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Modulated Branching Processes, Origins of Power Laws, and Queueing Duality

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Power law distributions have been repeatedly observed in a wide variety of socioeconomic, biological, and technological areas. In many of the observations, e.g., city populations and sizes of living organisms, the objects of interest evolve because of the replication of their many independent components, e.g., births and deaths of individuals and replications of cells. Furthermore, the rates of replications are often controlled by exogenous parameters causing periods of expansion and contraction, e.g., baby booms and busts, economic booms and recessions, etc. In addition, the sizes of these objects often have reflective lower boundaries, e.g., cities do not fall below a certain size, low-income individuals are subsidized by the government, companies are protected by bankruptcy laws, etc.

Hence, it is natural to propose reflected modulated branching processes as generic models for many of the preceding observations. Indeed, our main results show that the proposed mathematical models result in power law distributions under quite general polynomial Gärtner-Ellis conditions, the generality of which could explain the ubiquitous nature of power law distributions. In addition, on a logarithmic scale, we establish an asymptotic equivalence between the reflected branching processes and the corresponding multiplicative ones. The latter, as recognized by Goldie [Goldie, C. M. 1991. Implicit renewal theory and tails of solutions of random equations. *Ann. Appl. Probab.* **1**(1) 126–166], is known to be dual to queueing/additive processes. We emphasize this duality further in the generality of stationary and ergodic processes.

Key words: modulated branching processes; reflective/absorbing barriers; reflected multiplicative processes; proportional growth models; power law distributions; heavy tails; subexponential distributions; queueing processes; reflected additive random walks; Cramér large deviations; polynomial Gärtner-Ellis conditions
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1. Introduction. Power law distributions are found in a wide range of domains, ranging from socioeconomic to biological and technological areas. Specifically, these types of distributions describe the city populations, species area relationships, sizes of living organisms, values of companies, distributions of wealth, and, more recently, sizes of documents on the Web, visitor access patterns on websites, etc. Hence, one would expect that there exist universal mathematical laws that explain this ubiquitous nature of power law distributions. To this end, we propose a class of models termed modulated branching processes with reflective lower barriers that, under quite general *polynomial Gärtner-Ellis* conditions, result in power law distributions.

Empirical observations of power laws have a long history, starting from the discovery by Pareto [61] in 1896 that a plot of the logarithm of the number of incomes above a level against the logarithm of that level yields points close to a straight line, which is essentially equivalent to saying that the income distribution follows a power law. Hence, power law distributions are often called Pareto distributions; for more recent study on income distributions, see Champernowne [16], Mandelbrot [56], Dagum [21], and Reed [65, 66]. In a different context, early work by Arrhenius [4] in 1921 conjectured a power law relationship between the number of species and the census area, which was followed by Preston's prediction in Preston [64] that the slope on the log/log species area plot has a canonical value equal to 0.262. For additional information and measurements on species area relationships, see Connor and McCoy [18], Plotkin et al. [63], and Keeley [47]. Interestingly, there also exists a power law relationship between the rank of the cities and the population of the corresponding cities, proposed by Auerbach [8] in 1913 and later studied by Zipf [73] in 1949. This power law is also known as Zipf's law. Ever since, much attention on both empirical examinations and explanations of the city size distributions have been drawn (Zipf [73], Ioannides and Overman [37], Gabaix [28], Rosen and Resnick [69], Parr [62], Allen et al. [2]). Similar observations have been made for firm sizes (Amaral et al. [3]), language family sizes (Wichmann [72]), and even the gene family and protein statistics (Huynen and van Nimwegen [36], Rzhetsky and Gomez [70], Luscombe et al. [55], Brujic et al. [13]). It is perhaps even more surprising that many features of the Internet are governed by power laws, including the distribution of pages per website (Huberman and Adamic [34]), the page request distribution (Cunha et al. [20], Breslau et al. [12]), the file size distribution (Downey [25], Jelenković and Momčilović [39]), Ethernet traffic (Leland et al. [50]), World Wide Web traffic (Crovella and

Bestavros [19]), the number of visitors per website (Huberman and Adamic [35], Adamic and Huberman [1]), the distribution of scenes in video streams (Jelenković et al. [46]), and the distribution of the in-degrees and out-degrees in the Web graph as well as the physical network connectivity graph (Faloutsos et al. [26], Barabási et al. [9], Kumar et al. [49], Medina et al. [57]). In socioeconomic areas, in addition to income distributions, the fluctuations in stock prices have also been observed to be characterized by power laws (Gabaix et al. [29], Levy and Solomon [51]). This paragraph only exemplifies various observations of power laws; for a more complete survey, see Mitzenmacher [58].

Hence, these repeated empirical observations of power laws over a period of more than a hundred years strongly suggest that there exist general mathematical laws that govern these phenomena. In this regard, after carefully examining the situations that result in power laws, we discover that most of them are characterized by the following three features. First, in the vast majority of these observations, e.g., city populations and sizes of living organisms, the objects of interest evolve because of the replication of their many independent components, e.g., births and deaths of individuals and replications of cells. Second, the rate of replication of the many components is often controlled by exogenous parameters that cause periods of baby booms and busts, economic growths and recessions, etc. Third, the sizes of these objects often have lower boundaries, e.g., cities do not fall below a certain size, low-income individuals are subsidized by the government, companies are protected by bankruptcy laws, etc.

To capture the preceding features, it is natural to propose *modulated branching processes* (MBP) with reflective or absorbing barriers as generic models for many of the observations of power laws. Indeed, one of our main results, presented in Theorem 3.1, shows that MBPs with reflective barriers almost invariably produce power law distributions under quite general polynomial Gärtner-Ellis conditions. The generality of our results could explain the ubiquitous nature of power law distributions. Furthermore, an informal interpretation of our main results, stated in Theorems 3.1 and 3.2 of §3, suggests that alternating periods of expansions and contractions, e.g., economic booms and recessions, are primarily responsible for the appearance of power law distributions. Actually, Theorem 3.2 shows that the distribution of the reflected MBP decays faster than any power law if the conditional mean values of the branching process, given the environment, are smaller than one. From a mathematical perspective, we develop a novel sample path technique for analyzing reflected modulated branching processes because these objects appear new and the traditional methods for investigating branching processes (Athreya and Ney [7]) do not directly apply; furthermore, for traditional work on modulated branching processes without reflected boundaries (also known as branching processes in random environments), see Chapter 7 in Athreya and Ney [7]. A preliminary version of this work has appeared in the extended abstract in Jelenković and Tan [41].

Formal description of our reflected modulated branching process (RMBP) model is given in §2. In the singular case when the number of individuals born in each state of the modulating process is constant, our model reduces to a reflected multiplicative process. A rigorous connection (duality) between the reflected multiplicative processes (RMPs) and queueing theory was established in §5 of Goldie [32]; this duality was repeatedly observed and used later in, e.g., Sornette and Cont [71] and Gong et al. [33]. In §2.1, we further emphasize this duality in the context of stationary and ergodic processes. We would like to point out that this duality makes a vast literature on queueing theory directly applicable to the analysis of RMPs. As a direct consequence of this connection, in §2.1, we translate several well-known queueing results to the context of RMPs. Informally, these results show that the role that exponential distributions play in queueing theory (and in additive reflected random walks in general) is represented by power law distributions in the framework of RMPs/RMBPs. Furthermore, this relationship appears to reduce the debate on the relative importance of power law versus exponential distributions/models to the analogous question of the prevalence of proportional growth versus additive phenomena. Interestingly, the power law distribution satisfies the memoryless property in the multiplicative world, playing an equivalent role to the memoryless exponential distribution in the additive world. Indeed, if $\mathbb{P}[M > x] = x^{-\alpha}$, $\alpha > 0$, $x \geq 1$, then, for $x, y \geq 1$, we obtain $\mathbb{P}[M > xy \mid M > x] = \mathbb{P}[M > y]$.

Furthermore, this duality immediately implies and generalizes many of the prior results in the area of RMPs and power laws (see Levy and Solomon [51, 52, 53] and Sornette and Cont [71]). In addition, we would like to point out that the reflective nature of the barrier, assumed in the previous studies, is not essential for producing power law distributions. Indeed, one only needs a positive lower barrier, e.g., a porous, absorbing or reflective one—this is a natural condition because no physical object or socioeconomic one can approach zero arbitrarily close without repelling from it or simply disappearing. In many areas, objects of interest may not have a strictly reflecting barrier but rather they may have a porous one, e.g., cities may degenerate, bankruptcy protection may sometimes fail, and a company can be liquidated. In these cases, the power law effect follows from the well-known queueing results on the cycle maximum that we state briefly in §2.2. This observation presents a rigorous explanation for the previous study in Blank and Solomon [11] that argued heuristically on how multiplicative processes with absorbing barriers can result in power laws.

Here, we would like to point out that the stochastic difference equation ($M_{n+1} = J_n M_n + Q_n$) with random coefficients is closely related to RMPs and is known to produce power law distributions. It appears that the first rigorous study of this process was done by Kesten in 1973 (Kesten [48]); for later investigations of this model, see de Haan et al. [22], de Saporta [23], and the references therein. In addition, we refer the reader to Equations (1.1)–(1.6) of Goldie [32, p. 126] for other related stochastic recursions of multiplicative nature that produce power laws. For recent extensions of these results to recursions on trees, see Jelenković and Olvera-Cravioto [40].

Next, it is easy to see that RMBPs reduce to RMPs in the special case when a constant number of individuals are born in each state of the modulating process. However, our main result, Theorem 3.1, reveals a general asymptotic equivalence between the power law exponent of an RMBP and the corresponding RMP. In other words, Theorem 3.1 discovers the asymptotic insensitivity of the power law exponent on the conditional distributions of the reflected branching process beyond their conditional mean values. Furthermore, for the special case when the modulating process is independent and identically distributed (i.i.d.), we sharpen the result on the logarithmic asymptotics of Theorem 3.1 to the exact one in Theorem 4.3 by using the implicit renewal theory of Goldie [32].

In some domains, e.g., the growth of living organisms, the objects always grow (never shrink) up until a certain random time. Huberman and Adamic [34] also propose this model as an explanation of the growth dynamics of the World Wide Web by arguing that the observation time is an exponential random variable. This notion has been revisited in Reed [65] and generalized to a larger family of random processes observed at an exponential random time (Reed and Hughes [67]). In this regard, in §5.1.2, we study randomly stopped modulated branching processes and show, under more general conditions than the preceding studies, that the resulting variables follow power laws.

In regard to the previously mentioned situations with absorbing barriers, we discuss MBP with an absorbing barrier in §5.2 and argue that it leads to power law distributions as well. We conjecture that these types of models can be natural candidates for describing the bursts of requests at popular Internet websites, often referred to as hot spots.

Based on our new model, we discuss two related phenomena: truncated power laws and double Pareto distributions. We argue that one can obtain a truncated power law distribution by adding an upper barrier to RMBP, similar to the way truncated geometric distributions appear in queueing theory, e.g., the finite buffer $M/M/1$ queue. Furthermore, by the duality of RMBP and queueing theory, we give two new natural explanations of the origins of double Pareto distributions that have been observed in practice. In the queueing context, it has been shown that the tail of the queue-length distribution exhibits different decay rates in the heavy traffic and large deviation regime, respectively (Olvera-Cravioto et al. [60]); similar behavior of the queue-length distribution was attributed to the multiple time-scale arrivals in Jelenković and Lazar [38]. We claim that the preceding two mechanisms, when translated to the proportional growth context, provide natural explanations of the double Pareto distributions.

Finally, we would like to mention that there might be other nonmultiplicative mechanisms that result in power law distributions, e.g., the random typing model used to explain the power law distribution of frequencies of words in natural languages (Mitzenmacher [58]) as well as highly optimized tolerance studied in Carlson and Doyle [15]. Very recently, the new power law phenomenon in situations where jobs have to restart from the beginning after a failure was discovered in Fiorini et al. [27] and further studied in Asmussen et al. [6]; equivalently, in the communication context, retransmission-based protocols in data networks were shown to almost invariably lead to power laws and, in general, heavy tails in Jelenković and Tan [42, 43, 44]. For a recent survey on various mechanisms that result in power laws, see Mitzenmacher [58].

The rest of the paper is organized as follows. After introducing the modulated branching processes in §2, we study the duality between the queueing theory and the multiplicative processes with reflected barriers in §2.1 and absorbing barriers in §2.2, respectively. Then, we present our main results in §3 on the logarithmic asymptotics of the stationary distribution of the reflected modulated branching process and the corresponding multiplicative one. This is followed by the study of the exact asymptotics under the more restrictive conditions in §4. As further extensions, we discuss three related models in §5, i.e., randomly stopped processes in §5.1, modulated branching processes with absorbing barriers in §5.2, and truncated power laws in §5.3. Section 6 presents the majority of the technical proofs that have been deferred from the preceding sections for increased readability.

2. Reflected modulated branching processes. In this section, we formally describe our model. Let $\{J_n\}_{n>-\infty}$ be a stationary and ergodic modulating process that takes values in the positive integers. Define a family of independent, nonnegative, integer-valued random variables $\{B_n^i(j)\}_{i,\, j\geq 1}$, $-\infty < n < \infty$, which are independent of the modulating process $\{J_n\}$. In addition, for fixed j, variables $\{B(j), B_n^i(j)\}$ are identically distributed with $0 < \mu(j) \triangleq \mathbb{E}[B(j)] < \infty$.

Definition 2.1. A modulated branching process (MBP) $\{Z_n\}_{n=0}^{\infty}$ is recursively defined by

$$Z_{n+1} \triangleq \sum_{i=1}^{Z_n} B_n^i(J_n), \tag{1}$$

where the initial value $Z_0 < \infty$ is an integer-valued random variable. For increased clarity, we may explicitly write $\{Z_n^l\}$ when $Z_0 = l$.

Definition 2.2. For any $l \in \mathbb{N}$ and an integer-valued Λ_0, a reflected modulated branching process (RMBP) $\{\Lambda_n\}_{n=0}^{\infty}$ is recursively defined as

$$\Lambda_{n+1} \triangleq \max\left(\sum_{i=1}^{\Lambda_n} B_n^i(J_n), l\right). \tag{2}$$

Remark 2.1. These types of modulated branching processes with a reflecting barrier appear to be new and thus the traditional methods for the analysis of branching processes (Athreya and Ney [7]) do not seem to directly apply. For traditional work on modulated branching processes without the reflected boundaries (also known as branching processes in random environments), see Chapter 7 in Athreya and Ney [7].

Remark 2.2. A more general framework would be to define

$$Z_{n+1} = \int_0^{Z_n} B_n^t(J_n(t))\, d\nu(t) \tag{3}$$

for any real measure ν and similarly

$$\Lambda_{n+1} = \max\left(\int_0^{\Lambda_n} B_n^t(J_n(t))\, d\nu(t), l\right), \tag{4}$$

where $l > 0$ and $B_n^t(J_n(t))$ is ν-measurable. We refrain from this generalization because it introduces additional technical difficulties without much new insight.

Now, we present the basic limiting results on the convergence to stationarity of Z_n and Λ_n.

Lemma 2.1. *If $\mathbb{E}\log\mu(J_0) < 0$, then almost surely (a.s.) we have*

$$\lim_{n\to\infty} Z_n = 0.$$

Proof. For all $n \geq 1$, let $W_n = Z_n/\Pi_{n-1}^0$, where $\Pi_n^0 = \prod_{i=0}^n \mu(J_i)$. It is easy to check that W_n is a positive martingale with respect to the filtration $\mathcal{F}_n = \sigma(J_i, Z_i, 0 \leq i \leq n-1)$. Hence, by the martingale convergence theorem (see Theorem 35.5. of Billingsley [10]) a.s. as $n \to \infty$,

$$W_n \to W < \infty. \tag{5}$$

Next, because $\{J_n\}$ is stationary and ergodic, so is $\{\mu(J_n)\}$ and therefore a.s.

$$\frac{\log \Pi_{n-1}^0}{n} = \frac{1}{n}\sum_{i=0}^{n-1} \log\mu(J_i) \to \mathbb{E}\log\mu(J_0) < 0 \quad \text{as } n \to \infty.$$

Thus, $\Pi_{n-1}^0 \to 0$ as $n \to \infty$ a.s. This, recalling (5) and $Z_n = W_n \Pi_{n-1}^0$, finishes the proof. □

Next, let Z_{-n} be the number of individuals at time zero in an unrestricted branching process that starts at time $-n$ with l individuals defined on the same sequence $\{B_n^i(J_n)\}_{i\geq 1}$, $-\infty < n < \infty$; when needed for clarity, we will use the notation Z_{-n}^l to explicitly indicate the initial state l.

Lemma 2.2. *Assume $\mathbb{E}\log\mu(J_0) < 0$, then, for any a.s. finite initial condition Λ_0, Λ_n converges in distribution to*

$$\Lambda \stackrel{d}{=} \max_{n\geq 0} Z_{-n},$$

where $\stackrel{d}{=}$ stands for equality in distribution.

Proof. First, assuming $\Lambda_0 = l$, we observe that by stationarity of $\{J_n\}$,

$$\Lambda_1 = \max\left(\sum_{i=1}^{l} B_1^i(J_1), l\right) \stackrel{d}{=} \max(Z_{-1}, Z_0).$$

Then, by induction and stationarity, it is easy to show

$$\Lambda_n \stackrel{d}{=} \max(Z_{-n}, Z_{-(n-1)}, \ldots, Z_{-1}, Z_0), \tag{6}$$

where $Z_{-n}, \ldots, Z_0$ are defined on the same sequence $\{B_k^i(J_k)\}_{i\geq 1}$, $-n \leq k \leq 0$. Hence, by monotonicity, we obtain

$$\mathbb{P}[\Lambda_n > x] \to \mathbb{P}[\Lambda > x] \quad \text{as } n \to \infty.$$

Now, if $\Lambda_n^{\Lambda_0}$ is a process defined on the same sequence $\{B_n^i(J_n)\}$ with the initial condition $\Lambda_0 \geq l$, then it is easy to see that

$$\Lambda_n^{\Lambda_0} \geq \Lambda_n \geq l \quad \text{for all } n,$$

implying

$$\mathbb{P}[\Lambda_n^{\Lambda_0} > x] \geq \mathbb{P}[\Lambda_n > x]. \tag{7}$$

Next, if we define the stopping time τ to be the first time when $\Lambda_n^{\Lambda_0}$ hits the boundary l, then the preceding monotonicity implies that $\Lambda_n = \Lambda_n^{\Lambda_0}$ for all $n \geq \tau$. Using this observation, we obtain

$$\begin{aligned}\mathbb{P}[\Lambda_n^{\Lambda_0} > x] &= \mathbb{P}[\Lambda_n^{\Lambda_0} > x, \tau > n] + \mathbb{P}[\Lambda_n^{\Lambda_0} > x, \tau \leq n] \\ &= \mathbb{P}[\Lambda_n^{\Lambda_0} > x, \tau > n] + \mathbb{P}[\Lambda_n > x, \tau \leq n] \\ &\leq \mathbb{P}[\tau > n] + \mathbb{P}[\Lambda_n > x]. \end{aligned} \tag{8}$$

Next, by Lemma 2.1, τ is a.s. finite and thus by (7) and (8), we conclude

$$\lim_{n\to\infty} \mathbb{P}[\Lambda_n^{\Lambda_0} > x] = \lim_{n\to\infty} \mathbb{P}[\Lambda_n > x] = \mathbb{P}[\Lambda > x]. \quad \square$$

2.1. Reflected multiplicative processes and queueing duality. Note that in the special case $B_n^i(J_n) \equiv J_n$, reflected modulated branching processes reduce to reflected multiplicative processes with J_n being integer valued. In general, using the definition in (3), J_n can be relaxed to take any positive real values. Hence, in this subsection, we assume that $\{J_n\}_{-\infty<n<\infty}$ is a positive, stationary, and ergodic real-valued process.

Definition 2.3. For $l > 0$ and $M_0^\circ < \infty$, define a reflected multiplicative process (RMP) as

$$M_{n+1}^\circ = \max(M_n^\circ \cdot J_n, l), \quad n \geq 0. \tag{9}$$

The preceding RMP model was studied by Goldie in 1991 (Goldie [32]); for later considerations of this model, see Sornette and Cont [71], Levy and Solomon [52, 53], Gabaix [28], Gong et al. [33], Downey [25]. Goldie [32] also shows a direct connection (duality) between RMP and queueing theory in §5 of Goldie [32] for the case when $\{J_n\}$ is an i.i.d. sequence. Here, we study this duality further in the generality of stationary and ergodic processes.

Without loss of generality, we can assume $l = 1$ because we can always divide (9) by l and define $M_n^{\circ 1} = M_n^\circ / l$. Now, let $X_n = \log J_n$ and $Q_n = \log M_n^\circ$ with the standard conventions $\log 0 = -\infty$ and $e^{-\infty} = 0$. Then, for $l = 1$, Equation (9) is equivalent to

$$Q_{n+1} = \max(Q_n + X_n, 0), \tag{10}$$

which is the workload (waiting time) recursion in a single-server first-in-first-out (FIFO) queue.

Lemma 2.3. *If $\mathbb{E}\log J_n < 0$, then M_n° converges in distribution to an a.s. finite random variable M° that satisfies*

$$M^\circ \stackrel{d}{=} \sup_{n\geq 0} \Pi_n^\circ, \tag{11}$$

where $\Pi_0^\circ = 1$, $\Pi_n^\circ = \prod_{i=-n}^{-1} J_i$, $n \geq 1$.

Proof. By the classical result of Loynes [54], Q_n, defined by (10), converges in distribution to an a.s. finite stationary limit Q if $\mathbb{E}X_n = \mathbb{E}\log J_n < 0$. Furthermore,

$$Q \stackrel{d}{=} \sup_{n\geq 0} S_n,$$

where $S_0 = 0$ and $S_n = \sum_{i=-n}^{-1} X_i$. This implies the convergence in distribution of M_n to

$$M^\circ \stackrel{d}{=} e^{\sup_{n\geq 0} S_n} = \sup_{n\geq 0} e^{S_n} = \sup_{n\geq 0} \Pi_n^\circ. \quad \square$$

The following theorem is a direct corollary of Theorem 1 in Glynn and Whitt [31]; see also Theorem 3.8 in Chang [17]. For a more recent presentation, we refer the reader to Ganesh et al. [30].

Theorem 2.1. *Let $\{J_n\}_{n>-\infty}$ be a stationary and ergodic sequence of positive random variables. If there exists a function Ψ and positive constants α^* and ε^* such that*

(1) *$n^{-1}\log\mathbb{E}[(\Pi_n^\circ)^\alpha] \to \Psi(\alpha)$ as $n\to\infty$ for $|\alpha-\alpha^*| < \varepsilon^*$,*
(2) *Ψ is finite and differentiable in a neighborhood of α^* with $\Psi(\alpha^*) = 0$, $\Psi'(\alpha^*) > 0$, and*
(3) *$\mathbb{E}[(\Pi_n^\circ)^{\alpha^*+\varepsilon}] < \infty$ for $n\geq 1$ and some $\varepsilon > 0$,*

then

$$\lim_{x\to\infty} \frac{\log\mathbb{P}[M^\circ > x]}{\log x} = -\alpha^*. \tag{12}$$

Remark 2.3. We refer to conditions (1)–(3) as the *polynomial Gärtner-Ellis conditions* for the process $\{J_n\}$. Note that condition (2) can be relaxed such that Ψ is only differentiable at α^* and condition (3) can be weakened to $\varepsilon = 0$ (Glynn and Whitt [31]). Because conditions (2) and (3) are used for Theorem 3.1 in §3, we keep the current form to provide a unified framework. Also, it is worth noting that the multiplicative process Π_n° without the reflective boundary would essentially follow the lognormal distribution as was recently observed in Gong et al. [33] (this is similar to the fact that the unrestricted additive random walk is approximated well by normal distribution). However, we would like to reemphasize that the lower boundary l is not just a mathematical artifact but is a very natural condition because no physical object can approach zero arbitrarily close without either repelling (reflecting) from it or vanishing (absorbing); the absorbing boundary will be discussed in §2.2.

Here, we illustrate Theorem 2.1 by the following examples. Assume that $\{A_n\}, \{C_n\}$ are two mutually independent sequences, and let $J_n = e^{A_n - C_n}$. Then, the quantity $Q_n \triangleq \log M_n^\circ$, where M_n° is defined in (9), satisfies

$$Q_{n+1} = (Q_n + A_n - C_n)^+. \tag{13}$$

The first two examples assume that $\{A_n\}, \{C_n\}$ are two i.i.d. sequences; the third example takes $\{J_n\}$ to be a Markov chain; and, in the last example, $\{J_n\}$ is modulated by a Markov chain $\{X_n\}$.

Example 2.1. If $\{A_n\}, \{C_n\}$ follow exponential distributions, $\mathbb{P}[C_n > x] = e^{-\mu x}$, $\mathbb{P}[A_n > x] = e^{-\lambda x}$, and $\lambda < \mu$, then Q_n represents the waiting time in the $M/M/1$ queue. By Theorem 9.1 of Asmussen [5], the stationary waiting time in the $M/M/1$ queue is distributed as

$$\mathbb{P}[Q > x] = \frac{\lambda}{\mu} e^{-(\mu-\lambda)x}, \quad x \geq 0,$$

which equivalently yields a power law distribution for M°,

$$\mathbb{P}[M^\circ > x] = \mathbb{P}[Q > \log x] = \frac{\lambda}{\mu x^{\mu-\lambda}}, \quad x \geq 1$$

with power exponent $\alpha = \mu - \lambda$.

Example 2.2. If $\{A_n\}, \{C_n\}$ are two i.i.d Bernoulli processes with $\mathbb{P}[A_n = 1] = 1 - \mathbb{P}[A_n = 0] = p$, $\mathbb{P}[C_n = 1] = 1 - \mathbb{P}[C_n = 0] = q$, $p < q$. Then, the elementary queueing/Markov chain theory shows that the stationary distribution of Q_n, as defined in (13), is geometric $\mathbb{P}[Q \geq j] = (1-\rho)\rho^j$, $j \geq 0$, where $\rho = p(1-q)/(q(1-p)) < 1$. Therefore,

$$\mathbb{P}[M^\circ \geq x] = \mathbb{P}[Q \geq \log x] = \rho^{\lfloor \log x\rfloor}, \quad x \geq 1.$$

Because $\log x - 1 < \lfloor \log x \rfloor \leq \log x$, it is easy to conclude that

$$\frac{1}{x^{\log(1/\rho)}} \leq \mathbb{P}[M^\circ \geq x] < \frac{1}{\rho x^{\log(1/\rho)}}.$$

Example 2.3. If $\{J_n\}$ is a Markov chain taking values in a finite set Σ of positive reals and possessing an irreducible transition matrix $Q = (q(i,j))_{i,j\in\Sigma}$, then the function Ψ defined in Theorem 2.1 can be explicitly computed. To this end, define matrix Q_α with elements

$$q_\alpha(i,j) = q(i,j)j^\alpha, \quad i,j \in \Sigma.$$

By Theorem 3.1.2 of Dembo and Zeitouni [24], we have as $n \to \infty$

$$n^{-1} \log \mathbb{E}[(\Pi_n^\circ)^\alpha] \to \log(\mathrm{dev}(Q_\alpha)),$$

where $\mathrm{dev}(Q_\alpha)$ is the Perron-Frobenius eigenvalue of matrix Q_α. To illustrate this result, we take $\Sigma = \{u, d\}$ where $u = 1/d > 1$, and $q(d,u) = q$, $q(d,d) = 1-q$, $q(u,d) = p$, $q(u,u) = 1-p$ where $p > q$. It is easy to compute

$$Q_\alpha = \begin{pmatrix} (1-p)u^\alpha & pd^\alpha \\ qu^\alpha & (1-q)d^\alpha \end{pmatrix},$$

and, letting $\log(\mathrm{dev}(Q_\alpha)) = 0$, we obtain

$$\alpha^* = \frac{\log(1-q) - \log(1-p)}{\log u}.$$

Example 2.4 (Double Pareto). If $\{J_n \equiv J(X_n)\}$ is modulated by a Markov chain X_n, we argue that $\mathbb{P}[M^\circ > x]$ can have different asymptotic decay rates over multiple time scales. This phenomenon was investigated in Jelenković and Lazar [38] in the queueing context and formulated as Theorem 3 therein. To visualize this phenomenon, we study the following example. Consider a Markov process X_n of two states (say $\{1,2\}$) with transition probabilities $p_{12} = 1/5000$, $p_{21} = 1/10$, and $\mathbb{P}[J(1) = 1.2] = 1 - \mathbb{P}[J(1) = 0.6] = 0.5$, $\mathbb{P}[J(2) = 1.7] = 1 - \mathbb{P}[J(2) = 0.25] = 0.6$. The corresponding simulation result for 5×10^7 trials is presented in Figure 1. We observe from this figure a double Pareto distribution for M°, which provides a new explanation to the origins of double Pareto distributions as compared to the one in Reed and Jorgensen [68].

Remark 2.4. For reasons of simplicity, we have chosen $\{J_n\}$ in all of the preceding examples to be Markovian. However, Theorem 2.1 extends beyond the Markovian framework, e.g., $\{J_n\}$ can be a semi-Markov process where the periods of (sojourn) time that the process spends in a state are asymptotically exponential but not necessarily memoryless.

2.2. Multiplicative processes with absorbing barriers and cycle maximum. As briefly discussed in §1, we explained that the reflective nature of the barrier is not essential for producing power law distributions. Indeed, one only needs a positive lower barrier, e.g., porous, absorbing, or reflective, which is a natural condition because no physical objects or socioeconomic ones can approach zero arbitrarily close without repelling from it

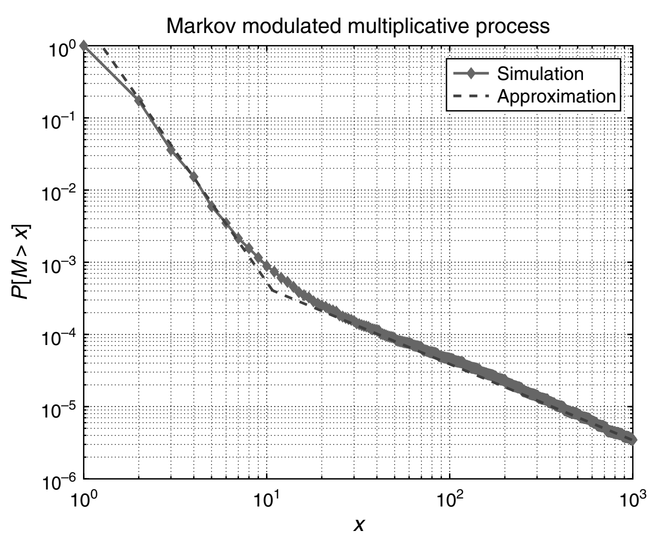

Figure 1. Illustration for Example 2.4 of the double Pareto distribution.

or simply disappearing. To illustrate the situations when the objects can vanish, we name a few examples, e.g., cities may degenerate, bankruptcy protection may sometimes fail, and a company can be liquidated. In these cases, the power law effect follows from the well-known queueing result on the cycle maximum that is stated in Theorem 2.2 (next). We also discuss in §5.2 a more complicated situation when newly generated objects in the system can arrive/appear or leave/disappear.

For a sequence of positive i.i.d. random variables $\{J, J_n\}_{n\geq 1}$, define the stopping time $\tau \triangleq \inf\{n\colon \prod_{i=1}^{n} J_i \leq 1, n \geq 1\}$ with the corresponding cycle maximum $M_\tau^\circ \triangleq \sup\{\prod_{i=1}^{n} J_i\colon 1 \leq n \leq \tau\}$.

Theorem 2.2. *If the sequence* $\{\log J_n\}_{n\geq 1}$ *is nonlattice, satisfying* $\mathbb{E}[J^{\alpha^*}] = 1$ *for some* $\alpha^* > 0$ *and* $(\mathbb{E}[J^\alpha])'|_{\alpha=\alpha^*} < \infty$, *then*

$$\lim_{x\to\infty} x^{\alpha^*} \mathbb{P}[M_\tau^\circ > x] = c > 0.$$

Proof. This result follows from Corollary 5.9 in Asmussen [5, p. 368]. □

3. Main results. This section presents our main results in Theorems 3.1 and 3.2. To avoid technical difficulties, we assume $\underline{\mu} \triangleq \inf_j \mu(j) > 0$. Recall that the process $\{J_n\}$ on positive integers is assumed to be stationary and ergodic, and define $\Pi_n = \prod_{i=-n}^{-1} \mu(J_i)$, $n \geq 1$, $\Pi_0 = l$, and $M = \sup_{n\geq 0} \Pi_n$. In this paper, we use the following standard notation. For any two real functions $a(t)$ and $b(t)$, we use $a(t) = o(b(t))$ to denote that $\lim_{t\to\infty}(a(t)/b(t)) = 0$, and $a(t) = O(b(t))$ to denote that $\overline{\lim}_{t\to\infty}(a(t)/b(t)) < \infty$; when needed for increased clarity, we may explicitly write $a(t) = o(b(t))$ as $t \to \infty$.

Theorem 3.1. *Assume that the process* $\{\mu(J_n)\}_{n>-\infty}$ *satisfies the polynomial Gärtner-Ellis conditions* (*conditions* (1)–(3) *of Theorem* 2.1 *with respect to* $\{\Pi_n\}_{n\geq 1}$), *and* $\sup_j \mathbb{E}[e^{\theta|B(j)-\mu(j)|}] < \infty$ *for some* $\theta > 0$. *Then,*

$$\lim_{x\to\infty} \frac{\log \mathbb{P}[\Lambda > x]}{\log x} = \lim_{x\to\infty} \frac{\log \mathbb{P}[M > x]}{\log x} = -\alpha^*. \tag{14}$$

Remark 3.1. Note that conditions (1) and (2) of Theorem 2.1 imply that there exists j such that $\mu(j) > 1$, because otherwise we have $\sup_\alpha \Psi(\alpha) \leq 0$, which would contradict $\Psi(\alpha^*) = 0$ and $\Psi'(\alpha^*) > 0$ in condition (2). The following theorem covers the opposite situation when the previous condition is not satisfied, i.e., $\sup_j \mu(j) < 1$.

Theorem 3.2. *If* $\sup_j \mu(j) < 1$ *and* $\sup_j \mathbb{E}[e^{\theta|B(j)-\mu(j)|}] < \infty$ *for some* $\theta > 0$, *then,*

$$\lim_{x\to\infty} \frac{\log \mathbb{P}[\Lambda > x]}{\log x} = -\infty. \tag{15}$$

Remark 3.2. Informally speaking, Theorems 3.1 and 3.2 show that the alternating periods of contractions and expansions, e.g., economic booms and recessions, are primarily responsible for the appearance of power law distributions. In other words, if there are no periods of expansions, i.e., the condition $\sup_j \mu(j) < 1$ of Theorem 3.2 is satisfied, then Λ has a tail that is lighter than any power law distribution. Furthermore, the first equality in (14) of Theorem 3.1 reveals a general asymptotic equivalence between the reflected modulated branching process and the corresponding reflected multiplicative process, showing that the power law exponent α^* is insensitive to the higher order distributional properties of $B(j)$ beyond the conditional mean $\mu(j)$.

Remark 3.3. A careful examination of the proofs reveals that the existence of a uniform upper bound of the exponential moments for $|B(j) - \mu(j)|$ could possibly be relaxed to $\sup_j \mathbb{E}[|B(j) - \mu(j)|^\alpha] < \infty$ for $\alpha > \alpha^*$. However, such an extension would considerably complicate the proofs. Furthermore, in most practical applications, the distributions of $\{B(j)\}$ are typically very concentrated. For the preceding reasons, we do not consider such extensions.

We present the proofs of Theorems 3.1 and 3.2 in §6.1.

4. Exact asymptotics. This section presents the exact asymptotics of the RMPs and RMBPs in the following two subsections, respectively.

4.1. Exact asymptotics of RMPs and the double Pareto phenomenon. The following two theorems essentially provide a new general explanation of the measured double Pareto phenomenon (e.g., see Mitzenmacher [59] and Reed and Jorgensen [68]) because they rely on two universal statistical laws, the first based on the large deviation theory and the second implied by the central limit theorem.

The theorems are direct translations from the corresponding queueing theory results. Theorem 4.1 is based on the large deviation result that studies the situation when M° is large, and Theorem 4.2 is derived from the heavy traffic approximation of the GI/GI/1 queue where we study the limiting behavior of a sequence of multiplicative processes with the multiplicative drift tending to one. Theorems 4.1 and 4.2 are corollaries of Theorem 5.2 in Chapter XIII and Theorem 7.1 in Chapter X of Asmussen [5], respectively.

For a sequence of positive i.i.d. random variables $\{J, J_n\}_{n>-\infty}$, let $\{S_n = \sum_{i=1}^n \log J_i\}_{n\geq 1}$ and $\tau_+ = \inf\{n \geq 1: S_n > 0\}$, and define G_+ to be the ladder height distribution $G_+(x) = \mathbb{P}[S_{\tau_+} \leq x, \tau_+ < \infty]$, $x > 0$, with $\|G_+\| = \mathbb{P}[\tau_+ < \infty] = \mathbb{P}[S_n > 0$ for some $n \geq 1]$. Similarly, as in §2.1, we assume without loss of generality that the absorbing barrier is equal to one.

THEOREM 4.1. *If the sequence $\{\log J_n\}_{n>-\infty}$ is nonlattice, satisfying $\mathbb{E}[J^{\alpha^*}] = 1$ for some $\alpha^* > 0$ and $(\mathbb{E}[J^\alpha])'|_{\alpha=\alpha^*} < \infty$, then*

$$\lim_{x\to\infty} x^{\alpha^*}\,\mathbb{P}[M^\circ > x] = \frac{1-\|G_+\|}{\alpha^* \int_0^\infty u e^{\alpha^* u} G_+(du)}.$$

PROOF. The result is a direct consequence of Theorem 5.3 in Chapter XIII of Asmussen [5]. □

REMARK 4.1. For the case when S_n is lattice valued, see Remark 5.4 of Chapter XIII on p. 366 of Asmussen [5].

Now, we study a sequence of multiplicative processes indexed by an integer k, where $J_n^{(k)}$, $S_n^{(k)}$, and $M^{(k)}$ are properly defined for all $k \geq 1$, in the limit as k goes to infinity.

THEOREM 4.2. *If $\{J^{(k)}, J_n^{(k)}\}_{n\geq 1}$ are positive and i.i.d. for each fixed k with $m_k \triangleq \mathbb{E}[\log J^{(k)}]$, $\sigma_k^2 \triangleq \mathrm{Var}[\log J^{(k)}]$, the random walks $\{S_n^{(k)} = \sum_{i=1}^n \log J_i^{(k)}\}_{n\geq 1}$ satisfy $m_k < 0$, $\lim_{k\to\infty} m_k = 0$, $\underline{\lim}_{k\to\infty} \sigma_k^2 > 0$, and the family of random variables $(\log J^{(k)})^2$ is uniformly integrable, then, for $y \geq 1$,*

$$\lim_{k\to\infty} \mathbb{P}\left[(M^{(k)})^{-m_k/\sigma_k^2} > y\right] = 1/y^2,$$

where $M^{(k)} = \sup_{n\geq 0} \prod_{i=1}^n J_i^{(k)}$ with the convention $\prod_{i=1}^0 J_i^{(k)} = 1$.

PROOF. Observing that $\log M^{(k)} = \sup_{n\geq 0} S_n^{(k)}$ with $S_0^{(k)} \equiv 0$ and using Theorem 7.1 in Chapter X in Asmussen [5, p. 287], we obtain, for $z \geq 0$,

$$\lim_{k\to\infty} \mathbb{P}\left[-\frac{m_k}{\sigma_k^2}\log M^{(k)} > z\right] = e^{-2z},$$

which, letting $z = \log y$, finishes the proof of Theorem 4.2. □

4.2. Exact asymptotics of reflected branching processes. In this subsection, assuming that $\{J, J_n\}_{n\geq 1}$ are i.i.d. and $\{\log \mu(J)\}$ is nonlattice, we will give exact asymptotics for RMBPs using the implicit renewal theorem of Goldie (1991); see Theorem 2.3 and Corollary 2.4 in Goldie [32]. To this end, let $\{B(j), B^i(j)\}_{i,j}$ be independent random variables that are independent of $\{J, J_n\}$ and satisfy $B^i(j) \stackrel{d}{=} B(j)$.

THEOREM 4.3. *If $\sup_j \mathbb{E}[e^{\theta|B(j)-\mu(j)|}] < \infty$ for some $\theta > 0$, $\mathbb{E}[\mu(J)^{\alpha^*}] = 1$ for some $\alpha^* > 0$ and $\mathbb{E}[\mu(J)^{\alpha^*+\delta}] < \infty$ for some $\delta > 0$, then,*

$$\lim_{x\to\infty} x^{\alpha^*}\,\mathbb{P}[\Lambda > x] = \frac{\mathbb{E}[(\Lambda^*)^{\alpha^*} - (\mu(J)\Lambda)^{\alpha^*}]}{\alpha^*\,\mathbb{E}[\mu(J)^{\alpha^*}\log \mu(J)]}, \tag{16}$$

where $\Lambda^* \triangleq \max(\sum_{i=1}^\Lambda B^i(J), l)$ and Λ is independent of J and $\{B^i(j)\}_{i,j}$.

The proof of Theorem 4.3 is presented in §6.2.

REMARK 4.2. The preceding result is implicit because the constant on the right-hand side of Equation (16) depends on Λ, which is what we are trying to compute. In principle, to derive the explicit exact asymptotics for RMBPs is a difficult problem because the asymptotic constant depends on the behavior around the boundary l. However, in the scaling region where the boundary l grows as well (albeit slowly), one can derive an explicit asymptotic characterization; see the extended Internet version of this paper (Jelenković and Tan [45]).

5. Discussion of related models. Based on the study of reflected modulated branching processes, we address three related models: randomly stopped processes, modulated branching processes with absorbing barriers, and truncated power laws.

5.1. Randomly stopped processes. In this subsection, we discuss randomly stopped multiplicative and branching processes, respectively.

5.1.1. Randomly stopped multiplicative processes. The following two theorems show that randomly stopped multiplicative processes and reflected multiplicative processes are closely related and are, to a certain extent, equivalent under more restrictive conditions. Following the approach of Chapter VIII of Asmussen [5], we study the ladder heights of a multiplicative process. For any RMP with i.i.d positive multiplicative increments, the random variable M°, as defined in Lemma 2.3, can be represented in terms of the ladder heights. To this end, define $\Pi_n^{\circ 0} \triangleq \prod_{i=1}^n J_i$, $n \geq 1$, $\Pi_0^{\circ 0} = 1$ and let $\{H_i\}_{i\geq 1}$ be the i.i.d. ascending ladder height process of the random walk $\{S_n = \sum_{i=1}^n \log J_i\}_{n\geq 1}$ with $\mathbb{P}[H_i \leq x] = G_+(x)/\|G_+\|$, $x \geq 0$, $\|G_+\| = G_+(\infty) < 1$, and $G_+(x)$ is the same as defined in front of Theorem 4.1; let $H_i^e \triangleq e^{H_i}$.

Theorem 5.1. *Suppose that $\{J, J_n\}_{n>-\infty}$ is a positive i.i.d. sequence with $\mathbb{E}[\log J] < 0$. Then,*

$$M^\circ \stackrel{d}{=} \prod_{i=1}^N H_i^e \tag{17}$$

with $\prod_{i=1}^0 H_i^e = 1$, where N is independent of $\{H_i^e\}_{i\geq 1}$ and satisfies $\mathbb{P}[N \geq n] = \|G_+\|^n$.

Proof. Based on the well-known ladder height representation (see Chapter VIII of Asmussen [5])

$$\log M^\circ \stackrel{d}{=} \sum_{i=1}^N H_i$$

with $\sum_{i=1}^0 H_i = 0$, where N is independent of $\{H_i\}$ with $\mathbb{P}[N \geq n] = \|G_+\|^n$, it immediately follows that

$$\mathbb{P}[M^\circ > x] = \mathbb{P}\left[e^{\sum_{i=1}^N H_i} > x\right] = \mathbb{P}\left[\prod_{i=1}^N H_i^e > x\right]. \quad \square$$

Conversely, we can prove that if the observation time N has an exponential tail, the stopped process $\Pi_N^{\circ 0}$ has a power law tail under quite general conditions as shown in Theorem 5.2 (next). Note that here we do not require $\{J_n\}$ to be an i.i.d. sequence.

Theorem 5.2. *Let N be an integer random variable independent of $\{J_n\}$ with*

$$\lim_{x\to\infty} \frac{\log \mathbb{P}[N > x]}{x} = -\lambda < 0.$$

For a positive ergodic and stationary process $\{J_n\}_{n\geq 0}$, if $n^{-1} \log \mathbb{E}[(\Pi_n^{\circ 0})^\alpha] \to \Psi(\alpha) < \infty$ as $n \to \infty$ in a neighborhood of $\alpha^* > 0$, $\Psi(\alpha)$ is differentiable at α^* with $\Psi(\alpha^*) = \lambda$, $\Psi'(\alpha^*) > 0$, and $\mathbb{E}[(\Pi_n^{\circ 0})^{\alpha^*}] < \infty$ for $n \geq 1$, then

$$\lim_{x\to\infty} \frac{\log \mathbb{P}[\Pi_N^{\circ 0} > x]}{\log x} = -\alpha^*. \tag{18}$$

Remark 5.1. Theorem 5.2 generalizes the previous results from Huberman and Adamic [34], Reed [65], and Reed and Hughes [67] where only i.i.d. multiplicative increments are considered.

Theorem 5.3 (next) shows that randomly stopped multiplicative processes and reflected multiplicative processes are basically equivalent under more restrictive conditions. This equivalence is established using classical results on the $M/GI/1$ queue. In this regard, we assume that $\{J_n\}_{n>-\infty}$ is an i.i.d. process, $\Pi_n^{\circ 0}$ is the corresponding multiplicative process, N is a geometric random variable that is independent of $\Pi_n^{\circ 0}$ with $\mathbb{P}[N \geq n] = \rho^n$, $0 < \rho < 1$, and $\bar{G}(t)$, $t \geq 0$ is a complementary distribution function (i.e., there exists a random variable $S \geq 0$ such that $\bar{G}(t) = \mathbb{P}[S \geq t]$).

THEOREM 5.3. *If a random variable N with $\mathbb{P}[N \geq n] = \rho^n$, $n \geq 0$, $0 < \rho < 1$ is independent of $\{J_n\}$, where J_1 satisfies*

$$\mathbb{P}[\log J_1 \leq x] = \int_0^x \bar{G}(y)\,dy \Big/ \int_0^\infty \bar{G}(y)\,dy, \quad x \geq 0$$

for some complementary distribution function $\bar{G}(\cdot)$. Then, we can always construct a stationary RMP such that $\Pi_N^{\circ 0} \stackrel{d}{=} M^\circ$. In addition, if $\bar{G}(\cdot)$ is nonlattice and $\int_0^\infty e^{\alpha^* y}\bar{G}(y)\,dy = \rho^{-1}\int_0^\infty \bar{G}(y)\,dy$, $\int_0^\infty y e^{\alpha^* y}\bar{G}(y)\,dy < \infty$ for $\alpha^* > 0$, then

$$\lim_{x\to\infty} x^{\alpha^*}\mathbb{P}[M^\circ > x] = \lim_{x\to\infty} x^{\alpha^*}\mathbb{P}[\Pi_N^{\circ 0} > x] = \frac{(1-\rho)\int_0^\infty \bar{G}(y)\,dy}{\alpha^*\rho\int_0^\infty y e^{\alpha^* y}\bar{G}(y)\,dy}.$$

The proofs of Theorems 5.2 and 5.3 are presented in §6.3.

5.1.2. Randomly stopped branching processes. In the following theorem, we extend Theorem 5.2 of §5.1.1 to the context of randomly stopped branching processes. Recall $\Pi_n^0 \triangleq \prod_{i=0}^n \mu(J_i)$.

THEOREM 5.4. *Suppose that N is independent of $B_n^i(j) \geq 1$ for all n, i, j. Then, under the same conditions as in Theorem 5.2 (replacing Π_n° by Π_n^0 therein) with $\mathbb{E}[(\Pi_n^0)^{\alpha^*}] < \infty$ for $n \geq 1$ and $\Psi(\alpha)$ being differentiable in a neighborhood of $\alpha^* > 0$, we obtain, for $\{Z_n\}_{n\geq 0}$ defined in (1) with a bounded initial value $Z_0 < z_0 < \infty$,*

$$\lim_{x\to\infty}\frac{\log \mathbb{P}[Z_N > x]}{\log x} = \lim_{x\to\infty}\frac{\log \mathbb{P}[\Pi_N^0 > x]}{\log x} = -\alpha^*.$$

The proof of Theorem 5.4 is based on arguments similar to the proof of Theorem 3.1, presented in §6.3.

5.2. Branching processes with absorbing barriers. For many dynamic processes (e.g., city sizes), quite often when the sizes of the objects fall below a threshold, the whole object disappears (e.g., urban decay). Therefore, it is natural to study branching processes with absorbing barriers. As discussed in §2.2, we know that a single object with an absorbing barrier can result in power law distributions based on the duality with the queueing cycle maximum. In this context, we can also study a more complicated situation where the newly generated objects can join the system and evolve together. This naturally models the arrivals to popular websites (hot spots) because information (news) is distributed according to a branching process, e.g., user A passes the information to B and C. Furthermore, B may inform D, etc. Empirical examination shows that Web requests follow power law distributions (see Huberman and Adamic [35] and Adamic and Huberman [1]). For a complete discussion of this model, please refer to the longer Internet version of this paper (Jelenković and Tan [45]).

5.3. Truncated power laws. Truncated power laws have been observed empirically in many practical situations where the studied objects have natural upper boundaries. Here, we want to point out that by using the duality between the modulated branching processes and the queueing theory, one easily obtains truncated power laws when both a lower and an upper barrier are added to the modulated branching process. To illustrate this point, recall that the $M/M/1/b$ queue with a finite buffer b results in a truncated geometric distribution for the number of customers in the queue. By the duality, it essentially follows that, in a proportional growth world with both a lower and an upper barrier, truncated power laws can naturally arise and play a similar role as that of truncated exponential/geometric distributions in an additive world. Prior related work on this subject can be found in Sornette and Cont [71].

6. Proofs.

6.1. Proofs of Theorems 3.1 and 3.2. The proof of Theorem 3.1, composed of the upper bound and the lower bound, and the proof of Theorem 3.2 are presented in the following three subsections, respectively.

6.1.1. Proof of Theorem 3.1: Upper bound. Because the proof is based on the change (increase) of boundary l, we denote this dependence explicitly as $\Lambda^l \equiv \Lambda$. According to Lemma 2.2, the initial value of $\{\Lambda_n\}$ has no impact on Λ; therefore, in this subsection, we simply assume that $\Lambda_0^l = l$. Before stating the proof of the upper bound, we establish preliminary Lemmas 6.1, 6.2, 6.3, and 6.4.

The first lemma shows that, most likely, the supremum of Z_n occurs for an index $n \leq x$.

Lemma 6.1. *For any $\beta > 0$, the branching process Z_n^l defined in (1) satisfies*

$$\sum_{n>x}^{\infty} \mathbb{P}[Z_n^l > x] = O\left(\frac{1}{x^\beta}\right) \quad \text{as } x \to \infty.$$

Proof. Similarly, as in the proof of Lemma 2.1, note that for $\Pi_{n-1}^0 = \prod_{i=0}^{n-1} \mu(J_i)$, the stochastic process $W_n = Z_n^l/\Pi_{n-1}^0$, $n \geq 1$ is a martingale with respect to the filtration $\mathscr{F}_n = \sigma(J_i, Z_i, 0 \leq i \leq n-1)$ that satisfies $\mathbb{E}[W_1] = l$. Therefore, recalling $\Pi_n = \prod_{i=-n}^{-1} \mu(J_i)$, we obtain, for any $\varepsilon > 0$,

$$\begin{aligned}\mathbb{P}[Z_n^l > x] &= \mathbb{P}[W_n \Pi_{n-1}^0 > x] = \mathbb{P}[(W_n e^{-\varepsilon n})(\Pi_{n-1}^0 e^{\varepsilon n}) > x] \\ &\leq \mathbb{P}[W_n e^{-\varepsilon n} > 1] + \mathbb{P}[\Pi_n e^{\varepsilon n} > x] \\ &\leq \mathbb{E}[W_n e^{-\varepsilon n}] + \mathbb{P}[\Pi_n e^{\varepsilon n} > x]. \end{aligned} \tag{19}$$

Next, using the martingale property $\mathbb{E}[W_n] = \mathbb{E}[W_1] = l$, we derive

$$\sum_{n>x}^{\infty} \mathbb{E}[W_n e^{-\varepsilon n}] = l \sum_{n>x}^{\infty} e^{-\varepsilon n} \leq \frac{l e^{-\varepsilon x}}{1 - e^{-\varepsilon}} = O\left(\frac{1}{x^\beta}\right) \quad \text{as } x \to \infty. \tag{20}$$

Then, recalling conditions (1) and (2) of Theorem 3.1 (or Theorem 2.1), we can choose $\delta, \varepsilon > 0$ small enough and n_0 large enough such that $\Psi(\alpha^* - \delta) + 2\varepsilon(\alpha^* - \delta) = -\zeta < 0$ and $n^{-1} \log \mathbb{E}[\Pi_n^{(\alpha^*-\delta)}] < \Psi(\alpha^* - \delta) + \varepsilon(\alpha^* - \delta)$ for $n > n_0$, which implies, for $x > n_0$,

$$\begin{aligned}\sum_{n>x}^{\infty} \mathbb{P}[\Pi_n e^{\varepsilon n} > x] &\leq \sum_{n>x}^{\infty} \frac{\mathbb{E}[\Pi_n^{(\alpha^*-\delta)}] e^{\varepsilon(\alpha^*-\delta)n}}{x^{(\alpha^*-\delta)}} \leq \sum_{n>x}^{\infty} \frac{e^{-\zeta n}}{x^{\alpha^*-\delta}} \\ &\leq \frac{e^{-\zeta x}}{(1 - e^{-\zeta}) x^{\alpha^*-\delta}} = O\left(\frac{1}{x^\beta}\right) \quad \text{as } x \to \infty. \end{aligned} \tag{21}$$

Finally, using (19), (20), and (21), we complete the proof. □

The following lemma relates Λ_n to the corresponding multiplicative process.

Lemma 6.2. *Let $\varepsilon > 0$ and Λ_n^l be the reflected branching process as defined in (2). Then, for $x \geq l$,*

$$\mathbb{P}[\Lambda_n^l > x] \leq \mathbb{P}\left[\max_{1 \leq j \leq n} \Pi_j (1+\varepsilon)^j > x/l\right] + n\,\mathbb{P}[\mathscr{B}_0^{l,\varepsilon}],$$

where $\Pi_j = \prod_{i=-j}^{-1} \mu(J_i)$ and $\mathscr{B}_n^{l,\varepsilon} = \bigcup_{j \geq l} \{\sum_{i=1}^{j} B_n^i(J_n) > j\mu(J_n)(1+\varepsilon)\}$ for any integer n.

Proof. From (6), we have

$$\Lambda_n^l \stackrel{d}{=} \max(Z_{-n}^l, Z_{-(n-1)}^l, \ldots, Z_{-1}^l, Z_0^l). \tag{22}$$

Next, let $Z_{-n}^l(k)$ be the branching process that starts at time $-n$ with l objects and is observed at time $k \geq -n$. Note that $Z_{-i}^l(-i) = l$, $Z_{-i}^l(0) \equiv Z_{-i}^l$ and

$$Z_{-j}^l = \sum_{i=1}^{Z_{-j}^l(-1)} B_{-1}^i(J_{-1})$$

for $j \geq 1$. Now, using the preceding observation, (22), and $Z_0^l = l$, we derive, for $x \geq l$,

$$\begin{aligned}\mathbb{P}[\Lambda_n^l > x] &\leq \mathbb{P}[\max(Z_{-n}^l, Z_{-(n-1)}^l, \ldots, Z_{-1}^l, l) > x, (\mathscr{B}_{-1}^{l,\varepsilon})^\complement] + \mathbb{P}[\mathscr{B}_{-1}^{l,\varepsilon}] \\ &\leq \mathbb{P}[\max(Z_{-n}^l(-1)(1+\varepsilon)\mu(J_{-1}), \ldots, Z_{-2}^l(-1)(1+\varepsilon)\mu(J_{-1}), l(1+\varepsilon)\mu(J_{-1})) > x, (\mathscr{B}_{-1}^{l,\varepsilon})^\complement] + \mathbb{P}[\mathscr{B}_{-1}^{l,\varepsilon}] \\ &\leq \mathbb{P}\left[\left\{(1+\varepsilon)\mu(J_{-1}) \max_{2 \leq i \leq n}(Z_{-i}^l(-1)) > x\right\} \cup \{\mu(J_{-1})(1+\varepsilon) > x/l\}\right] + \mathbb{P}[\mathscr{B}_{-1}^{l,\varepsilon}], \end{aligned}$$

where $\mathscr{X}^\complement$ denotes the complement of $\mathscr{X}$.

Then, intersecting with event $\mathscr{B}_{-2}^{l,\varepsilon}$ and using

$$Z_{-j}^l(-1) = \sum_{i=1}^{Z_{-j}^l(-2)} B_{-2}^i(J_{-2})$$

for $j \geq 2$, one easily obtains

$$\mathbb{P}[\Lambda_n^l > x] \leq \mathbb{P}\Big[\Big\{(1+\varepsilon)^2\mu(J_{-2})\mu(J_{-1})\max_{3\leq i\leq n}(Z_{-i}^l(-2)) > x\Big\} \cup \{\max(\mu(J_{-2})\mu(J_{-1})(1+\varepsilon)^2, \mu(J_{-1})(1+\varepsilon)) > x/l\}\Big] + \mathbb{P}[\mathscr{B}_{-2}^{l,\varepsilon}] + \mathbb{P}[\mathscr{B}_{-1}^{l,\varepsilon}],$$

which, continuing the induction and using $\mathbb{P}[\mathscr{B}_i^{l,\varepsilon}] = \mathbb{P}[\mathscr{B}_0^{l,\varepsilon}]$ for all i, finishes the proof. □

Now, we show that the "error" event $\mathscr{B}_0^{l,\varepsilon}$ in Lemma 6.2 has a negligible probability for large l relative to any power law distribution.

Lemma 6.3. *If* $\sup_j \mathbb{E}[e^{\theta|B(j)-\mu(j)|}] < \infty$, $\theta > 0$ *and* $\underline{\mu} = \inf_j \mu(j) > 0$, *then, setting* $l_x = \lfloor x^\delta \rfloor$, $\delta > 0$ *in the definition of* $\mathscr{B}_0^{l_x,\varepsilon}$ *in Lemma 6.2, we obtain, for any* $\beta > 0$,

$$\mathbb{P}[\mathscr{B}_0^{l_x,\varepsilon}] = O\left(\frac{1}{x^\beta}\right) \quad \text{as } x \to \infty.$$

Proof. First, we derive

$$\begin{aligned} P(n) &\triangleq \mathbb{P}\left[\sum_{i=1}^n B_0^i(J_0) > \mu(J_0)(1+\varepsilon)n\right] \\ &\leq \mathbb{P}\left[\sum_{i=1}^n (B_0^i(J_0) - \mu(J_0)) > \varepsilon\underline{\mu}n\right] \\ &\leq \mathbb{E}\big[\big(\mathbb{E}\big[e^{\zeta(B(J)-\mu(J))} \mid J\big]\big)^n\big]e^{-\zeta\varepsilon\underline{\mu}n}, \quad \zeta > 0, \end{aligned}$$

which, using the elementary inequality $e^t \leq 1 + t + t^2 e^{|t|}/2$, $t \in \mathbb{R}$ and setting $t = B(J) - \mu(J)$, yields

$$P(n) \leq \mathbb{E}\left[\left(1 + \frac{\zeta^2}{2}\mathbb{E}[(B(J)-\mu(J))^2 e^{\zeta|B(J)-\mu(J)|}|J]\right)^n\right]e^{-\zeta\varepsilon\underline{\mu}n}.$$

For any $\eta > 0$ and large enough n such that $\zeta = \eta\log n/(\varepsilon\underline{\mu}n) < \theta$, the assumption $\sup_j \mathbb{E}[e^{\theta|B(j)-\mu(j)|}] < \infty$ implies

$$\mathbb{E}[(B(J)-\mu(J))^2 e^{\zeta|B(J)-\mu(J)|}|J] < C < \infty,$$

which yields

$$P(n) \leq \left(1 + \frac{C(\eta\log n)^2}{2(\varepsilon\underline{\mu}n)^2}\right)^n n^{-\eta} = O\left(\frac{1}{n^\eta}\right). \tag{23}$$

Therefore, choosing $\eta = 1 + \beta/\delta$ in (23), we obtain, for $l_x = \lfloor x^\delta \rfloor$, $\delta > 0$ and any $\beta > 0$, as $x \to \infty$,

$$\mathbb{P}[\mathscr{B}_0^{l_x,\varepsilon}] \leq \sum_{i=l_x}^\infty P(n) \leq O\left(\sum_{n=\lfloor x^\delta\rfloor}^\infty \frac{1}{n^\eta}\right) = O\left(\frac{1}{x^\beta}\right). \quad \square$$

The following lemma allows us to increase the lower barrier in order to prove the upper bound.

Lemma 6.4. *Assume that* $\Lambda_n^{l_1}$ *and* $\Lambda_n^{l_2}$ *are defined on the same sequence* $\{B_n^j(J_n)\}$ *with initial conditions* l_1 *and* l_2, *respectively. If* $l_1 \geq l_2$, *then, for all* $n \geq 0$,

$$\Lambda_n^{l_1} \geq \Lambda_n^{l_2}.$$

Proof. The result holds trivially for $n = 0$. Now, we prove the result using induction. Suppose that it is true for all $0 \leq k \leq n$, and for $k = n+1$,

$$\Lambda_{n+1}^{l_1} = \max\left(\sum_{i=1}^{\Lambda_n^{l_1}} B_n^i(J_n), l_1\right) \geq \max\left(\sum_{i=1}^{\Lambda_n^{l_2}} B_n^i(J_n), l_2\right) = \Lambda_{n+1}^{l_2},$$

which implies that Lemma 6.4 is true for all $n \geq 0$. □

Now, we are ready to complete the proof of the upper bound.

Proof of the Upper Bound of Theorem 3.1. Choosing $l_x = \lfloor x^\varepsilon \rfloor \geq l, 0 < \varepsilon < 1$, using Lemma 6.4 and then Lemma 6.2, we derive

$$\begin{aligned}
\mathbb{P}[\Lambda^l > x] &= \mathbb{P}\left[\sup_{j\geq 1} Z^l_{-j} > x\right] \leq \mathbb{P}[\Lambda^l_{\lfloor x \rfloor} > x] + \mathbb{P}\left[\sup_{j>x} Z^l_{-j} > x\right] \\
&\leq \mathbb{P}[\Lambda^{l_x}_{\lfloor x \rfloor} > x] + \sum_{j>x} \mathbb{P}[Z^l_j > x] \\
&\leq \mathbb{P}\left[\sup_{j\geq 1} \Pi_j (1+\varepsilon)^j > x^{1-\varepsilon}\right] + x\,\mathbb{P}[\mathscr{B}^{l_x,\,\varepsilon}_0] + \sum_{j>x} \mathbb{P}[Z^l_j > x] \\
&\triangleq I_1(x) + I_2(x) + I_3(x).
\end{aligned} \tag{24}$$

Now, define a new process $\{\mu^\varepsilon(J_n) = \mu(J_n)(1+\varepsilon)\}_{n\geq 1}$ and $\Pi^\varepsilon_n = \prod_{i=-n}^{-1} \mu^\varepsilon(J_i)$. Then, for ε small enough, we have

(1) $n^{-1} \log \mathbb{E}(\Pi^\varepsilon_n)^\alpha \to \Psi^\varepsilon(\alpha) = \Psi(\alpha) + \alpha \log(1+\varepsilon)$ as $n \to \infty$ for $|\alpha - \alpha^*| < \varepsilon^*$,

(2) Ψ^ε is finite in a neighborhood of α^*_ε, $\alpha^*_\varepsilon < \alpha^*$, and differentiable at α^*_ε with $\Psi(\alpha^*_\varepsilon) + \alpha^*_\varepsilon \log(1+\varepsilon) = 0$, $\Psi'(\alpha^*_\varepsilon) > 0$, and

(3) $\mathbb{E}[(\Pi^\varepsilon_n)^{\alpha^*_\varepsilon}] < \infty$ for $n \geq 1$.

Therefore, by Theorem 2.1, we obtain

$$\lim_{x\to\infty} \frac{\log \mathbb{P}[\sup_{i\geq 1} \Pi_i (1+\varepsilon)^i > x^{1-\varepsilon}]}{\log x} = -(1-\varepsilon)\alpha^*_\varepsilon, \tag{25}$$

which, in conjunction with Lemmas 6.1 and 6.3, yields

$$I_2(x) + I_3(x) = o(I_1(x)). \tag{26}$$

Then, combining (24), (25), and (26) yields

$$\frac{\log \mathbb{P}[\Lambda^l > x]}{\log x} \leq \frac{\log((1+o(1))I_1(x))}{\log x} \longrightarrow -(1-\varepsilon)\alpha^*_\varepsilon \quad \text{as } x \to \infty.$$

Because $\Psi^\varepsilon(\alpha)$ is continuous in a neighborhood of α^* in both α and ε, we derive

$$\lim_{\varepsilon\to 0} \alpha^*_\varepsilon = \alpha^*,$$

implying

$$\overline{\lim_{x\to\infty}} \frac{\log \mathbb{P}[\Lambda > x]}{\log x} \leq -\alpha^*. \quad \square \tag{27}$$

6.1.2. Proof of Theorem 3.1: Lower bound. In order to prove the lower bound, we need to establish the following three lemmas. Specifically, Corollary 6.1 allows us to obtain a lower bound for Λ while, perhaps somewhat counterintuitively, increasing the lower barrier l.

Lemma 6.5. *Let $\{\Lambda^{y_1}_n\}$ and $\{\Lambda^{y_2}_n\}$ be defined on the same modulating sequence $\{J_n\}_{n\geq 0}$ and independent random variables $\{B^{i,1}_n(j), B^{i,2}_n(j)\}$, respectively, with $B^{i,k}_n(j)$ identically distributed for fixed j. Then,*

$$\Lambda^{y_1+y_2}_n \overset{d}{\leq} \Lambda^{y_1}_n + \Lambda^{y_2}_n,$$

where $\Lambda^{y_1}_n$ and $\Lambda^{y_2}_n$ are conditionally independent given $\{J_n\}_{n\geq 0}$, and $\overset{d}{\leq}$ stands for inequality in distribution.

Proof. We use induction to prove Lemma 6.5. Starting with $n = 1$, we obtain

$$\begin{aligned}
\Lambda^{y_1+y_2}_1 &= \max\left(\sum_{i=1}^{y_1+y_2} B^i_0(J_0), y_1+y_2\right) \overset{d}{=} \max\left(\sum_{i=1}^{y_1} B^{i,1}_0(J_0) + \sum_{i=1}^{y_2} B^{i,2}_0(J_0), y_1+y_2\right) \\
&\leq \max\left(\sum_{i=1}^{y_1} B^{i,1}_0(J_0), y_1\right) + \max\left(\sum_{i=1}^{y_2} B^{i,2}_0(J_0), y_2\right) \\
&\overset{d}{=} \Lambda^{y_1}_1 + \Lambda^{y_2}_1
\end{aligned}$$

because, for any x_1, x_2, y_1, y_2,

$$\max(x_1 + x_2, y_1 + y_2) \le \max(x_1, y_1) + \max(x_2, y_2).$$

The proof is completed by induction in n:

$$\Lambda_{n+1}^{y_1+y_2} = \max\left(\sum_{i=1}^{\Lambda_n^{y_1+y_2}} B_1^i(J_n), y_1 + y_2\right) \overset{d}{\le} \max\left(\sum_{i=1}^{\Lambda_n^{y_1}+\Lambda_n^{y_2}} B_1^i(J_n), y_1 + y_2\right)$$
$$\overset{d}{\le} \Lambda_{n+1}^{y_1} + \Lambda_{n+1}^{y_2}. \quad \square$$

Next, a straightforward application of Lemma 6.5 yields the following corollary.

Corollary 6.1. *If $\{\Lambda_{n,j}^1\}_{1\le j\le y}$ are conditionally i.i.d. copies of Λ_n^1 given $\{J_i\}_{1\le i\le n}$, then*

$$\Lambda_n^y \overset{d}{\le} \sum_{j=1}^{y} \Lambda_{n,j}^1.$$

Now, we show that the supremum of Π_i occurs most likely for small indexes $i \le O(\log x)$.

Lemma 6.6. *Assume that condition* (1) *of Theorem* 3.1 *is satisfied, then, for $0 \le \varepsilon < 1$ and any $\beta > 0$, there exists $h > 0$ such that, when $x \to \infty$,*

$$\mathbb{P}\left[\sup_{i>h\log x} \Pi_i (1-\varepsilon)^i > x\right] = O\left(\frac{1}{x^\beta}\right).$$

Proof. Using condition (1) of Theorem 3.1, we can choose $0 < \alpha < \alpha^*$ with $n^{-1}\log \mathbb{E}[\Pi_n^\alpha] \to \Psi(\alpha) < 0$ and n_0 large enough such that $\mathbb{E}[\Pi_n^\alpha] < \zeta^n$ for some $0 < \zeta < 1$ and all $n > n_0$. Thus, for $h = -\beta/\log\zeta > 0$ and $x > e^{n_0/h}$,

$$\mathbb{P}\left[\sup_{i>h\log x} \Pi_i (1-\varepsilon)^i > x\right] \le \sum_{i>h\log x}^{\infty} \mathbb{P}[\Pi_i > x] \le \sum_{i>h\log x}^{\infty} \frac{\mathbb{E}[\Pi_i^\alpha]}{x^\alpha} \le \sum_{i>h\log x}^{\infty} \frac{\zeta^i}{x^\alpha} = O\left(\frac{1}{x^\beta}\right). \quad \square$$

Finally, the last lemma shows that $\sum_{i=1}^{j} B_n^i(J_n)$ cannot deviate too much from $j\mu(J_n)$ for large j.

Lemma 6.7. *Under the assumptions of Lemma* 6.3, *any $0 < \delta,\ \varepsilon < 1$, and $\mathscr{C}_n^{l,\varepsilon} \triangleq \bigcup_{j\ge l}\{\sum_{i=1}^{j} B_n^i(J_n) < j\mu(J_n)(1-\varepsilon)\}$, we obtain, for any $\beta > 0$,*

$$\mathbb{P}\left[\mathscr{C}_0^{\lfloor x^\delta\rfloor,\varepsilon}\right] = O\left(\frac{1}{x^\beta}\right).$$

Proof. The proof of Lemma 6.7 is basically the same as Lemma 6.3. Observe

$$P(n) \triangleq \mathbb{P}\left[\sum_{i=1}^{n} B_0^i(J_0) < \mu(J_0)(1-\varepsilon)n\right]$$
$$\le \mathbb{P}\left[\sum_{i=1}^{n} (B_0^i(J_0) - \mu(J_0)) < -\varepsilon\mu(J_0)n\right]$$
$$\le \mathbb{P}\left[\sum_{i=1}^{n} (\mu(J_0) - B_0^i(J_0)) > \varepsilon\underline{\mu} n\right].$$

Then, using similar large deviation arguments as in deriving (23), we can prove, for any $\beta > 0$,

$$\mathbb{P}\left[\mathscr{C}_0^{\lfloor x^\delta\rfloor,\varepsilon}\right] = O\left(\frac{1}{x^\beta}\right). \quad \square$$

Next, we can complete the proof of the lower bound of Theorem 3.1.

Proof of the Lower Bound of Theorem 3.1. First, using Corollary 6.1, we obtain, for any integer $y \geq 1$,

$$\mathbb{P}[\Lambda_n^l > x] \geq \mathbb{P}[\Lambda_n^1 > x] = \frac{y\,\mathbb{P}[\Lambda_n^1 > x]}{y} \geq \frac{\mathbb{P}[\sum_{j=1}^{y}\Lambda_{n,j}^1 > yx]}{y} \geq \frac{\mathbb{P}[\Lambda_n^y > yx]}{y}. \tag{28}$$

Now, using (6) similarly as in the proof of Lemma 6.2, for $0<\varepsilon<1$ and $\mathscr{C}_n^{l,\varepsilon}$ defined in Lemma 6.7, we derive

$$\begin{aligned}
\mathbb{P}[\Lambda_n^y > yx] &\geq \mathbb{P}\left[\max_{0\leq i\leq n}(Z_{-i}^y) > yx, \left(\bigcup_{i=-n}^{-1}\mathscr{C}_i^{y,\varepsilon}\right)^{\complement}\right]\\
&\geq \mathbb{P}\left[\sup_{1\leq i\leq n}\Pi_i(1-\varepsilon)^i > x, \left(\bigcup_{i=-n}^{-1}\mathscr{C}_i^{y,\varepsilon}\right)^{\complement}\right]\\
&\geq \mathbb{P}\left[\sup_{1\leq i\leq n}\Pi_i(1-\varepsilon)^i > x\right] - n\,\mathbb{P}[\mathscr{C}_0^{y,\varepsilon}]\\
&\geq \mathbb{P}\left[\sup_{i\geq 1}\Pi_i(1-\varepsilon)^i > x\right] - \mathbb{P}\left[\sup_{i>n}\Pi_i(1-\varepsilon)^i > x\right] - n\,\mathbb{P}[\mathscr{C}_0^{y,\varepsilon}]\\
&\triangleq I_1(x) - I_2(x) - I_3(x).
\end{aligned} \tag{29}$$

Note that $\{I_j(x)\}_{1\leq j\leq 3}$ here are different from those in (24).

Next, similarly as in the proof of the upper bound, define a new process $\{\mu_\varepsilon(J_n) = \mu(J_n)(1-\varepsilon)\}_{n\geq 1}$ and let $\Pi_n^\varepsilon = \prod_{i=-n}^{-1}\mu_\varepsilon(J_i)$. Then, for ε small enough, we have

(1) $n^{-1}\log\mathbb{E}(\Pi_n^\varepsilon)^\alpha \to \Psi^\varepsilon(\alpha) = \Psi(\alpha) + \alpha\log(1-\varepsilon)$ as $n\to\infty$ for $|\alpha-\alpha^*| < \varepsilon^*$,

(2) $\Psi^\varepsilon(\alpha)$ is finite in a neighborhood of α_ε^*, $\alpha_\varepsilon^* > \alpha^*$ and differentiable at α_ε^* with $\Psi(\alpha_\varepsilon^*) + \alpha_\varepsilon^*\log(1-\varepsilon) = 0$, $\Psi'(\alpha_\varepsilon^*) > 0$, and

(3) $\mathbb{E}[(\Pi_n^\varepsilon)^{\alpha_\varepsilon^*}] < \infty$ for $n\geq 1$.

Therefore, by Theorem 2.1, we obtain

$$\lim_{x\to\infty}\frac{\log\mathbb{P}[\sup_{i\geq 1}\Pi_i(1-\varepsilon)^i > x]}{\log x} = -\alpha_\varepsilon^*. \tag{30}$$

Now, setting $y = \lfloor x^\delta\rfloor$, $0<\delta<1$, $n=\lfloor x\rfloor$ in (28), (29), and using Lemmas 6.6 and 6.7, it is easy to see that

$$I_2(x) + I_3(x) = o(I_1(x)),$$

which, by (28) and (29), yields

$$\begin{aligned}
\log\mathbb{P}[\Lambda > x] &\geq \log\mathbb{P}[\Lambda_n^l > x]\\
&\geq \log(I_1(x) - I_2(x) - I_3(x)) - \delta\log x\\
&= \log((1-o(1))I_1(x)) - \delta\log x.
\end{aligned}$$

From the preceding inequality and (30), we obtain

$$\varliminf_{x\to\infty}\frac{\log\mathbb{P}[\Lambda > x]}{\log x} \geq -\alpha_\varepsilon^* - \delta. \tag{31}$$

Because $\Psi^\varepsilon(\alpha)$ is continuous in a neighborhood of α^* in both α and ε, we have $\lim_{\varepsilon\to 0}\alpha_\varepsilon^* = \alpha^*$. Then, passing $\varepsilon, \delta\to 0$ in (31) completes the proof of the lower bound, which, in conjunction with (27), finishes the proof of Theorem 3.1. □

6.1.3. Proof of Theorem 3.2. Using the same arguments as in deriving (24) in the proof of the upper bound of Theorem 3.1, we obtain, for $l_x = \lfloor x\rfloor \geq l$ and $0<\varepsilon<1$,

$$\begin{aligned}
\mathbb{P}[\Lambda > x] &\leq \mathbb{P}[\Lambda_{\lfloor x\rfloor}^l > x] + \mathbb{P}\left[\sup_{j>x} Z_{-j}^l > x\right]\\
&\leq \mathbb{P}[\Lambda_{\lfloor x\rfloor}^{l_x} > x] + \sum_{j>x}\mathbb{P}[Z_j^l > x]\\
&\leq \mathbb{P}\left[\sup_{j\geq 1}\Pi_j(1+\varepsilon)^j > 1\right] + x\,\mathbb{P}\left[\mathscr{B}_0^{l_x,\varepsilon}\right] + \sum_{j>x}^{\infty}\mathbb{P}[Z_j^l > x]\\
&\triangleq I_1(x) + I_2(x) + I_3(x).
\end{aligned} \tag{32}$$

Recalling $\Pi_j = \prod_{i=-1}^{-j} \mu(J_i)$ and noting $\sup_j \mu(j) < 1$, we can choose $\varepsilon > 0$ such that $\sup_j \mu(j)(1+\varepsilon) < 1$, which implies $I_1(x) = 0$. Additionally, by Lemma 6.3, we obtain $I_2(x) = O(x^{-\beta})$ for all $\beta > 0$.

Next, using similar arguments as in deriving (19) in the proof of Lemma 6.1, we obtain, for $\varepsilon > 0$ and $j \geq 1$,

$$\mathbb{P}[Z_j^l > x] \leq \mathbb{E}[W_j e^{-\varepsilon j}] + \mathbb{P}[\Pi_j e^{\varepsilon j} > x],$$

which, recalling $\sup_j \mu(j) < 1$ and choosing ε small enough such that $\mathbb{P}[\Pi_j e^{\varepsilon j} > x] = 0$ for $x > 1$, yields

$$I_3(x) \leq \sum_{j > \lfloor x \rfloor}^{\infty} \mathbb{E}[W_j e^{-\varepsilon j}] = l \sum_{j > \lfloor x \rfloor}^{\infty} e^{-\varepsilon j} = O(e^{-\varepsilon x}).$$

Finally, combining (32) and the bounds on $I_1(x)$, $I_2(x)$, and $I_3(x)$ finishes the proof. □

6.2. Proof of Theorem 4.3. In order to prove Theorem 4.3, we first derive the following lemma.

LEMMA 6.8. *Under the assumptions of Theorem 4.3, there exists $\gamma > 0$ such that*

$$\mathbb{E}\left[\left|\sum_{i=1}^{\Lambda}(B^i(J) - \mu(J))\right|^{\alpha^*+\gamma}\right] < \infty.$$

PROOF. We observe that, for $x > 0$,

$$\mathbb{P}\left[\left|\sum_{i=1}^{\Lambda}(B^i(J) - \mu(J))\right| > x\right] = \mathbb{P}\left[\sum_{i=1}^{\Lambda}(B^i(J) - \mu(J)) > x\right] + \mathbb{P}\left[\sum_{i=1}^{\Lambda}(\mu(J) - B^i(J)) > x\right] \triangleq I_1 + I_2.$$

We choose γ and δ such that $0 < \gamma < \alpha^*$ and $\gamma < \delta < \alpha^*$. To evaluate I_1 for $0 < \varepsilon < (\alpha^* - \delta)/2$, we set $0 < \beta \triangleq (\delta + \varepsilon)/(\alpha^* - \varepsilon) < 1$ and obtain

$$I_1 \leq \mathbb{P}[\Lambda > x^{1+\beta}] + \mathbb{P}\left[\sum_{i=1}^{\Lambda}(B^i(J) - \mu(J)) > x, \Lambda \leq x^{1+\beta}\right], \tag{33}$$

which, recalling that Theorem 3.1 implies $\mathbb{P}[\Lambda > x] = O(x^{-(\alpha^* - \varepsilon)})$, results in

$$\mathbb{P}[\Lambda > x^{1+\beta}] = O(x^{-(\alpha^* + \delta)}). \tag{34}$$

Now, we study the second probability on the right-hand side of (33). Using the fact that J is independent of Λ and applying Chernoff bound, we obtain, for $\zeta > 0$,

$$\begin{aligned}\mathbb{P}\left[\sum_{i=1}^{\Lambda}(B^i(J) - \mu(J)) > x, \Lambda \leq x^{1+\beta}\right] &= \sum_{n=l}^{\lfloor x^{1+\beta} \rfloor} \mathbb{P}[\Lambda = n]\mathbb{P}\left[\sum_{i=1}^{n}(B^i(J) - \mu(J)) > x\right] \\ &\leq \sum_{n=l}^{\lfloor x^{1+\beta} \rfloor} \mathbb{P}[\Lambda = n]\mathbb{E}[(\mathbb{E}[e^{\zeta(B(J)-\mu(J))} \mid J])^n]e^{-\zeta x}. \end{aligned} \tag{35}$$

Then, setting $t = \zeta(B(J) - \mu(J))$ in (35), using $e^t \leq 1 + t + t^2 e^{|t|}/2$, $t \in \mathbb{R}$, and observing that $\mathbb{E}[B(J) - \mu(J)] = 0$, (35) is further upper bounded by

$$\sum_{n=l}^{\lfloor x^{1+\beta} \rfloor} \mathbb{P}[\Lambda = n]\mathbb{E}\left[\left(1 + \frac{\zeta^2}{2}\mathbb{E}[(B(J) - \mu(J))^2 e^{\zeta|B(J)-\mu(J)|} \mid J]\right)^n\right]e^{-\zeta x}.$$

Next, $\sup_j \mathbb{E}[e^{\theta|B(j)-\mu(j)|}] < \infty$ implies $\mathbb{E}[(B(J) - \mu(J))^2 e^{\zeta|B(J)-\mu(J)|} \mid J] < C < \infty$, $\zeta < \theta$. Hence, for x large, we have $\zeta = (\alpha^* + \delta)\log x/x < \theta$, which implies that (35) is bounded by

$$\sum_{n=l}^{\lfloor x^{1+\beta} \rfloor} \mathbb{P}[\Lambda = n]\left(1 + \frac{C((\alpha^* + \delta)\log x)^2}{2x^2}\right)^{x^{1+\beta}} x^{-(\alpha^*+\delta)} = O(x^{-(\alpha^*+\delta)}) \tag{36}$$

because $\beta < 1$. Combining (34), (35), and (36) proves

$$I_1 = O(x^{-(\alpha^*+\delta)}). \tag{37}$$

Using the same approach as in proving (37), we can also show

$$I_2 = O(x^{-(\alpha^*+\delta)}),$$

which implies

$$\mathbb{P}\left[\left|\sum_{i=1}^{\Lambda}(B^i(J)-\mu(J))\right| > x\right] = O(x^{-(\alpha^*+\delta)}).$$

Therefore, because $\delta > \gamma$,

$$\mathbb{E}\left[\left|\sum_{i=1}^{\Lambda}(B^i(J)-\mu(J))\right|^{\alpha^*+\gamma}\right] = \int_0^\infty \mathbb{P}\left[\left|\sum_{i=1}^{\Lambda}(B^i(J)-\mu(J))\right|^{\alpha^*+\gamma} > x\right]dx$$
$$= O\left(\int_1^\infty x^{-(\alpha^*+\delta)(\alpha^*+\gamma)}\,dx\right) < \infty,$$

which finishes the proof of Lemma 6.8. □

Now, we proceed with proving Theorem 4.3.

Proof of Theorem 4.3. The proof is based on Corollary 2.4 in Goldie [32] for which it is sufficient to show

$$I \triangleq \mathbb{E}\left[\left|\left(\max\left(\sum_{i=1}^{\Lambda}B^i(J), l\right)\right)^{\alpha^*} - (\mu(J)\Lambda)^{\alpha^*}\right|\right] < \infty. \tag{38}$$

In order to prove the preceding inequality, we will use the following elementary inequality (see Equation (9.27) in Goldie [32]) for $x, y \geq 0$:

$$|x^\alpha - y^\alpha| \leq \begin{cases} |x-y|^\alpha, & 0 < \alpha \leq 1 \\ \alpha|x-y|(x^{\alpha-1}+y^{\alpha-1}), & 1 < \alpha < \infty. \end{cases} \tag{39}$$

Also, using the inequality $|\max(x, y) - z| \leq y + |x - z|$ for $x, y, z \geq 0$, we know

$$I \leq l^{\alpha^*} + \mathbb{E}\left[\left|\left(\sum_{i=1}^{\Lambda}B^i(J)\right)^{\alpha^*} - (\mu(J)\Lambda)^{\alpha^*}\right|\right]. \tag{40}$$

First, we prove the case when $0 < \alpha^* \leq 1$. By (39), (40), and Lemma 6.8, we obtain

$$I \leq l^{\alpha^*} + \mathbb{E}\left[\left|\sum_{i=1}^{\Lambda}(B^i(J)-\mu(J))\right|^{\alpha^*}\right] < \infty. \tag{41}$$

Next, we prove the case when $\alpha^* > 1$. Applying (39) and (40), we obtain

$$\begin{aligned} I &\leq l^{\alpha^*} + \mathbb{E}\left[\left|\left(\sum_{i=1}^{\Lambda}B^i(J)\right)^{\alpha^*} - (\mu(J)\Lambda)^{\alpha^*}\right|\right] \\ &\leq l^{\alpha^*} + \alpha^*\,\mathbb{E}\left[\left|\sum_{i=1}^{\Lambda}(B^i(J)-\mu(J))\right|\left|\sum_{i=1}^{\Lambda}B^i(J)\right|^{\alpha^*-1}\right] \\ &\quad + \alpha^*\,\mathbb{E}\left[\left|\sum_{i=1}^{\Lambda}(B^i(J)-\mu(J))\right|\left|\Lambda\mu(J)\right|^{\alpha^*-1}\right] \\ &\triangleq l^{\alpha^*} + I_1 + I_2. \end{aligned} \tag{42}$$

For I_1, we use Hölder's inequality to obtain, for $0 < \varepsilon < 1$,

$$\begin{aligned} I_1 &\leq \alpha^*\,\mathbb{E}\left[\left|\sum_{i=1}^{\Lambda}(B^i(J)-\mu(J))\right|^{\alpha^*/(1-\varepsilon)}\right]^{(1-\varepsilon)\alpha^*} \mathbb{E}\left[\left(\sum_{i=1}^{\Lambda}B^i(J)\right)^{((\alpha^*-1)\alpha^*)/(\alpha^*+\varepsilon-1)}\right]^{(\alpha^*+\varepsilon-1)/\alpha^*} \\ &\leq \alpha^*\,\mathbb{E}\left[\left|\sum_{i=1}^{\Lambda}(B^i(J)-\mu(J))\right|^{\alpha^*/(1-\varepsilon)}\right]^{(1-\varepsilon)/\alpha^*} \mathbb{E}\left[\Lambda^{((\alpha^*-1)\alpha^*)/(\alpha^*+\varepsilon-1)}\right]^{(\alpha^*+\varepsilon-1)/\alpha^*}, \end{aligned} \tag{43}$$

where the last inequality uses the fact that

$$\sum_{i=1}^{\Lambda} B^i(J) \le \max\left(\sum_{i=1}^{\Lambda} B^i(J), l\right) \stackrel{d}{=} \Lambda.$$

Now, Theorem 3.1 implies that

$$\mathbb{E}[\Lambda^{((\alpha^*-1)\alpha^*)/(\alpha^*+\varepsilon-1)}] < \infty,$$

which, using (43), choosing ε small enough, and applying Lemma 6.8, results in

$$I_1 < \infty. \tag{44}$$

Using the same argument as in proving (43) and noting that Λ and $\mu(J)$ are independent from each other, we obtain

$$I_2 \le \alpha^* \mathbb{E}\left[\left|\sum_{i=1}^{\Lambda}(B^i(J)-\mu(J))\right|^{\alpha^*/(1-\varepsilon)}\right]^{(1-\varepsilon)/\alpha^*} \mathbb{E}[\Lambda^{((\alpha^*-1)\alpha^*)/(\alpha^*+\varepsilon-1)}]^{(\alpha^*+\varepsilon-1)/\alpha^*} \cdot \mathbb{E}[\mu(J)^{((\alpha^*-1)\alpha^*)/(\alpha^*+\varepsilon-1)}]^{(\alpha^*+\varepsilon-1)\alpha^*} < \infty,$$

which, in conjunction with (44), proves (38) and finishes the proof of Theorem 4.3. □

6.3. Proofs of Theorems 5.2, 5.3, and 5.4.

Proof of Theorem 5.2. First, we prove the *upper bound*. For a fixed α that is in the neighborhood of α^* and $0<\varepsilon<\lambda$, there exists n_ε such that $\mathbb{E}[(\Pi_n^{\circ 0})^\alpha] < e^{(\Psi(\alpha)+\varepsilon)n}$ and $e^{-(\lambda-\varepsilon)n} > \mathbb{P}[N\ge n] > e^{-(\lambda+\varepsilon)n}$ for all $n\ge n_\varepsilon$. Because $\Psi(\alpha^*)=\lambda$ and $\Psi'(\alpha^*)>0$, we can choose $\delta,\varepsilon>0$ small enough such that $\Psi(\alpha^*-\delta)-\lambda+2\varepsilon = -\xi<0$. Thus, noting that N is independent of $\Pi_n^{\circ 0}$, we obtain

$$\begin{aligned}\mathbb{P}[\Pi_n^{\circ 0}>x] &= \sum_{n=1}^{\infty}\mathbb{P}[N=n]\mathbb{P}[\Pi_n^{\circ 0}>x]\\ &\le \sum_{n=1}^{n_\varepsilon}\mathbb{P}[N=n]\mathbb{P}[\Pi_n^{\circ 0}>x] + \sum_{n=n_\varepsilon}^{\infty}\mathbb{P}[N\ge n]\mathbb{P}[\Pi_n^{\circ 0}>x]\\ &\le \sum_{n=1}^{n_\varepsilon}\mathbb{P}[N=n]\frac{\mathbb{E}[(\Pi_n^{\circ 0})^{\alpha^*}]}{x^{\alpha^*}} + \sum_{n=n_\varepsilon}^{\infty} e^{-(\lambda-\varepsilon)n}\frac{\mathbb{E}[(\Pi_n^{\circ 0})^{\alpha^*-\delta}]}{x^{\alpha^*-\delta}}\\ &\le O\left(\frac{1}{x^{\alpha^*}}\right) + \frac{1}{x^{\alpha^*-\delta}}\sum_{n=n_\varepsilon}^{\infty} e^{-\xi n},\end{aligned}$$

which implies

$$\overline{\lim_{x\to\infty}}\frac{\log\mathbb{P}[\Pi_N^{\circ 0}>x]}{\log x} = -\alpha^*+\delta.$$

Passing $\delta\to 0$ in the preceding equality completes the proof of the upper bound.

Next, we prove the *lower bound* by applying the standard exponential change of measure argument. For $0<3\varepsilon<\lambda$, $\delta>2\varepsilon/(\lambda-3\varepsilon)$, and $\log x > n_\varepsilon$, and recalling that $e^{-(\lambda-\varepsilon)n} > \mathbb{P}[N\ge n] > e^{-(\lambda+\varepsilon)n}$, we obtain, for large x,

$$\begin{aligned}\mathbb{P}\left[\frac{(1+\delta)\log x}{\Psi'(\alpha^*)} \le N \le \frac{(1+2\delta)\log x}{\Psi'(\alpha^*)}\right] &\ge e^{-((\lambda+\varepsilon)(1+\delta)\log x)/(\Psi'(\alpha^*))} - e^{-((\lambda-\varepsilon)(1+2\delta)\log x)/(\Psi'(\alpha^*))}\\ &\ge (1-\varepsilon)e^{-((\lambda+\varepsilon)(1+\delta)\log x)(\Psi'(\alpha^*))}\end{aligned}$$

because $(\lambda+\varepsilon)(1+\delta) < (\lambda-\varepsilon)(1+2\delta)$ by our choice of δ. This implies that there exists $\delta\le\zeta\le 2\delta$ such that $n_x = \lceil(1+\zeta)(\log x)/\Psi'(\alpha^*)\rceil$ satisfies

$$\mathbb{P}[N=n_x] \ge \frac{(1-\varepsilon)\Psi'(\alpha^*)e^{-(\lambda+\varepsilon)(1+\delta)\log x/\Psi'(\alpha^*)}}{\delta\log x}. \tag{45}$$

Therefore, using (45) and denoting $\log J_i$ by X_i, we obtain

$$\begin{aligned}\mathbb{P}[\Pi_n^{\circ 0} > x] &\geq \mathbb{P}[N = n_x]\,\mathbb{P}\left[\sum_{i=1}^{n_x} \log J_i > \log x\right]\\ &\geq \frac{(1-\varepsilon)\Psi'(\alpha^*)e^{-(\lambda+\varepsilon)(1+\delta)\log x/\Psi'(\alpha^*)}}{\delta \log x}\,\mathbb{P}\left[\sum_{i=1}^{n_x} X_i > \frac{\Psi'(\alpha^*)}{1+\delta} n_x\right]. \end{aligned} \tag{46}$$

Next, we perform an exponential change of measure for the probability on the right-hand side of (46). Let $\mathbb{P}_n^*$ be the probability measure on $\mathbb{R}^n$ defined by the probability measure $\mathbb{P}$ of the stationary and ergodic process $\{X_i\}_{i\geq 1}$:

$$\mathbb{P}_n^*(dx_1, \dots, dx_n) = e^{\alpha^* S_n - \Psi_n(\alpha^*)}\,\mathbb{P}(dx_1, \dots, dx_n),$$

where $S_n = \sum_{i=1}^n X_i$ and $\Psi_n(\alpha) \triangleq \log \mathbb{E}[e^{\alpha S_n}]$, satisfying $n^{-1}\Psi_n(\alpha) \to \Psi(\alpha)$ in the neighborhood of α^*. Thus,

$$\begin{aligned}\mathbb{P}\left[\sum_{i=1}^n X_i > \frac{\Psi'(\alpha^*)}{1+\delta} n\right] &= \mathbb{E}_n^*\left[e^{-\alpha^* S_n + \Psi_n(\alpha^*)}\,\mathbf{1}\left(S_n > \frac{\Psi'(\alpha^*)}{1+\delta} n\right)\right]\\ &\geq \mathbb{E}_n^*\left[e^{-\alpha^* S_n + \Psi_n(\alpha^*)}\,\mathbf{1}\left(\left|\frac{S_n}{n} - \Psi'(\alpha^*)\right| < \frac{\Psi'(\alpha^*)\delta}{1+\delta}\right)\right]\\ &\geq e^{-\alpha^*((1+2\delta)\Psi'(\alpha^*))/(1+\delta)n + \Psi_n(\alpha^*)}\,\mathbb{P}_n^*\left[\left|\frac{S_n}{n} - \Psi'(\alpha^*)\right| < \frac{\Psi'(\alpha^*)\delta}{1+\delta}\right]. \end{aligned} \tag{47}$$

Then, by Claim 1 on p. 17 of Bucklew [14], we know that

$$\mathbb{P}_n^*\left[\left|\frac{S_n}{n} - \Psi'(\alpha^*)\right| < \frac{\Psi'(\alpha^*)\delta}{1+\delta}\right] \to 1 \quad \text{as } x \to \infty,$$

which, using (46) and setting $n = n_x$ in (47), yields

$$\varliminf_{x\to\infty} \frac{\log \mathbb{P}[\Pi_n^{\circ 0} > x]}{\log x} \geq -\frac{(\lambda+\varepsilon)(1+\delta)}{\Psi'(\alpha^*)} - \frac{\alpha^*(1+2\delta)^2}{1+\delta} + \frac{(1+\delta)\Psi(\alpha^*)}{\Psi'(\alpha^*)}.$$

Finally, passing $\varepsilon, \delta \to 0$ in the preceding equality and noting $\Psi(\alpha^*) = \lambda$, we prove the lower bound. □

Proof of Theorem 5.3. We give a constructive proof based on the connection (duality) between the $M/GI/1$ queue and the geometrically stopped multiplicative process.

Consider the $M/GI/1$ queue with the service distribution $\mathbb{P}[S \geq t] = \bar{G}(t), t \geq 0$ and Poisson arrivals of rate $\lambda = \rho/\mathbb{E}[S], \mathbb{E}[S] < \infty$. Then, by the Pollaczeck-Khinchine formula (see, e.g., Theorem 5.7 on p. 237 of Asmussen [5]), the stationary workload Q of this $M/GI/1$ queue is equal in distribution to $\sum_{i=1}^N H_i$, where $N, \{H_i\}_{i\geq 1}$ are independent with $\mathbb{P}[N \geq n] = \rho^n, n \geq 0$ and

$$\mathbb{P}[H_i \leq x] = \frac{\int_0^x \mathbb{P}[S \geq s]\,ds}{\mathbb{E}[S]} = \frac{\int_0^x \bar{G}(s)\,ds}{\int_0^\infty \bar{G}(s)\,ds} = \mathbb{P}[\log J_i \leq x], \quad x \geq 0,$$

where the last equality follows from the assumption. Now, using the preceding observation, we show that there exists an RMP such that $M^\circ = e^Q$ satisfies

$$\mathbb{P}[M^\circ > x] = \mathbb{P}[Q > \log x] = \mathbb{P}\left[\sum_{i=1}^N H_i > \log x\right] = \mathbb{P}\left[\sum_{i=1}^N \log J_i > \log x\right] = \mathbb{P}[\Pi_N^{\circ 0} > x], \tag{48}$$

which proves the first claim of Theorem 5.3.

Next, using the additional assumptions of Theorem 5.3, it is easy to show that the Cramér-Lundberg condition for the $M/GI/1$ queue $\mathbb{E}(e^{\alpha^* S})\lambda/(\lambda + \alpha^*) = 1$ is satisfied and, thus, by applying Theorem 5.3 in Chapter XIII and Theorem 5.7 in Chapter VIII of Asmussen [5]), we obtain

$$\lim_{x\to\infty} \mathbb{P}[M^\circ > x]x^{\alpha^*} = \frac{(1-\rho)\int_0^\infty \bar{G}(y)\,dy}{\alpha^*\rho \int_0^\infty y e^{\alpha^* y}\bar{G}(y)\,dy},$$

which, by (48), completes the proof. □

PROOF OF THEOREM 5.4. The second equality is implied by Theorem 5.2, and we only need to prove the first one. We begin with proving the *upper bound*. Recalling the definition of $\mathcal{B}_n^{l,\varepsilon}$ in Lemma 6.2 and, for $n \geq 1$, $0 < \varepsilon$, $\xi < 1$, choosing $x^{\xi} > z_0 > Z_0$, we obtain

$$\begin{aligned}\mathbb{P}[Z_n^{Z_0} > x] &\leq \mathbb{P}\left[Z_n^{\lfloor x^{\xi}\rfloor} > x\right]\\ &\leq \mathbb{P}\left[Z_n^{\lfloor x^{\xi}\rfloor} > x, \bigcap_{i=0}^{n-1}(\mathcal{B}_i^{\lfloor x^{\xi}\rfloor,\varepsilon})^{\complement}\right] + \mathbb{P}\left[\bigcup_{i=0}^{n-1}\mathcal{B}_i^{\lfloor x^{\xi}\rfloor,\varepsilon}\right]\\ &\leq \mathbb{P}[\Pi_n^0(1+\varepsilon)^n > x^{1-\xi}] + n\,\mathbb{P}[\mathcal{B}_0^{\lfloor x^{\xi}\rfloor,\varepsilon}],\end{aligned}$$

which, by the independence of N and $\{B_n^i(j), J_n\}$, implies

$$\mathbb{P}[Z_N > x] \leq \mathbb{P}\left[\Pi_N^0(1+\varepsilon)^N > x^{1-\xi}\right] + \mathbb{E}[N]\,\mathbb{P}\left[\mathcal{B}_0^{\lfloor x^{\xi}\rfloor,\varepsilon}\right]. \tag{49}$$

Next, define a new process $\{\Pi_n^{\varepsilon} = \Pi_n^0(1+\varepsilon)^n\}$. It is easy to see that, for ε small enough, the sequence $\{\Pi_n^{\varepsilon}\}$ satisfies $n^{-1}\log\mathbb{E}[(\Pi_n^{\varepsilon})^{\alpha}] \to \Psi(\alpha) + \alpha\log(1+\varepsilon)$. Therefore, by Theorem 5.2, we obtain

$$\lim_{x\to\infty}\frac{\log\mathbb{P}[\Pi_N^0(1+\varepsilon)^N > x^{1-\xi}]}{\log x} = -(1-\xi)\alpha_{\varepsilon}^*, \tag{50}$$

where α_{ε}^* satisfies $\Psi(\alpha_{\varepsilon}^*) + \alpha_{\varepsilon}^*\log(1+\varepsilon) = 0$. Combining (49), (50), and Lemma 6.3, we obtain

$$\overline{\lim_{x\to\infty}}\frac{\log\mathbb{P}[Z_N > x]}{\log x} \leq -(1-\xi)\alpha_{\varepsilon}^*,$$

which, passing $\varepsilon, \xi \to 0$, completes the proof of the upper bound.

Now, we prove the *lower bound*. Let $\{Z_{n,j}^1\}$ be i.i.d. copies of $\{Z_n^1\}$ given the common modulating process $\{J_n\}$. Then, noting that $Z_n^y \stackrel{d}{=} \sum_{j=1}^{y} Z_{n,j}^1$ for integer y and using the union bound, we derive, for $0 < \xi < 1$, $n \geq 0$,

$$\mathbb{P}[Z_n > x] \geq \frac{\lfloor x^{\xi}\rfloor}{x^{\xi}}\mathbb{P}[Z_n^1 > x] \geq \frac{1}{x^{\xi}}\mathbb{P}\left[Z_n^{\lfloor x^{\xi}\rfloor} > x\lfloor x^{\xi}\rfloor\right].$$

Hence, recalling the definition of $\mathcal{C}_n^{l,\varepsilon}$ in Lemma 6.7, we obtain

$$\begin{aligned}\mathbb{P}[Z_n > x] &\geq \frac{1}{x^{\xi}}\mathbb{P}\left[Z_n^{\lfloor x^{\xi}\rfloor} > x\lfloor x^{\xi}\rfloor, \bigcap_{i=0}^{n-1}\left(\mathcal{C}_i^{\lfloor x^{\xi}\rfloor,\xi}\right)^{\complement}\right]\\ &\geq \frac{1}{x^{\xi}}\left(\mathbb{P}[\Pi_n^0(1-\xi)^n > x] - n\,\mathbb{P}\left[\mathcal{C}_0^{\lfloor x^{\xi}\rfloor,\xi}\right]\right),\end{aligned}$$

which, by the independence of N and $\{B_n^i(j), J_n\}$, yields

$$\mathbb{P}[Z_N > x] \geq \frac{1}{x^{\xi}}\left(\mathbb{P}[\Pi_N^0(1-\xi)^N > x] - \mathbb{E}[N]\,\mathbb{P}\left[\mathcal{C}_0^{\lfloor x^{\xi}\rfloor,\xi}\right]\right).$$

Then, by using the same approach as in the proof of the upper bound and Lemma 6.7, we can easily show that

$$\underline{\lim_{x\to\infty}}\frac{\log\mathbb{P}[Z_N > x]}{\log x} \geq -\alpha^*.$$

Finally, combining the upper bound and the lower bound, we finish the proof. □

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