcal{T}$, and A_n, $n \geq 0$, to denote the set of all individuals in the nth generation of $\mathcal{T}$, $A_0 = \{\emptyset\}$. Let Z_n be the number of individuals in the nth generation, that is, $Z_n = |A_n|$, where $|\cdot|$ denotes the cardinality of a set; in particular, $Z_0 = 1$.

Next, let $\mathbb{N}_+ = \{1, 2, 3, \dots\}$ be the set of positive integers and let $U = \bigcup_{k=0}^{\infty} (\mathbb{N}_+)^k$ be the set of all finite sequences $\mathbf{i} = (i_1, i_2, \dots, i_n) \in U$, where by convention $\mathbb{N}_+^0 = \{\emptyset\}$ contains the null sequence $\emptyset$. To ease the exposition, for a sequence $\mathbf{i} = (i_1, i_2, \dots, i_k) \in U$ we write $\mathbf{i}|n = (i_1, i_2, \dots, i_n)$, provided $k \geq n$, and $\mathbf{i}|0 = \emptyset$ to denote the index truncation at level n, $n \geq 0$. Also, for $\mathbf{i} \in A_1$ we simply use the notation $\mathbf{i} = i_1$, that is, without the parenthesis. Similarly, for $\mathbf{i} = (i_1, \dots, i_n)$ we will use $(\mathbf{i}, j) = (i_1, \dots, i_n, j)$ to denote the index concatenation operation, if $\mathbf{i} = \emptyset$, then $(\mathbf{i}, j) = j$.

We iteratively construct the tree as follows. Let N be the number of individuals born to the root node $\emptyset$, $N_\emptyset = N$, and let $\{N_\mathbf{i}\}_{\mathbf{i} \in U}$ be iid copies of N. Define now

$$A_1 = \{i \in \mathbb{N} : 1 \leq i \leq N\}, \quad A_n = \{(\mathbf{i}, i_n) \in U : \mathbf{i} \in A_{n-1}, 1 \leq i_n \leq N_\mathbf{i}\}. \tag{2.1}$$

It follows that the number of individuals $Z_n = |A_n|$ in the nth generation, $n \geq 1$, satisfies the branching recursion

$$Z_n = \sum_{\mathbf{i} \in A_{n-1}} N_\mathbf{i}.$$

Now, we construct the weighted branching tree $\mathcal{T}_C$ as follows. Let $\{(N_\mathbf{i}, C_{(\mathbf{i},1)}, C_{(\mathbf{i},2)}, \dots)\}_{\mathbf{i} \in U}$ be a sequence of iid copies of $(N, C_1, C_2, \dots)$. $N_\emptyset$ determines the number of nodes in the first generation of of $\mathcal{T}$ according to (2.1), and each node in the first generation is then assigned its corresponding vector $(N_i, C_{(i,1)}, C_{(i,2)}, \dots)$ from the iid sequence defined above. In general, for $n \geq 2$, to each node $\mathbf{i} \in A_{n-1}$ we assign its corresponding $(N_\mathbf{i}, C_{(\mathbf{i},1)}, C_{(\mathbf{i},2)}, \dots)$ from the sequence and construct $A_n = \{(\mathbf{i}, i_n) \in U : \mathbf{i} \in A_{n-1}, 1 \leq i_n \leq N_\mathbf{i}\}$. For each node in $\mathcal{T}_C$ we also define the weight $\Pi_{(i_1,\dots,i_n)}$ via the recursion

$$\Pi_{i_1} = C_{i_1}, \qquad \Pi_{(i_1,\dots,i_n)} = C_{(i_1,\dots,i_n)} \Pi_{(i_1,\dots,i_{n-1})}, \quad n \geq 2,$$

where $\Pi = 1$ is the weight of the root node. Note that the weight $\Pi_{(i_1,\dots,i_n)}$ is equal to the product of all the weights $C_{(\cdot)}$ along the branch leading to node $(i_1, \dots, i_n)$, as depicted in Figure 1.

3. Rate of convergence in the Implicit Renewal Theorem on trees

In this section we present an extension of Corollary 3.4 in [7]. Similarly as in [10], the key observation that facilitates this generalization is the following lemma which shows that a certain measure on a tree is a matrix product measure; its proof can be found in [10]. For the case of positive weights, a similar observation was made for a scalar measure in [5]. Throughout the paper we use the standard convention $0^\alpha \log 0 = 0$ for all $\alpha > 0$.

Let $\mathbf{F} = (F_{ij})$ be an $n \times n$ matrix whose elements are finite nonnegative measures concentrated on $\mathbb{R}$. The convolution $\mathbf{F} * \mathbf{G}$ of two such matrices is the matrix with elements $(\mathbf{F} * \mathbf{G})_{ij} \triangleq \sum_{k=1}^n F_{ik} * G_{kj}$, $i, j = 1, \dots, n$, where $F_{ik} * G_{kj}$ is the convolution of individual measures.

Definition 3.1. *A matrix renewal measure is the matrix of measures*

$$\mathbf{U} = \sum_{k=0}^{\infty} \mathbf{F}^{*k},$$

where $\mathbf{F}^{*1} = \mathbf{F}$, $\mathbf{F}^{*(k+1)} = \mathbf{F}^{*k} * \mathbf{F} = \mathbf{F} * \mathbf{F}^{*k}$, $\mathbf{F}^{*0} = \delta_0 \mathbf{I}$, δ_0 *is the point measure at 0, and* $\mathbf{I}$ *is the identity* $n \times n$ *matrix.*

Lemma 3.2. *Let* $\mathcal{T}_C$ *be the weighted branching tree defined by the vector* $(N, C_1, C_2, \dots)$, *where* $N \in \mathbb{N} \cup \{\infty\}$ *and the* $\{C_i\}$ *are real-valued. For any* $n \in \mathbb{N}$ *and* $\mathbf{i} \in A_n$, *let* $V_\mathbf{i} = \log |\Pi_\mathbf{i}|$ *and* $X_\mathbf{i} = \operatorname{sgn}(\Pi_\mathbf{i})$; $V_\emptyset \equiv 0$, $X_\emptyset \equiv 1$. *For* $\alpha > 0$ *define the measures*

$$\mu_n^{(+)}(dt) = e^{\alpha t} E\left[\sum_{\mathbf{i} \in A_n} 1(X_\mathbf{i} = 1, V_\mathbf{i} \in dt) \right],$$

$$\mu_n^{(-)}(dt) = e^{\alpha t} E\left[\sum_{\mathbf{i} \in A_n} 1(X_\mathbf{i} = -1, V_\mathbf{i} \in dt) \right],$$

for $n = 0, 1, 2, \dots$, *and let* $\eta_\pm(dt) = \mu_1^{(\pm)}(dt)$. *Suppose that* $E\left[\sum_{i=1}^N |C_i|^\alpha \log |C_i|\right] \geq 0$ *and* $E\left[\sum_{i=1}^N |C_i|^\alpha\right] = 1$. *Then,* $(\eta_+ + \eta_-)(\cdot)$ *is a probability measure on* $\mathbb{R}$ *that places no mass at* $-\infty$, *and has mean*

$$\int_{-\infty}^{\infty} u\, \eta_+(du) + \int_{-\infty}^{\infty} u\, \eta_-(du) = E\left[\sum_{j=1}^N |C_j|^\alpha \log |C_j| \right].$$

Furthermore, if we let $\boldsymbol{\mu}_n = (\mu_n^{(+)}, \mu_n^{(-)})$, $\mathbf{e} = (1, 0)$ *and* $\mathbf{F} = \begin{pmatrix} \eta_+ & \eta_- \\ \eta_- & \eta_+ \end{pmatrix}$, *then*

$$\boldsymbol{\mu}_n = (\mu_n^{(+)}, \mu_n^{(-)}) = (1, 0) \begin{pmatrix} \eta_+ & \eta_- \\ \eta_- & \eta_+ \end{pmatrix}^{*n} = \mathbf{e} \mathbf{F}^{*n}, \tag{3.1}$$

where $\mathbf{F}^{*n}$ *denotes the nth matrix convolution of* $\mathbf{F}$ *with itself.*

In what follows, $\widehat{\nu}(s) = \int_{-\infty}^{\infty} e^{sx} \nu(ds)$ denotes the Laplace transform of measure ν. If $\mathbf{F}$ is a matrix of measures, then $\widehat{\mathbf{F}}(s)$ is the corresponding matrix of Laplace transforms.

Assumption 3.3. *Suppose the matrix of measures*

$$\mathbf{F} = \begin{pmatrix} \eta_+ & \eta_- \\ \eta_- & \eta_+ \end{pmatrix},$$

satisfies that for some $\epsilon > 0$, the equation

$$(1 - \widehat{\eta}_+(s))^2 - (\widehat{\eta}_-(s))^2 = 0$$

has no roots different from zero on the strip $\{s \in \mathbb{C} : 0 \leq \mathcal{R}s \leq \epsilon\}$, and that there exists an integer $m \geq 1$ such that the Laplace transform of the singular part of $\mathbf{F}^{*m}$, denoted $\widehat{\mathbf{F}}_s^{*m}(\theta)$ has spectral radius strictly smaller than one for $\theta \in \{0, \epsilon\}$.

Theorem 3.4. *Let $(N, C_1, C_2, \dots)$ be a random vector, where $N \in \mathbb{N} \cup \{\infty\}$ and the $\{C_i\}$ are real-valued. Suppose $\mathbf{F}$ satisfies Assumption 3.3 for some $\epsilon > 0$. Assume further that $0 < E\left[\sum_{j=1}^N |C_j|^\alpha \log |C_j|\right] < \infty$, $E\left[\sum_{j=1}^N |C_j|^\alpha\right] = 1$, $E\left[\sum_{j=1}^N |C_j|^\alpha (\log |C_j|)^2\right] < \infty$, $E\left[\sum_{j=1}^N |C_j|^\gamma\right] < \infty$ for some $0 \leq \gamma < \alpha$, and that R is independent of $(N, C_1, C_2, \dots)$.*

a) If $\{C_i\} \geq 0$ a.s., $E[((R)^+)^\beta] < \infty$ for any $0 < \beta < \alpha$, and for $\sigma \in \{0, \epsilon\}$,

$$\int_0^\infty \left| P(R > t) - E\left[\sum_{j=1}^N 1(C_j R > t)\right] \right| t^{\alpha+\sigma-1} dt < \infty, \tag{3.2}$$

or, respectively, $E[((R^-)^\beta] < \infty$ for any $0 < \beta < \alpha$, and for $\sigma \in \{0, \epsilon\}$,

$$\int_0^\infty \left| P(R < -t) - E\left[\sum_{j=1}^N 1(C_j R < -t)\right] \right| t^{\alpha+\sigma-1} dt < \infty, \tag{3.3}$$

then

$$|t^\alpha P(R > t) - H_+| = o\left(t^{-\epsilon}\right), \qquad t \to \infty,$$

or, respectively,

$$|t^\alpha P(R < -t) - H_-| = o\left(t^{-\epsilon}\right), \qquad t \to \infty,$$

where $0 \leq H_\pm < \infty$ are given by

$$H_\pm = \frac{1}{E\left[\sum_{j=1}^N |C_j|^\alpha \log |C_j|\right]} \int_0^\infty v^{\alpha-1} \left(P((\pm 1)R > v) - E\left[\sum_{j=1}^N 1((\pm 1)C_j R > v)\right]\right) dv.$$

b) If $P(C_j < 0) > 0$ for some $j \geq 1$, $E[|R|^\beta] < \infty$ for any $0 < \beta < \alpha$, and both (3.2) and (3.3) are satisfied, then

$$|t^\alpha P(R > t) - H| = o\left(t^{-\epsilon}\right) \quad \text{and} \quad |t^\alpha P(R < -t) - H| = o\left(t^{-\epsilon}\right), \qquad t \to \infty,$$

where $0 \leq H = (H_+ + H_-)/2 < \infty$ is given by

$$H = \frac{1}{2E\left[\sum_{j=1}^N |C_j|^\alpha \log |C_j|\right]} \int_0^\infty v^{\alpha-1} \left(P(|R| > v) - E\left[\sum_{j=1}^N 1(|C_j R| > v)\right]\right) dv.$$

Remark 3.5. *(i) Note that when $N \equiv 1$, then (3.2) and (3.3) only need to hold for $\sigma = \epsilon$, since in this case*

$$\int_0^\infty |P(\pm R > t) - P(\pm CR > t)| \, t^{\alpha-1} dt \leq \int_0^1 t^{\alpha-1} dt + \int_1^\infty |P(\pm R > t) - P(\pm CR > t)| \, t^{\alpha+\epsilon-1} dt < \infty,$$

which is equivalent to conditions (3.7) and (3.9) in Theorems 3.2 and 3.3 of [7]. Furthermore, for $N \equiv 1$, our condition $E\left[C^\alpha (\log C)^2\right] < \infty$ is slightly weaker than $E[C^{\alpha+\epsilon}] < \infty$ in [7].

Lemma 3.6. *Let $\alpha, b > 0$ and $0 \leq H < \infty$. Suppose that for some $\epsilon > 0$*

$$\left| t^{-b} \int_0^t b v^{\alpha+b-1} P(R > v) dv - H \right| = o\left(t^{-\epsilon}\right)$$

as $t \to \infty$. Then,

$$\left| t^{\alpha} P(R > t) - H \right| = o\left(t^{-\epsilon}\right)$$

as $t \to \infty$.

Proof. Fix $\delta \in (0, 1/2)$ and note that, as $t \to \infty$,

$$\begin{aligned}
P(R > t) b t^{\alpha+b} \cdot \frac{(1+\delta)^{\alpha+b} - 1}{\alpha+b} &\geq \int_t^{(1+\delta)t} b v^{\alpha+b-1} P(R > v)\, dv \\
&= ((1+\delta)t)^b \left(((1+\delta)t)^{-b} \int_0^{(1+\delta)t} b v^{\alpha+b-1} P(R > v)\, dv - H \right) \\
&\quad - t^b \left(t^{-b} \int_0^t b v^{\alpha+b-1} P(R > v)\, dv - H \right) + H((1+\delta)t)^b - H t^b \\
&= H t^b \left((1+\delta)^b - 1 \right) + o\left(t^{b-\epsilon}\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
P(R > t) b t^{\alpha+b} \cdot \frac{1 - (1-\delta)^{\alpha+b}}{\alpha+b} &\leq \int_{(1-\delta)t}^{t} b v^{\alpha+b-1} P(R > v)\, dv \\
&= -((1-\delta)t)^b \left(((1-\delta)t)^{-b} \int_0^{(1-\delta)t} b v^{\alpha+b-1} P(R > v)\, dv - H \right) \\
&\quad + t^b \left(t^{-b} \int_0^t b v^{\alpha+b-1} P(R > v)\, dv - H \right) - H((1-\delta)t)^b + H t^b \\
&= H t^b \left(1 - (1-\delta)^b \right) + o\left(t^{b-\epsilon}\right).
\end{aligned}$$

It follows that

$$H \left(\frac{(\alpha+b)((1+\delta)^b - 1)}{b((1+\delta)^{\alpha+b} - 1)} - 1 \right) + o\left(t^{-\epsilon}\right) \leq P(R > t) t^{\alpha} - H \leq H \left(\frac{(\alpha+b)\left(1 - (1-\delta)^b\right)}{b(1 - (1-\delta)^{\alpha+b})} - 1 \right) + o\left(t^{-\epsilon}\right).$$

Now choose $\delta = t^{-2\epsilon}$ and use the fact that $(1 \pm \delta)^c = 1 \pm c\delta + O(\delta^2)$ as $\delta \to 0$ to obtain

$$\frac{(\alpha+b)((1 \pm \delta)^b - 1)}{b((1 \pm \delta)^{\alpha+b} - 1)} - 1 = \frac{(\alpha+b)(b\delta + O(\delta^2))}{b((\alpha+b)\delta + O(\delta^2))} - 1 = \frac{1 + O(\delta)}{1 + O(\delta)} - 1 = O(\delta) = o\left(t^{-\epsilon}\right)$$

as $t \to \infty$. □

Proof of Theorem 3.4. Define the measures η_+ and η_- according to Lemma 3.2 and let

$$g_+(t) = e^{\alpha t} \left(P(R > e^t) - E\left[\sum_{j=1}^{N} 1(C_j R > e^t) \right] \right),$$

$$g_-(t) = e^{\alpha t}\left(P(R < -e^t) - E\left[\sum_{j=1}^N 1(C_j R < -e^t) \right] \right),$$

$$\text{and} \qquad r(t) = e^{\alpha t} P(R > e^t).$$

Fix $b > \epsilon > 0$ and define the operator

$$\breve{f}(t) = \int_{-\infty}^t b e^{-b(t-u)} f(u)\, du.$$

Now, the same arguments used in the proof of Theorem 3.4 in [10], lead to

$$\breve{r}(t) = \mathbf{e}\left(\mathbf{U} * \breve{\mathbf{g}}\right)(t), \tag{3.4}$$

where $\mathbf{e} = (1, 0)$, $\breve{\mathbf{g}} = (\breve{g}_+, \breve{g}_-)^T$, $\mathbf{U} = \sum_{k=0}^\infty \mathbf{F}^{*k}$, and $\mathbf{F} = \begin{pmatrix} \eta_+ & \eta_- \\ \eta_- & \eta_+ \end{pmatrix}$.

Next, we proceed to verify the assumptions of Theorem 2 in [12].

Define $\varphi(t) = e^{\epsilon t^+}$ and note that

$$r_1 \triangleq \lim_{t\to-\infty} \frac{\log \varphi(t)}{t} = 0 \qquad \text{and} \qquad r_2 \triangleq \lim_{t\to\infty} \frac{\log \varphi(t)}{t} = \epsilon.$$

We will now show that provided the assumptions (3.2) and (3.3) hold, $\breve{\mathbf{g}}$ satisfies the following properties:

a) $\breve{\mathbf{g}} \in L_1(\mathbb{R})$
b) $\breve{\mathbf{g}}(t)\varphi(t) \in L_\infty(\mathbb{R})$
c) $\breve{\mathbf{g}}(t)\varphi(t) \to 0$ as $|t| \to \infty$ outside of a set of Lebesgue measure zero
d) $\varphi(t) \int_t^\infty |\breve{\mathbf{g}}(x)| dx \to 0$ as $t \to \infty$ and $\varphi(t) \int_{-\infty}^t |\breve{\mathbf{g}}(x)| dx \to 0$ as $t \to -\infty$.

For part a), note that by (3.2) and (3.3) we know that $g_\pm \in L_1(\mathbb{R})$, so by Lemma 9.2 from [7], $\breve{g}_\pm$ is directly Riemann integrable, and in particular, $\breve{\mathbf{g}} \in L_1(\mathbb{R})$.

For part b), note that

$$\begin{aligned}
\breve{g}_\pm(t)\varphi(t) &= b e^{-bt+\epsilon t^+} \int_{-\infty}^t e^{bu} g_\pm(u) du \\
&= b e^{-bt+\epsilon t^+} \int_{-\infty}^t e^{(b+\alpha)u} \left(P((\pm 1)R > e^u) - E\left[\sum_{j=1}^N 1((\pm 1)C_j R > e^u) \right] \right) du \\
&= b e^{-bt+\epsilon t^+} \int_0^{e^t} v^{b+\alpha-1} \left(P((\pm 1)R > v) - E\left[\sum_{j=1}^N 1((\pm 1)C_j R > v) \right] \right) dv.
\end{aligned}$$

Since for $0 \le v \le e^t$ we have $v^b \le e^{(b-\epsilon)t} v^\epsilon$, it follows that

$$\sup_{t\ge 0} |\breve{g}_\pm(t)\varphi(t)| \le \sup_{t \ge 0} b \int_0^{e^t} v^{\alpha+\epsilon-1} \left| P((\pm 1)R > v) - E\left[\sum_{j=1}^N 1((\pm 1)C_j R > v) \right] \right| dv < \infty,$$

by (3.2) and (3.3). For the supremum over the negative reals note that since $0 \le v \le e^t$, then $v^b \le e^{bt}$, hence

$$\sup_{t<0} |\breve{g}_\pm(t)\varphi(t)| \le \sup_{t<0} b \int_0^{e^t} v^{\alpha-1} \left| P((\pm 1)R > v) - E\left[\sum_{j=1}^N 1((\pm 1)C_j R > v) \right] \right| dv < \infty.$$

To verify c) for $t \to \infty$, note that if

$$\int_0^\infty v^{\alpha+\epsilon-1} \left| P((\pm 1)R > v) - E\left[\sum_{j=1}^N 1((\pm 1)C_j R > v) \right] \right| dv < \infty,$$

then we trivially have $\lim_{t\to\infty} \breve{g}_\pm(t)\varphi(t) = 0$; if it is infinite we can apply L'Hôpital rule to obtain

$$\begin{aligned}
\lim_{t\to\infty} \breve{g}_\pm(t)\varphi(t) &\leq \lim_{t\to\infty} \frac{b e^{(b+\alpha-1)t} \left| P((\pm 1)R > e^t) - E\left[\sum_{j=1}^N 1((\pm 1)C_j R > e^t)\right] \right| e^t}{(b-\epsilon)e^{(b-\epsilon)t}} \\
&= \frac{b}{b-\epsilon} \lim_{t\to\infty} e^{(\alpha+\epsilon)t} \left| P((\pm 1)R > e^t) - E\left[\sum_{j=1}^N 1((\pm 1)C_j R > e^t) \right] \right|,
\end{aligned}$$

which is zero by (3.2) and (3.3). That $\lim_{t\to-\infty} \breve{g}_\pm(t)\varphi(t) = 0$ follows from the estimates given above. And for part d) note that for $t \geq 0$,

$$\begin{aligned}
\varphi(t) \int_t^\infty |\breve{g}_\pm(x)| dx &= e^{\epsilon t} \int_t^\infty \left| b e^{-bx} \int_{-\infty}^x e^{(b+\alpha)u} \left(P((\pm 1)R > e^u) - E\left[\sum_{j=1}^N 1((\pm 1)C_j R > e^u) \right] \right) du \right| dx \\
&\leq b e^{\epsilon t} \int_t^\infty e^{-bx} \int_{-\infty}^x e^{(b+\alpha)u} \left| P((\pm 1)R > e^u) - E\left[\sum_{j=1}^N 1((\pm 1)C_j R > e^u) \right] \right| du\, dx \\
&= b e^{\epsilon t} \int_{-\infty}^\infty \int_{t\vee u}^\infty e^{-bx} e^{(b+\alpha)u} \left| P((\pm 1)R > e^u) - E\left[\sum_{j=1}^N 1((\pm 1)C_j R > e^u) \right] \right| dx\, du \\
&= e^{\epsilon t} \int_{-\infty}^\infty e^{-b(t\vee u)} e^{(b+\alpha)u} \left| P((\pm 1)R > e^u) - E\left[\sum_{j=1}^N 1((\pm 1)C_j R > e^u) \right] \right| du \\
&= e^{-(b-\epsilon)t} \int_0^{e^t} v^{b+\alpha-1} \left| P((\pm 1)R > v) - E\left[\sum_{j=1}^N 1((\pm 1)C_j R > v) \right] \right| dv \\
&\quad + e^{\epsilon t} \int_{e^t}^\infty v^{\alpha-1} \left| P((\pm 1)R > v) - E\left[\sum_{j=1}^N 1((\pm 1)C_j R > v) \right] \right| dv \\
&\leq e^{-(b-\epsilon)t} \int_0^{e^{t/2}} v^{b+\alpha-1} \left| P((\pm 1)R > v) - E\left[\sum_{j=1}^N 1((\pm 1)C_j R > v) \right] \right| dv \qquad (3.5) \\
&\quad + \int_{e^{t/2}}^\infty v^{\alpha+\epsilon-1} \left| P((\pm 1)R > v) - E\left[\sum_{j=1}^N 1((\pm 1)C_j R > v) \right] \right| dv, \qquad (3.6)
\end{aligned}$$

where in the last inequality we split the range of integration of the first integral into $[0, e^{t/2}]$ and $[e^{t/2}, e^t]$ and used the inequalities $v^b \leq e^{(b-\epsilon)t} v^\epsilon$ for $e^{t/2} \leq v \leq e^t$ and $e^{\epsilon t} \leq v^\epsilon$ for $v \geq e^t$. The integral in (3.6) converges to zero as $t \to \infty$ since it is the tail of a finite integral; the integral in (3.5) is bounded by

$$e^{-\frac{(b-\epsilon)}{2}t} \int_0^\infty v^{\alpha+\epsilon-1} \left| P((\pm 1)R > v) - E\left[\sum_{j=1}^N 1((\pm 1)C_j R > v) \right] \right| dv,$$

which also converges to zero as $t \to \infty$. Similarly, for $t < 0$,

$$\begin{aligned}
\varphi(t) \int_{-\infty}^{t} |\breve{g}_{\pm}(x)| dx &\leq b \int_{-\infty}^{t} e^{-bx} \int_{-\infty}^{x} e^{(b+\alpha)u} \left| P((\pm 1)R > e^u) - E\left[\sum_{j=1}^{N} 1((\pm 1)C_j R > e^u) \right] \right| du\, dx \\
&= \int_{-\infty}^{t} e^{(b+\alpha)u} \left| P((\pm 1)R > e^u) - E\left[\sum_{j=1}^{N} 1((\pm 1)C_j R > e^u) \right] \right| \left(e^{-bu} - e^{-bt} \right) du \\
&= \int_{0}^{e^t} v^{b+\alpha-1} (v^{-b} - e^{-bt}) \left| P((\pm 1)R > v) - E\left[\sum_{j=1}^{N} 1((\pm 1)C_j R > v) \right] \right| dv \\
&\leq \int_{0}^{e^t} v^{\alpha-1} \left| P((\pm 1)R > v) - E\left[\sum_{j=1}^{N} 1((\pm 1)C_j R > v) \right] \right| dv \to 0
\end{aligned}$$

as $t \to -\infty$.

We split the rest of the proof into the two different cases.

Case a): $C_i \geq 0$ for all i.

For this case we have $\eta_- \equiv 0$, from where it follows that

$$\mathbf{eU} = (1, 0) \sum_{k=0}^{\infty} \begin{pmatrix} \eta_+ & 0 \\ 0 & \eta_+ \end{pmatrix}^{*k} = (1, 0) \begin{pmatrix} \sum_{i=1}^{\infty} \eta_+^{*k} & 0 \\ 0 & \sum_{k=0}^{\infty} \eta_+^{*k} \end{pmatrix} = \left(\sum_{k=0}^{\infty} \eta_+^{*k}, 0 \right),$$

which in turn implies that

$$\breve{r}(t) = \sum_{k=0}^{\infty} (\breve{g}_+ * \eta_+^{*k})(t).$$

We can then think of this case as a standard one dimensional problem by renaming $\mathbf{F} = \eta_+$ and $\mathbf{U} = \sum_{k=0}^{\infty} \eta_+^{*k}$. The "matrix" $\mathbf{F}(\mathbb{R})$ is clearly irreducible and its spectral radius $\rho[\mathbf{F}(\mathbb{R})] = 1$ (since η_+ is a probability measure in this case). Also,

$$\int_{-\infty}^{\infty} x \mathbf{F}(dx) = \int_{-\infty}^{\infty} x \eta_+(dx) = E\left[\sum_{j=1}^{N} |C_j|^\alpha \log |C_j| \right] \triangleq \mu \in (0, \infty).$$

We now note that

$$\begin{aligned}
\int_{-\infty}^{0} (1 + |x|)^2 \varphi(x) \eta_+(dx) &= \int_{-\infty}^{0} (1 + |x|)^2 e^{\alpha x} E\left[\sum_{i=1}^{N} 1(\log |C_i| \in dx) \right] \\
&= E\left[\sum_{i=1}^{N} \int_{0}^{\infty} (1 + |x|)^2 e^{\alpha x} 1(\log |C_i| \in dx) \right] \qquad \text{(by Fubini's theorem)} \\
&= E\left[\sum_{i=1}^{N} |C_i|^\alpha \left(1 + |\log |C_i||\right)^2 \right],
\end{aligned}$$

which is finite by assumption. This observation, by the remarks preceding Theorem 2 in [12], implies that $T^2 \mathbf{F} \in S(\varphi)$, where for any finite complex-valued measure ν, $T\nu$ is defined as the $\sigma-$finite measure with density $v(x; \nu) \triangleq \nu((x, \infty))$ for $x \geq 0$ and $v(x; \nu) \triangleq -\nu((-\infty, x])$ for $x < 0$, and $S(\varphi)$ is the collection of all complex-valued measures κ such that $\int_{-\infty}^{\infty} \varphi(x) |\kappa|(dx) < \infty$, with $|\kappa|$ the total variation of κ.

Then, by Theorem 2 in [12],

$$\left|\breve{r}(t) - \frac{1}{\mu}\int_0^\infty \breve{g}_+(x)dx\right| = \left|\mathbf{U} * \breve{g}_+(t) - \frac{1}{\mu}\int_0^\infty \breve{g}_+(x)dx\right| = o\left(e^{-\epsilon t}\right)$$

as $t \to \infty$.

To derive the result for $P(R < -t)$, follow the same steps leading to (3.4) in the proof of Theorem 3.4 in [10] but starting with a telescoping sum for $P(-R > e^t)$ instead, and defining $r(t) = e^{\alpha t}P(R < -e^t)$. Using the same arguments given above then gives

$$\left|\breve{r}(t) - \frac{1}{\mu}\int_0^\infty \breve{g}_-(x)dx\right| = \left|\mathbf{U} * \breve{g}_-(t) - \frac{1}{\mu}\int_0^\infty \breve{g}_-(x)dx\right| = o\left(e^{-\epsilon t}\right)$$

as $t \to \infty$.

We have thus shown that

$$\begin{aligned}
&\left|\int_{-\infty}^t be^{-b(t-s)}e^{\alpha s}P(\pm R > e^s)ds - \frac{1}{\mu}\int_0^\infty \int_{-\infty}^x be^{-b(x-s)}g_\pm(s)\,ds\,dx\right| \\
&= \left|e^{-bt}\int_0^{e^t} bv^{\alpha+b-1}P(\pm R > v)dv - \frac{1}{\mu}\int_{-\infty}^\infty g_\pm(s)\,ds\right| \\
&= \left|e^{-bt}\int_0^{e^t} bv^{\alpha+b-1}P(\pm R > v)dv - H_\pm\right| = o\left(e^{-\epsilon t}\right),
\end{aligned}$$

as $t \to \infty$, where

$$\begin{aligned}
H_\pm &\triangleq \frac{1}{\mu}\int_{-\infty}^\infty g_\pm(s)\,ds \\
&= \frac{1}{\mu}\int_{-\infty}^\infty e^{\alpha t}\left(P(\pm R > e^t) - E\left[\sum_{j=1}^N 1((\pm 1)C_jR > e^t)\right]\right)dt \\
&= \frac{1}{\mu}\int_0^\infty v^{\alpha-1}\left(P(\pm R > v) - E\left[\sum_{j=1}^N 1((\pm 1)C_jR > v)\right]\right)dv.
\end{aligned}$$

Therefore, by Lemma 3.6, we obtain

$$|t^\alpha P(\pm R > t) - H_\pm| = o\left(t^{-\epsilon}\right)$$

as $t \to \infty$.

Case b): $P(C_j < 0) > 0$ for some $j \geq 1$.

For this case we have that η_- is nonzero. Also, note that the matrix

$$\mathbf{F}(\mathbb{R}) = \begin{pmatrix} E\left[\sum_{j=1}^N |C_j|^\alpha 1(X_j = 1)\right] & E\left[\sum_{j=1}^N |C_j|^\alpha 1(X_j = -1)\right] \\ E\left[\sum_{j=1}^N |C_j|^\alpha 1(X_j = -1)\right] & E\left[\sum_{j=1}^N |C_j|^\alpha 1(X_j = 1)\right]\end{pmatrix} \triangleq \begin{pmatrix} q & 1-q \\ 1-q & q\end{pmatrix}$$

is irreducible and has eigenvalues $\{1, 2q-1\}$, and therefore spectral radius $\rho[\mathbf{F}(\mathbb{R})] = 1$. Moreover, $(1,1)$ and $(1,1)^T$ are left and right eigenvalues, respectively, of $\mathbf{F}(\mathbb{R})$ corresponding to eigenvalue one, and by assumption,

$$(1,1)\int_{-\infty}^\infty x\mathbf{F}(dx)\begin{pmatrix}1\\1\end{pmatrix} = 2\left(\int_{-\infty}^\infty x\eta_+(dx) + \int_{-\infty}^\infty x\eta_-(dx)\right) = 2E\left[\sum_{j=1}^N |C_j|^\alpha \log|C_j|\right] = 2\mu \in (0,\infty).$$

Also, similarly as in the nonnegative case, we have

$$\int_{-\infty}^{0} (1+|x|)^2 \varphi(x) \eta_{\pm}(dx) = E\left[\sum_{i=1}^{N} |C_i|^\alpha \left(1+|\log|C_i|\right)^2 1(X_i = \pm 1)\right],$$

which is finite by assumption. And from the remarks preceding Theorem 2 in [12], we have that $T^2\mathbf{F} \in S(\varphi)$. Then, by Theorem 2 in [12],

$$\left|\mathbf{U} * \breve{\mathbf{g}}(t) - \frac{(1,1)^T(1,1)}{2\mu} \int_{-\infty}^{\infty} \breve{\mathbf{g}}(x)dx\right| = \left|\mathbf{U} * \breve{\mathbf{g}}(t) - \frac{1}{2\mu}\begin{pmatrix} \int_{-\infty}^{\infty} (\breve{g}_+(u) + \breve{g}_-(u))du \\ \int_{-\infty}^{\infty} (\breve{g}_+(u) + \breve{g}_-(u))du \end{pmatrix}\right| = o\left(e^{-\epsilon t}\right)$$

as $t \to \infty$. Hence, it follows from $\breve{r}(t) = \mathbf{e}\mathbf{U} * \breve{\mathbf{g}}(t)$ that

$$\left|\breve{r}(t) - \frac{1}{2\mu}\int_{-\infty}^{\infty} (\breve{g}_+(u) + \breve{g}_-(u))du\right| = \left|e^{-bt}\int_0^{e^t} bv^{\alpha+b-1} P(R > v)dv - \frac{1}{2}(H_+ + H_-)\right| = o\left(e^{-\epsilon t}\right)$$

as $t \to \infty$. Let $H = (H_+ + H_-)/2$, then by Lemma 3.6

$$|t^\alpha P(R > t) - H| = o\left(t^{-\epsilon}\right)$$

as $t \to \infty$.

To derive the result for $P(R < -t)$ simply start by defining $r(t) = e^{\alpha t} P(-R > e^t)$, which in this case leads to the same result as above, that is,

$$|t^\alpha P(R < -t) - H| = o\left(t^{-\epsilon}\right)$$

as $t \to \infty$.

Finally, we note, by using the representations for H_+ and H_- from Case a), that

$$\begin{aligned} H &= \frac{1}{2\mu}\int_0^\infty v^{\alpha-1}\left(P(R > v) - E\left[\sum_{j=1}^{N} 1(C_j R > v)\right]\right) dv \\ &\quad + \frac{1}{2\mu}\int_0^\infty v^{\alpha-1}\left(P(R < -v) - E\left[\sum_{j=1}^{N} 1(C_j R < -v)\right]\right) dv \\ &= \frac{1}{2\mu}\int_0^\infty v^{\alpha-1}\left(P(|R| > v) - E\left[\sum_{j=1}^{N} 1(|C_j R| > v)\right]\right) dv. \end{aligned}$$

□

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