Modulated Branching Processes, Origins of Power Laws and Queueing Duality

Predrag R. Jelenković Jian Tan
Department of Electrical Engineering
Columbia University, New York, NY 10027
{predrag, jiantan}@ee.columbia.edu

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Abstract

Power law distributions have been repeatedly observed in a wide variety of socioeconomic, biological and technological areas, including distributions of: wealth, species-area relationships, populations of cities, values of companies, sizes of living organisms and, more recently, documents and visitors on the Web, etc. In the vast majority of these observations, e.g., city populations and sizes of living organisms, the objects of interest evolve due to the replication of their many independent components, e.g., births-deaths of individuals and replications of cells. Furthermore, the rates of replication of the many components are often controlled by exogenous parameters causing periods of expansion and contraction, e.g., baby booms and busts, economic booms and recessions, etc. In addition, the sizes of these objects often either have reflective lower boundaries, e.g., cities do not fall bellow a certain size, low income individuals are subsidized by the government, companies are protected by bankruptcy laws, etc; or have porous/absorbing lower boundaries, e.g., cities may degenerate, bankruptcy protections may fail and companies can be liquidated.

Hence, it is natural to propose reflected modulated branching processes as generic models for many of the preceding observations of power laws that are typically observed in proportional growth environments. Indeed, our main results show that the proposed mathematical models result in power law distributions under quite general *polynomial Gärtner-Ellis* conditions. The generality of our results could explain the ubiquitous nature of power law distributions. Furthermore, an informal interpretation of our main results suggests that alternating periods of expansion and reduction, e.g., economic booms and recessions, are primarily responsible for the appearance of power law distributions.

Our results also establish a general asymptotic equivalence between the reflected branching processes and the corresponding multiplicative processes. Furthermore, in the course of our analysis, we observe a duality between the reflected multiplicative processes and queueing theory. Essentially, this duality demonstrates that the power law distributions play an equivalent role for reflected multiplicative processes as the exponential/geometric distributions do in queueing analysis.

Keywords: Modulated branching processes, reflective/absorbing barriers, reflected multiplicative processes, proportional growth models, power law distributions, heavy tails, subexponential distributions, queueing processes, reflected additive random walks, Cramér large deviations, polynomial Gärtner-Ellis conditions.

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1 Introduction

Power law distributions are found in a wide range of domains, ranging from socioeconomic to biological and technological areas. Specifically, these types of distributions describe the city populations, species-area relationships, sizes of living organisms, value of companies, distributions of wealth, and more recently, sizes of documents on the Web, visitor access patterns on Web sites, etc. Hence, one would expect that there exist universal mathematical laws that explain this ubiquitous nature of power law distributions. To this end, we propose a class of models, termed modulated branching processes with reflective or absorbing lower barriers that, under quite general *polynomial Gärtner-Ellis* conditions, result in power law distributions.

Empirical observations of power laws have a long history, starting from the discovery by Pareto [58] in 1897 that a plot of the logarithm of the number of incomes above a level against the logarithm of that level yields points close to a straight line, which is essentially equivalent to saying that the income distribution follows a power law. Hence, power law distributions are often called Pareto distributions; for more recent study on income distributions see [16, 52, 21, 62, 63]. In a different context, early work by Arrhenius [4] in 1921 conjectured a power law relationship between the number of species and the census area, which was followed by Preston's prediction in [61] that the slope on the log/log species-area plot has a canonical value equal to 0.262; for additional information and measurements on species-area relationships see [18, 60, 44]. Interestingly, there also exists a power law relationship between the rank of the cities and the population of the corresponding cities. This was proposed by Auerbach [8] in 1913 and later studied by Zipf [71], after whom power law is also known as Zipf's law. Ever since, much attention on both empirical examinations and explanations of the city size distributions have been drawn [71, 34, 26, 66, 59, 2]. Similar observations have been made for firm sizes [3], language family sizes [70], and even the gene family and protein statistics [33, 67, 51, 13]. It is maybe even more surprising that many features of the Internet are governed by power laws, including the distribution of pages per Web site [31], the page request distribution [20, 12], the file size distribution [23, 37], Ethernet LAN traffic [46], World Wide Web traffic [19], the number of visitors per Web site [32, 1], the distribution of scenes in MPEG video streams [36] and the distribution of the indegrees and outdegrees in the Web graph as well as the physical network connectivity graph [24, 9, 45, 53]. In socio-economic areas, in addition to income distributions, the fluctuations in stock prices have also been observed to be characterized by power laws [27, 47]. This paragraph only exemplifies various observations of power laws; for a more complete survey see [54].

Hence, these repeated empirical observations of power laws, over a period of more than a hundred years, strongly suggest that there exist general mathematical laws that govern these phenomena. In this regard, after carefully examining the situations that result in power laws, we discover that most of them are characterized by the following three features. First, in the vast majority of these observations, e.g., city populations and sizes of living organisms, the objects of interest evolve due to the replication of their many independent components, e.g., birth-deaths of individuals and replications of cells. Secondly, the rate of replication of the many components is often controlled by exogenous parameters causing periods of baby booms and busts, economic growths and recessions, etc. Thirdly, the sizes of these objects often have lower boundaries, e.g., cities do not fall bellow a certain size, low income individuals are subsidized by the government, companies are protected by bankruptcy laws, etc.

In order to capture the preceding features, it is natural to propose *modulated branching processes* (MBP) with reflective or absorbing barriers as generic models for many of the observations of power laws. Indeed, one of our main results, presented in Theorem 3, shows that MBPs with reflective barriers almost invariably produce power law distributions under quite general *polynomial Gärtner-Ellis* conditions. The generality of our results could explain the ubiquitous nature of power law distributions. Furthermore, an informal interpretation of our main results, stated in Theorems 3 and 4 of Section 3, suggests that alternating periods of expansions and contractions, e.g., economic booms and recessions, are primarily responsible for the appearance of power law distributions. Actually, Theorem 4 shows that the distribution of the reflected MBP is exponentially bounded if the process always contracts. From a mathematical perspective, we develop a novel mathematical technique for analyzing reflected modulated branching processes since these objects appear new and the traditional methods for investigating branching processes [7] do not directly apply; a preliminary version of this work has appeared in the extended abstract in [39].

Formal description of our reflected modulated branching process (RMBP) model is given in Section 2. In the singular case when the number of individuals born in each state of the modulating process is constant, our model reduces to a reflected multiplicative process. In Subsection 2.1 we establish a rigorous connection (duality) between the reflected multiplicative processes (RMPs) and queueing theory. We would like to point out that this duality, although a minor point of our paper, makes a vast literature on queueing theory directly applicable to the analysis of RMPs. As a direct consequence of this connection, in Subsection 2.1 we translate several well known queueing results to the context of RMPs. Informally, these results show that the role which exponential distributions play in queueing theory, and in additive reflected random walks in general, is represented by power low distributions in the framework of RMPs/RMBPs. Furthermore, this relationship appears to reduce the debate on the relative importance of power law versus exponential distributions/models to the analogous question of the prevalence of proportional growth versus additive phenomena. Interestingly, the power law distribution satisfies the *memoryless property* in the multiplicative world, playing an equivalent role to the memoryless exponential distribution in the additive world. Indeed, if $\mathbb{P}[M > x] = x^{-\alpha}, \alpha > 0, x \geq 1$, then, for $x, y \geq 1$, we obtain $\mathbb{P}[M > xy|M > x] = \mathbb{P}[M > y]$.

Furthermore, this duality immediately implies and generalizes many of the prior results in the area of RMPs and power laws. Some of these prior results include the work of Levy and Solomon that appears to be the first to show how power laws can be obtained by adding a reflection condition to a multiplicative process [48, 49, 47]; this was further analyzed by Sornette and Cont in [69], where a formal connection with the classical theory of i.i.d. additive random walks was established. Furthermore, we would like to point out that the reflective nature of the barrier, assumed in the previous studies, is not essential for producing power law distributions. Indeed, one only needs a positive lower barrier, e.g., porous, absorbing or reflective one, which is a natural condition since no physical object or socioeconomic one can approach zero arbitrarily close without repelling from it or simply disappearing. In many areas, objects of interest may not have a strictly reflecting barrier, but rather a porous one, e.g., cities may degenerate, bankruptcy protection may sometimes fail and a company can be liquidated. In these cases, the power law effect follows from the well-known queueing results on cycle maximum that we briefly stated in Subsection 2.2. This observation presents a rigorous explanation for the previous study in [11] that argued heuristically how multiplicative processes with absorbing barriers can result in power laws.

In addition, while the reduction of RMBPs to RMPs is apparent in the special case when constant number of individuals are born in each state of the modulating process, our main result, Theorem 3, reveals a deeper general asymptotic equivalence between the power law exponent of a RMBP and the corresponding RMP. In other words, Theorem 3 discovers the asymptotic insensitivity of the power law exponent on the conditional distributions of the reflected branching process beyond their conditional mean values.

In some domains, e.g., the growth of living organisms, the objects always grow (basically never shrink) up until a certain random time. Huberman and Adamic [31] also propose this model as an explanation of the growth dynamics of the World Wide Web by arguing that the observation time is an exponential random variable. This notion has been revisited in [62] and generalized to a larger family of random processes observed at an exponential random time [64]. In this regard, in Subsection 5.1.2, we study randomly stopped modulated branching processes and show, under more general conditions than the preceding studies, that the resulting variables follow power laws.

In regard to the previously mentioned situations with absorbing barriers, we study MBP in Subsection 5.2 with an absorbing barrier and show that it leads to power law distributions as well. The result, under somewhat more restrictive conditions, is basically a direct corollary of Theorem 3 on RMBPs. We argue that these types of models can be natural candidates for describing the bursts of requests at popular Internet Web sites, often referred to as hotspots.

Based on our new model, we discuss two related phenomena: truncated power laws and double Pareto distributions. We argue that one can obtain a truncated power law distribution by adding an upper barrier to RMBP, similarly as the truncated geometric distributions appear in queueing theory, e.g., finite buffer $M/M/1$ queue. Furthermore, by the duality of RMBP and queueing theory, we give two new natural explanations of the origins of double Pareto distributions that have been observed in practice. In the queueing context, it has been shown that the tail of the queue length distribution exhibits different decay rates in the heavy-traffic and large deviation regime, respectively [57]; similar behavior of the queue length distribution was attributed to the multiple time scale arrivals in [35]. We claim that the preceding two mechanisms, when translated to the proportional growth context, provide natural explanations of the double Pareto distributions.

Finally, we would like to mention that there might be other mechanisms that result in power law distributions, e.g., the randomly typing model used to explain the power law distribution of frequencies of words in natural languages [54] and the highly optimized tolerance studied in [15]. Very recently, the new power law phenomenon in the situations where jobs have to restart from the beginning after a failure was discovered in [25] and further studied in [68, 6]; equivalently in the communication context, the retransmission based protocols in data networks were shown to almost invariably lead to power laws and, in general, heavy tails in [38, 41, 40, 43, 42]. For a recent survey on various mechanisms that result in power laws see [54].

The rest of the paper is organized as follows. After introducing the modulated branching processes in Section 2, we study the duality between the queueing theory and the multiplicative processes with reflected barriers in Subsection 2.1 and absorbing barriers in Subsection 2.2, respectively. Then, we present our main results in Section 3 on the logarithmic asymptotics of the stationary distribution of the reflected modulated branching process and the corresponding multiplicative one, which is followed by the study of the exact asymptotics under the more restrictive conditions in Section 4. As further extensions, we discuss three related models in Section 5, i.e., randomly stopped processes in Subsection 5.1, modulated branching processes with absorbing barriers in Subsection 5.2 and truncated power laws in Subsection 5.3. In the end, Section 6 presents some of the technical proofs that have been deferred from the preceding sections.

2 Reflected Modulated Branching Processes

In this section we formally describe our model. Let $\{J_n\}_{n>-\infty}$ be a stationary and ergodic modulating process that takes values in positive integers. Define a family of independent, nonnegative, integer-valued random variables $\{B_n^i(j)\}, -\infty < i, j, n < \infty$, which are independent of the modulating process $\{J_n\}$. In addition, for fixed j, variables $\{B(j), B_n^i(j)\}$ are identically distributed with $\mu(j) \triangleq \mathbb{E}[B(j)] < \infty$.

Definition 1. *A Modulated Branching Process (MBP) $\{Z_n\}_{n=0}^{\infty}$ is recursively defined by*

$$Z_{n+1} \triangleq \sum_{i=1}^{Z_n} B_n^i(J_n), \tag{1}$$

where the initial value Z_0 is a positive integer. For increased clarity, we may explicitly write $\{Z_n^l\}$ when $Z_0 = l$.

Definition 2. *For any $l \in \mathbb{N}$ and an integer valued Λ_0, a Reflected Modulated Branching Process (RMBP) $\{\Lambda_n\}_{n=0}^{\infty}$ is recursively defined as*

$$\Lambda_{n+1} \triangleq \max\left(\sum_{i=1}^{\Lambda_n} B_n^i(J_n),\ l \right). \tag{2}$$

Remark 1. These types of modulated branching processes with a reflecting barrier appear to be new and, thus, the traditional methods for the analysis of branching processes [7] do not seem to directly apply.

Remark 2. A more general framework would be to define

$$Z_{n+1} = \int_0^{Z_n} B_n^t(J_n(t)) d\nu(t), \tag{3}$$

for any real measure ν, and, similarly,

$$\Lambda_{n+1} = \max\left(\int_0^{\Lambda_n} B_n^t(J_n(t)) d\nu(t),\ l \right), \tag{4}$$

where $l > 0$ and $B_n^t(J_n(t))$ is ν-measurable. We refrain from this generalization since it introduces additional technical difficulties without much new insight.

Now, we present the basic limiting results on the convergence to stationarity of Z_n and Λ_n.

Lemma 1. *If $\mathbb{E} \log \mu(J_0) < 0$, then a.s., we have*

$$\lim_{n \to \infty} Z_n = 0.$$

Proof. For all $n \geq 1$, let $W_n = Z_n / \Pi_{n-1}^0$, where $\Pi_n^0 = \prod_{i=0}^n \mu(J_i)$. It is easy to check that W_n is a positive martingale with respect to the filtration $\mathcal{F}_n = \sigma(J_i, Z_i, 0 \leq i \leq n-1)$. Hence, by the martingale convergence theorem (see Theorem 35.5. of [10]), a.s., as $n \to \infty$,

$$W_n \to W < \infty.$$

Next, since $\{J_n\}$ is stationary and ergodic, so is $\{\mu(J_n)\}$, and therefore, a.s.,

$$\frac{\log \Pi_{n-1}^0}{n} = \frac{1}{n} \sum_{i=0}^{n-1} \log \mu(J_i) \to \mathbb{E} \log \mu(J_0) < 0 \quad \text{as } n \to \infty.$$

Thus, $\Pi_{n-1}^0 \to 0$ as $n \to \infty$, which, by recalling $Z_n = W_n \Pi_{n-1}^0$, finishes the proof. ☐

Next, let Z_{-n} be the number of individuals at time 0 in an unrestricted branching process that starts at time $-n$ with l individuals; when needed for clarity, we will use the notation Z^l_{-n} to explicitly indicate the initial state l.

Lemma 2. *Assume* $\mathbb{E}\log\mu(J_0) < 0$*, then, for any a.s. finite initial condition* Λ_0*,* Λ_n *converges in distribution to*

$$\Lambda \stackrel{d}{=} \max_{n\geq 0} Z_{-n}.$$

Proof. First, assume that $\Lambda_0 = l$ and let Z^k_n be the number of individuals at time n in an unrestricted branching process that starts at time k with l individuals. Then, by stationarity of $\{J_n\}$, we have $Z^k_n \stackrel{d}{=} Z_{k-n}$. Clearly,

$$\Lambda_1 = \max\left(\sum_{i=1}^{l} B^j_1(J_1), l\right) \stackrel{d}{=} \max\{\, Z_{-1}, Z_0 \,\},$$

and, by induction and stationarity, it is easy to show

$$\Lambda_n \stackrel{d}{=} \max(\, Z_{-n}, Z_{-(n-1)}, \cdots, Z_{-1}, Z_0 \,),$$

which, by monotonicity, yields

$$\mathbb{P}[\Lambda_n > x] \to \mathbb{P}[\Lambda > x] \;\text{ as } n\to\infty.$$

Now, if $\Lambda^{\Lambda_0}_n$ is a process defined on the same sequence $\{B^i_n(J_n)\}$ with the initial condition $\Lambda_0 \geq l$, then, it is easy to see that

$$\Lambda^{\Lambda_0}_n \geq \Lambda_n \geq l, \quad \text{for all } n,$$

implying

$$\mathbb{P}[\Lambda^{\Lambda_0}_n > x] \geq \mathbb{P}[\Lambda_n > x]. \tag{5}$$

Next, we define the stopping time τ to be the first time when $\Lambda^{\Lambda_0}_n$ hits the boundary l, then, the preceding monotonicity implies that $\Lambda_n = \Lambda^{\Lambda_0}_n$ for all $n \geq \tau$. Using this observation, we obtain

$$\begin{aligned}\mathbb{P}[\Lambda^{\Lambda_0}_n > x] &= \mathbb{P}[\Lambda^{\Lambda_0}_n > x, \tau > n] + \mathbb{P}[\Lambda^{\Lambda_0}_n > x, \tau < n]\\ &\leq \mathbb{P}[\Lambda^{\Lambda_0}_n > x, \tau > n] + \mathbb{P}[\Lambda_n > x, \tau < n]\\ &\leq \mathbb{P}[\tau > n] + \mathbb{P}[\Lambda_n > x]. \end{aligned}\tag{6}$$

Next, by Lemma 1, τ is a.s. finite and, thus, by (5) and (6), we conclude

$$\lim_{n\to\infty}\mathbb{P}[\Lambda_n > x] = \lim_{n\to\infty}\mathbb{P}[\Lambda^{\Lambda_0}_n > x] = \mathbb{P}[\Lambda > x].$$

□

2.1 Reflected Multiplicative Processes and Queueing Duality

Note that in the special case $B^i_n(J_n) \equiv J_n$, reflected modulated branching processes reduce to reflected multiplicative processes with J_n being integer valued. In general, by using the definition in (3), J_n can be relaxed to take any positive real values. Hence, in this subsection we assume that $\{J_n\}_{n\geq 0}$ is a positive, real valued process.

Definition 3. *For $l > 0$ and $M_0 < \infty$, define a Reflected Multiplicative Process (RMP) as*

$$M_{n+1} = \max(\, M_n \cdot J_n \,,\, l\,), \quad n \geq 0. \tag{7}$$

RMP has been previously proposed and studied in literature [69, 49, 48, 26, 23] as the explanation of the origin of power laws. In this section we show a direct connection (duality) between RMP and queuing theory, by which most of the previously obtained results on RMP follow directly from the well-known queuing results.

Without loss of generality we can assume $l = 1$, since we can always divide (7) by l and define $M_n^1 = M_n/l$. Now, let $X_n = \log J_n$ and $Q_n = \log M_n$ with the standard conventions $\log 0 = -\infty$ and $e^{-\infty} = 0$. Then, for $l = 1$, equation (7) is equivalent to

$$Q_{n+1} = \max(Q_n + X_n, 0), \tag{8}$$

which is the workload (waiting-time) recursion in a single server (FIFO) queue.

Lemma 3. *If $\mathbb{E} \log J_n < 0$, then M_n converges in distribution to an a.s. finite random variable M that satisfies*

$$M \stackrel{d}{=} \sup_{n\geq 0} \Pi_n, \tag{9}$$

where $\Pi_0 = 1, \Pi_n = \prod_{i=-n}^{-1} J_i, n \geq 1$.

Proof. By the classical result of Loynes [50], Q_n, defined by (8), converges to an a.s. finite stationary limit Q if $\mathbb{E}X_n = \mathbb{E}\log J_n < 0$ and, furthermore,

$$Q \stackrel{d}{=} \sup_{n\geq 0} S_n,$$

where $S_0 = 0$ and $S_n = \sum_{i=-n}^{-1} X_i$. This implies the convergence of M_n and

$$M \stackrel{d}{=} e^{\sup_{n\geq 0} S_n} = \sup_{n\geq 0} e^{S_n} = \sup_{n\geq 0} \Pi_n.$$

□

The following theorem is a direct corollary of Theorem 1 in [29]; see also Theorem 3.8 in [17] and, for a more recent presentation, we refer the reader to [28].

Theorem 1. *Let $\{J_n\}_{n\geq 1}$ be a stationary and ergodic sequence of positive random variables. If there exists a function Ψ and positive constants α^* and ε^* such that*

1) $n^{-1} \log \mathbb{E}[(\Pi_n)^\alpha] \to \Psi(\alpha)$ as $n \to \infty$ for $|\, \alpha - \alpha^* \,| < \varepsilon^*$,

2) Ψ is finite and differentiable in a neighborhood of α^* with $\Psi(\alpha^*) = 0$, $\Psi'(\alpha^*) > 0$, and

3) $\mathbb{E}\left[(\Pi_n)^{\alpha^*+\varepsilon}\right] < \infty$, for $n \geq 1$ and some $\varepsilon > 0$,

then

$$\lim_{x\to\infty} \frac{\log \mathbb{P}[M > x]}{\log x} = -\alpha^*. \tag{10}$$

Remark 3. We refer to conditions 1) − 3) as the *polynomial Gr̈tner-Ellis conditions*. Note that condition 2) can be relaxed such that Ψ is only differentiable at α^* and condition 3) can be weakened to $\varepsilon = 0$ [29]. Since these two conditions are necessary for Theorem 3 in Section 3 to hold, we keep the current form to provide a unified framework. Also, it is worth noting that the multiplicative process Π_n without the reflective boundary would essentially follow the lognormal distribution, as it was recently observed in [30] (this is similar to the fact that the unrestricted additive random walk is approximated well by Normal distribution). However, we would like to reemphasize that the lower boundary l is not just a mathematical artifact, but a very natural condition since no physical object can approach zero arbitrarily close without either repelling (reflecting) from it or vanishing (absorbing); the absorbing boundary will be discussed in the following Subsection 2.2.

Remark 4. For the case when the sequence $\{J_n\}$ is i.i.d., the connection between reflected multiplicative processes and the classical theory of additive random walks was earlier observed in [69], where it is shown that α^* satisfies $\mathbb{E}\left[{J_1}^{\alpha^*}\right] = 1$. However, our observed equivalence with the queueing theory, especially in the general stationary and ergodic framework, appears novel.

Here, we illustrate the preceding theorem by the following examples. Assume that $\{A_n\}, \{C_n\}$ are two mutually independent sequences, and let $J_n = e^{A_n - C_n}$. Then the quantity $Q_n \triangleq \log M_n$, where M_n is defined in (7), satisfies

$$Q_{n+1} = (Q_n + A_n - C_n)^+. \tag{11}$$

The first two examples assume that $\{A_n\}, \{C_n\}$ are two i.i.d. sequences, the third example takes $\{J_n\}$ to be a Markov chain, and in the last example, $\{J_n\}$ is modulated by a Markov chain $\{X_n\}$.

Example 1. If $\{A_n\}, \{C_n\}$ follow exponential distributions, $\mathbb{P}[C_n > x] = e^{-\mu x}$, $\mathbb{P}[A_n > x] = e^{-\lambda x}$ and $\lambda < \mu$, then Q_n represents the waiting time in a $M/M/1$ queue. By Theorem 9.1 of [5], the stationary waiting time in a $M/M/1$ queue is distributed as

$$\mathbb{P}[Q > x] = \frac{\lambda}{\mu} e^{-(\mu-\lambda)x}, \quad x \geq 0,$$

which equivalently yields a power law distribution for M,

$$\mathbb{P}[M > x] = \mathbb{P}[Q > \log x] = \frac{\lambda}{\mu x^{\mu-\lambda}}, \quad x \geq 1$$

with power exponent $\alpha = \mu - \lambda$.

Example 2. If $\{A_n\}, \{C_n\}$ are two i.i.d Bernoulli processes with $\mathbb{P}[A_n = 1] = 1 - \mathbb{P}[A_n = 0] = p$, $\mathbb{P}[C_n = 1] = 1 - \mathbb{P}[C_n = 0] = q$, $p < q$. Then, the elementary queueing/Markov chain theory shows that the stationary distribution of Q_n, as defined in (11), is geometric $\mathbb{P}[Q \geq j] = (1-\rho)\rho^j, j \geq 0$, where $\rho = p(1-q)/q(1-p) < 1$. Therefore,

$$\mathbb{P}[M \geq x] = \mathbb{P}[Q \geq \log x] = \rho^{\lfloor \log x \rfloor}, \quad x \geq 1.$$

Since $\log x - 1 < \lfloor \log x \rfloor \leq \log x$, it is easy to conclude that

$$\frac{1}{x^{\log(1/\rho)}} \leq \mathbb{P}[M \geq x] < \frac{1}{\rho x^{\log(1/\rho)}}.$$

Example 3. If $\{J_n\}$ is a Markov chain taking values in a finite set Σ and possessing an irreducible transition matrix $Q = (q(i,j))_{i,j\in\Sigma}$, then the function Ψ defined in Theorem 1 can be explicitly computed. Define matrix Q_α with elements

$$q_\alpha(i,j) = q(i,j)j^\alpha, \quad i,j \in \Sigma.$$

By Theorem 3.1.2 of [22], we have as $n \to \infty$,

$$n^{-1} \log \mathbb{E}[(\Pi_n)^\alpha] \to \log\left(\text{dev}(Q_\alpha)\right),$$

where $\text{dev}(Q_\alpha)$ is the Perron-Frobenius eigenvalue of matrix Q_α. To illustrate this result, we take $\Sigma = \{u, d\}$ where $u = 1/d > 1$, and $q(d,u) = q, q(d,d) = 1-q, q(u,d) = p, q(u,u) = 1-p$ where $p > q$. It is easy to compute

$$Q_\alpha = \begin{pmatrix} (1-p)u^\alpha & pd^\alpha \\ qu^\alpha & (1-q)d^\alpha \end{pmatrix},$$

and, by letting $\log\left(\text{dev}(Q_\alpha)\right) = 0$, we obtain

$$\alpha^* = \frac{\log(1-q) - \log(1-p)}{\log u}.$$

Example 4 (double Pareto)**.** If $\{J_n \equiv J(X_n\}$ is modulated by a Markov chain X_n, we argue that $\mathbb{P}[M > x]$ can have different asymptotic decay rates over multiple time scales. This phenomenon was investigated in [35] in the queueing context and formulated as Theorem 3 therein. To visualize this phenomena, we study the following example. Consider a Markov process X_n of two states (say $\{1,2\}$) with transition probabilities $p_{12} = 1/5000$, $p_{21} = 1/10$, and $\mathbb{P}[J(1) = 1.2] = 1 - \mathbb{P}[J(1) = 0.6] = 0.5$, $\mathbb{P}[J(2) = 1.7] = 1 - \mathbb{P}[J(2) = 0.25] = 0.6$. The corresponding simulation result for 5×10^7 trials is presented in Figure 1. We observe from this figure a double Pareto distribution for M, which provides a new explanation to the origins of double Pareto distributions as compared to the one in [65].

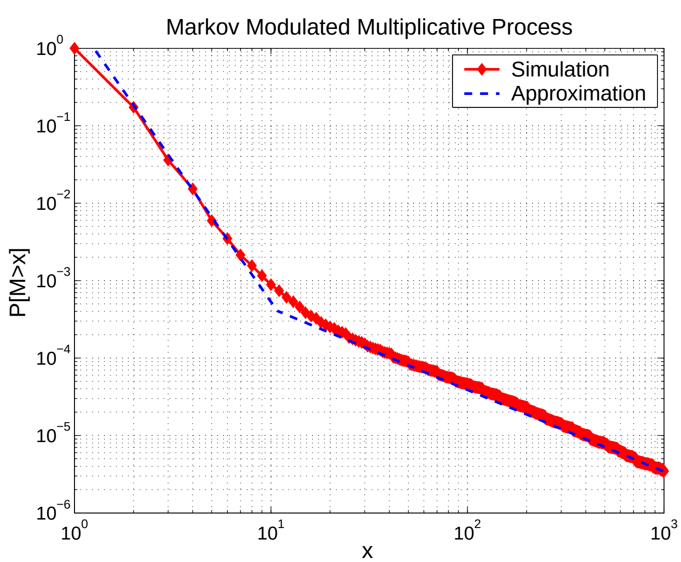

Figure 1: Illustration for Example 4 of the double Pareto distribution.

2.2 Multiplicative Processes with Absorbing Barriers and Cycle Maximum

As briefly discussed in the introduction, we explained that the reflective nature of the barrier is not essential for producing power law distributions. Indeed, one only needs a positive lower barrier, e.g., porous, absorbing or reflective one, which is a natural condition since no physical objects or socioeconomic ones can approach zero arbitrarily close without repelling from it or simply disappearing. To illustrate the situations when the objects can vanish, we name a few examples, e.g., cities may degenerate, bankruptcy protection may sometimes fail and a company can be liquidated. In these cases, the power law effect follows from the well-known queueing result on cycle maximum that is stated in Theorem 2 below. We also discuss in Subsection 5.2 a more complicated situation when newly generated objects in the system can arrive/appear or leave/disappear.

Following the notation from Chapter VIII of [5], for a sequence of positive i.i.d. random variables $\{J_n\}_{n\geq 1}$, denote by G_+ the ladder height distribution of the random walk $\{S_n = \sum_{i=1}^n \log J_i\}_{n\geq 1}$ with $\|G_+\| = \mathbb{P}[S_n \leq 0 \text{ for all } n \geq 1]$, and define the stopping time $\tau \triangleq \inf\{n : S_n \leq 0, n \geq 1\}$ with the corresponding cycle maximum $M_\tau \triangleq \sup\{\prod_{i=1}^n J_i : 1 \leq n \leq \tau\}$; here we assume, without loss of generality, that the absorbing barrier is equal to 1.

Theorem 2. *If the sequence $\{\log J_n\}_{n\geq 1}$ is nonlattice, satisfying $\mathbb{E}[\log J_1] < 0$, $\mathbb{E}[J_1^{\alpha^*}] = 1$ and $\alpha^* > 0$, then*

$$\lim_{x\to\infty} \mathbb{P}[M_\tau > x]x^{\alpha^*} = \frac{(1 - \|G_+\|)\left(1 - \mathbb{E}\left[e^{-\alpha^* S_\tau}\right]\right)}{\alpha^* \int_0^\infty x e^{\alpha^* x} G_+(dx)}.$$

Proof. The result is a direct consequence of Corollary 5.9 on p. 368 of [5]. □

3 Main Results

This section presents our main results in Theorems 3 and 4. In this regard, we define $\bar{B} \triangleq \sup_k B(k)$ and, to avoid technical difficulties, assume $\underline{\mu} \triangleq \inf_j \mu(j) > 0$. With a small abuse of notation, as compared to the preceding Subsection 2.1, we redefine here $\Pi_n = \prod_{i=-n}^{-1} \mu(J_i)$, $n \geq 1$, $\Pi_0 = l$ and $M = \sup_{n\geq 0} \Pi_n$. In this paper we use the following standard notation. For any two real functions $a(t)$ and $b(t)$, we use $a(t) = o(b(t))$ to denote that $\lim_{t\to\infty} a(t)/b(t) = 0$, and $a(t) = O(b(t))$ to denote that $\overline{\lim}_{t\to\infty} a(t)/b(t) < \infty$; when needed for increased clarity, we may explicitly write $a(t) = o(b(t))$ as $t \to \infty$.

Theorem 3. *Assume that the process $\{\Pi_n\}$ satisfies the polynomial Gärtner-Ellis conditions (conditions $1) - 3)$ of Theorem 1), and $\mathbb{E}[e^{\theta \bar{B}}] < \infty$ for some $\theta > 0$, then,*

$$\lim_{x\to\infty} \frac{\log \mathbb{P}[\Lambda > x]}{\log x} = \lim_{x\to\infty} \frac{\log \mathbb{P}[M > x]}{\log x} = -\alpha^*. \tag{12}$$

Remark 5. Note that conditions 1) and 2) of Theorem 1 imply that there exists j such that $\mu(j) > 1$, since otherwise we have $\sup_\alpha \Psi(\alpha) \leq 0$, which would contradict $\Psi(\alpha^*) = 0$ and $\Psi'(\alpha^*) > 0$ in condition 2). The following theorem covers the opposite situation when the previous condition is not satisfied, i.e., $\sup_j \mu(j) < 1$.

Theorem 4. *If $\sup_j \mu(j) < 1$ and $\mathbb{E}[e^{\theta \bar{B}}] < \infty$ for $\theta > 0$, then, $\mathbb{P}[\Lambda > x] = O\left(e^{-\xi x}\right)$ for some $\xi > 0$, implying*

$$\lim_{x\to\infty} \frac{\log \mathbb{P}[\Lambda > x]}{\log x} = -\infty. \tag{13}$$

Remark 6. Informally speaking, these two theorems show that the alternating periods of contractions and expansions, e.g., economic booms and recessions, are primarily responsible for the appearance of power law distributions; in other words, if there are no periods of expansions, i.e., the condition $\sup_j \mu(j) < 1$ of Theorem 4 is satisfied, then Λ has an exponentially bounded tail that is lighter than any power law distribution. Furthermore, the first equality in (12) of Theorem 3 reveals a general asymptotic equivalence between the reflected modulated branching process and the corresponding reflected multiplicative process, showing that the power law exponent α^* is insensitive to the high order distributional properties of $B(j)$ beyond the conditional mean $\mu(j)$.

Remark 7. A careful examination of the proofs reveals that the existence of the exponential moments for $\bar{B}$ could possibly be relaxed to $\mathbb{E}\left[\bar{B}^\alpha\right] < \infty$ for $\alpha > \alpha^*$. However, such an extension would considerably complicate the proofs. Furthermore, in most practical applications the distributions of $\{B(j)\}$ are typically very concentrated. For the preceding reasons, we do not consider such extensions.

In the following subsections, we present the **proofs** of Theorems 3 and 4.

3.1 Proof of Theorem 3

The proof of Theorem 3 is composed of the upper bound and the lower bound that are presented in the following two subsections, respectively.

3.1.1 Upper Bound

Since the proof is based on the change (increase) of boundary l, we denote this dependence explicitly as $\Lambda^l \equiv \Lambda$. According to Lemma 2, the initial value of $\{\Lambda_n\}$ has no impact on Λ and, therefore, in this subsection we simply assume that $\Lambda_0^l = l$. Before stating the proof of the upper bound, we establish some necessary lemmas.

The first lemma shows that, most likely, the supremum of Z_n occurs for an index $n \leq x$.

Lemma 4. *For any $\beta > 0$, the branching process Z_n^l defined in (1) satisfies,*

$$\sum_{n>x}^{\infty} \mathbb{P}\left[Z_n^l > x\right] = o\left(\frac{1}{x^\beta}\right) \quad \text{as } x \to \infty.$$

Proof. Similarly as in the proof of Lemma 1, note that for $\Pi_{n-1}^0 = \prod_{i=0}^{n-1} \mu(J_i)$, the stochastic process $W_n = Z_n^l/\Pi_{n-1}^0, n \geq 1$ is a martingale with respect to the filtration $\mathcal{F}_n = \sigma(J_i, Z_i, 0 \leq i \leq n-1)$ that satisfies $\mathbb{E}[W_1] = 1$. Therefore, by recalling $\Pi_n = \prod_{i=-n}^{-1} \mu(J_i)$, we obtain, for any $\varepsilon > 0$,

$$\begin{aligned} \mathbb{P}[Z_n^l > x] &= \mathbb{P}[W_n \Pi_{n-1} > x] = \mathbb{P}[(W_n e^{-\varepsilon n})(\Pi_{n-1}^0 e^{\varepsilon n}) > x] \\ &\leq \mathbb{P}[W_n e^{-\varepsilon n} > 1] + \mathbb{P}[\Pi_n e^{\varepsilon n} > x] \\ &\leq \mathbb{E}[W_n e^{-\varepsilon n}] + \mathbb{P}[\Pi_n e^{\varepsilon n} > x]. \end{aligned} \tag{14}$$

Next, by using the martingale property $\mathbb{E}[W_n] = \mathbb{E}[W_1] = 1$, we derive

$$\sum_{n>x}^{\infty} \mathbb{E}[W_n e^{-\varepsilon n}] = \sum_{n>x}^{\infty} e^{-\varepsilon n} \leq \frac{e^{-\varepsilon x}}{1 - e^{-\varepsilon}} = o\left(\frac{1}{x^\beta}\right) \quad \text{as } x \to \infty. \tag{15}$$

Then, recalling condition 1) of Theorem 3, we can choose $\delta, \varepsilon > 0$ small enough and n_0 large enough such that $\Psi(\alpha^*-\delta)+2\varepsilon(\alpha^*-\delta) = -\zeta < 0$ and $n^{-1}\log\mathbb{E}\left[\Pi_n^{(\alpha^*-\delta)}\right] < \Psi(\alpha^*-\delta)+\varepsilon(\alpha^*-\delta)$ for $n > n_0$, which implies, for $x > n_0$,

$$\begin{aligned}\sum_{n>x}^{\infty}\mathbb{P}[\Pi_n e^{\varepsilon n} > x] &\le \sum_{n>x}^{\infty}\frac{\mathbb{E}\left[\Pi_n^{(\alpha^*-\delta)}\right]e^{\varepsilon(\alpha^*-\delta)n}}{x^{(\alpha^*-\delta)}} \le \sum_{n>x}^{\infty}\frac{e^{-\zeta n}}{x^{\alpha^*-\delta}} \\ &\le \frac{e^{-\zeta x}}{(1-e^{-\zeta})x^{\alpha^*-\delta}} = o\left(\frac{1}{x^{\beta}}\right) \text{ as } x\to\infty. \qquad (16)\end{aligned}$$

Finally, by using (14), (15) and (16), we complete the proof. □

The following lemma relates Λ_n to the corresponding multiplicative process.

Lemma 5. *Let $\varepsilon > 0$ and Λ_n^l be the reflected branching process, as defined in (2), then*

$$\mathbb{P}\left[\Lambda_n^l > x\right] \le \mathbb{P}\left[\max_{1\le j\le n}\Pi_j(1+\varepsilon)^j > x/l\right] + n\mathbb{P}\left[\mathcal{B}_0^{l,\varepsilon}\right],$$

where $\Pi_j = \prod_{i=-1}^{-j}\mu(J_i)$ and $\mathcal{B}_n^{l,\varepsilon} = \bigcup_{j\ge l}\{\ \sum_{i=1}^{j} B_n^i(J_n) > j\mu(J_n)(1+\varepsilon)\ \}$.

Proof. Observe that

$$\begin{aligned}\mathbb{P}\left[\Lambda_n^l > x\right] &= \mathbb{P}\left[\Lambda_n^l > x, (\mathcal{B}_{n-1}^{l,\varepsilon})^C\right] + \mathbb{P}\left[\Lambda_n^l > x, \mathcal{B}_{n-1}^{l,\varepsilon}\right] \\ &\le \mathbb{P}\left[\left\{\sum_{i=1}^{\Lambda_{n-1}^l} B_{n-1}^i(J_{n-1}) > x\right\} \bigcup \left\{\sum_{i=1}^{l} B_{n-1}^i(J_{n-1}) > x\right\},\ \left(\mathcal{B}_{n-1}^{l,\varepsilon}\right)^C\right] + \mathbb{P}\left[\mathcal{B}_{n-1}^{l,\varepsilon}\right] \\ &\le \mathbb{P}\left[\left\{\ \Lambda_{n-1}^l\mu(J_{n-1})(1+\varepsilon) > x\ \right\} \bigcup \left\{\ \mu(J_{n-1})(1+\varepsilon) > x/l\ \right\}\right] + \mathbb{P}\left[\mathcal{B}_0^{l,\varepsilon}\right] \\ &\le \mathbb{P}\left[\left\{\ \Lambda_{n-1}^l\mu(J_{n-1})(1+\varepsilon) > x\ \right\} \bigcup \left\{\ \mu(J_{n-1})(1+\varepsilon) > x/l\ \right\},\ \left(\mathcal{B}_{n-2}^{l,\varepsilon}\right)^C\right] \\ &\quad + \mathbb{P}\left[\mathcal{B}_{n-2}^{l,\varepsilon}\right] + \mathbb{P}\left[\mathcal{B}_0^{l,\varepsilon}\right] \\ &\le \mathbb{P}\Big[\left\{\ \Lambda_{n-2}^l\mu(J_{n-1})\mu(J_{n-2})(1+\varepsilon)^2 > x\ \right\} \\ &\quad \bigcup \left\{\ \max\left(\ \mu(J_{n-1})\mu(J_{n-2})(1+\varepsilon)^2,\ \mu(J_{n-1})(1+\varepsilon)\ \right) > x/l\ \right\}\Big] + 2\mathbb{P}\left[\mathcal{B}_0^{l,\varepsilon}\right],\end{aligned}$$

where $\mathcal{B}^C$ is the complement of set $\mathcal{B}$. Now, by continuing this inductive argument one can easily obtain

$$\mathbb{P}\left[\Lambda_n^l > x\right] \le \mathbb{P}\left[\max_{1\le j\le n}(1+\varepsilon)^j\prod_{i=1}^{j}\mu(J_{n-i}) > x/l\right] + n\mathbb{P}\left[\mathcal{B}_0^{l,\varepsilon}\right],$$

which, by stationarity of $\{\mu(J_n)\}$, yields

$$\mathbb{P}\left[\Lambda_n^l > x\right] \le \mathbb{P}\left[\max_{1\le j\le n}\Pi_j(1+\varepsilon)^j > x/l\right] + n\mathbb{P}\left[\mathcal{B}_0^{l,\varepsilon}\right].$$

□

Now, we show that the "error" event $\mathcal{B}_0^{l,\varepsilon}$ in the preceding lemma has a negligible probability for large l relative to any power law distribution.

Lemma 6. *By setting $l_x = \lfloor x^\delta \rfloor$, $0 < \delta < 1$ in the definition of $\mathcal{B}_0^{l_x,\varepsilon}$ in Lemma 5, we obtain $\mathbb{P}\left[\mathcal{B}_0^{l_x,\varepsilon}\right] = O\left(e^{-\xi x^\delta}\right)$ for some $\xi > 0$, implying that for any $\beta > 0$,*

$$\mathbb{P}[\mathcal{B}_0^{l_x,\varepsilon}] = o\left(\frac{1}{x^\beta}\right) \quad as \quad x \to \infty.$$

The **proof** of this lemma is presented in Subsection 6.1.

The following lemma allows us to increase the lower barrier in order to prove the upper bound.

Lemma 7. *Assume that $\Lambda_n^{l_1}$ and $\Lambda_n^{l_2}$ are defined on the same sequence $\{B_n^j(J_n)\}$ with initial conditions l_1 and l_2, respectively. If $l_1 \geq l_2$, then, for all $n \geq 0$,*

$$\Lambda_n^{l_1} \geq \Lambda_n^{l_2}.$$

Proof. The result holds trivially for $n = 0$. Now we prove the result using induction. Suppose that it is true for all $0 \leq k \leq n$, and for $k = n+1$,

$$\Lambda_{n+1}^{l_1} = \max\left(\sum_{i=1}^{\Lambda_n^{l_1}} B_n^i(J_n),\ l_1\right) \geq \max\left(\sum_{i=1}^{\Lambda_n^{l_2}} B_n^i(J_n),\ l_2\right) = \Lambda_{n+1}^{l_2},$$

which implies the lemma is true for all $n \geq 0$. □

Now, we are ready to complete the proof of the upper bound.

Proof (of the upper bound of Theorem 3): Choosing $l_x = \lfloor x^\varepsilon \rfloor \geq l, 0 < \varepsilon < 1$, using Lemma 7 and then Lemma 5, we derive

$$\begin{aligned}
\mathbb{P}\left[\Lambda^l > x\right] &= \mathbb{P}\left[\sup_{j\geq 1} Z_{-j}^l > x\right] \leq \mathbb{P}\left[\Lambda_{\lfloor x\rfloor}^l > x\right] + \mathbb{P}\left[\sup_{j>x} Z_{-j}^l > x\right] \\
&\leq \mathbb{P}\left[\Lambda_{\lfloor x\rfloor}^{l_x} > x\right] + \sum_{j>x}\mathbb{P}\left[Z_j^l > x\right] \\
&\leq \mathbb{P}\left[\sup_{j\geq 1}\Pi_j(1+\varepsilon)^j > x^{1-\varepsilon}\right] + x\mathbb{P}\left[\mathcal{B}_0^{l_x,\varepsilon}\right] + \sum_{j>x}\mathbb{P}\left[Z_j^l > x\right] \\
&\triangleq I_1(x) + I_2(x) + I_3(x). \qquad (17)
\end{aligned}$$

Now, define a new process $\{\mu^\varepsilon(J_n) = \mu(J_n)(1+\varepsilon)\}_{n\geq 1}$ and $\Pi_n^\varepsilon = \prod_{i=-n}^{-1}\mu^\varepsilon(J_i)$. Then, for ε small enough, we have

1) $n^{-1}\log\mathbb{E}(\Pi_n^\varepsilon)^\alpha \to \Psi^\varepsilon(\alpha) = \Psi(\alpha) + \alpha\log(1+\varepsilon)$ as $n \to \infty$ for $\mid \alpha - \alpha^* \mid < \varepsilon^*$,
2) Ψ^ε is finite in a neighborhood of α_ε^*, $\alpha_\varepsilon^* < \alpha^*$, and differentiable at α_ε^* with $\Psi(\alpha_\varepsilon^*) + \alpha_\varepsilon^*\log(1+\varepsilon) = 0$, $\Psi'(\alpha_\varepsilon^*) > 0$, and
3) $\mathbb{E}\left[(\Pi_n^\varepsilon)^{\alpha_\varepsilon^*}\right] < \infty$ for $n \geq 1$.

Therefore, by Theorem 1, we obtain

$$\lim_{x\to\infty}\frac{\log\mathbb{P}[\sup_{i\geq 1}\Pi_i(1+\varepsilon)^i > x^{1-\varepsilon}]}{\log x} = -(1-\varepsilon)\alpha_\varepsilon^*, \qquad (18)$$

which, in conjunction with Lemma 4 and Lemma 6, yields

$$I_2(x) + I_3(x) = o(I_1(x)). \tag{19}$$

Then, combining (17), (18) and (19) yields

$$\frac{\log \mathbb{P}[\Lambda^l > x]}{\log x} \leq \frac{\log((1+o(1))I_1(x))}{\log x} \longrightarrow \alpha_\varepsilon^* \quad \text{as } x \to \infty.$$

Since $\Psi^\varepsilon(\alpha)$ is continuous in a neighborhood of α^* in both α and ε, we derive

$$\lim_{\varepsilon \to 0} \alpha_\varepsilon^* = \alpha^*,$$

implying,

$$\overline{\lim_{x\to\infty}} \frac{\log \mathbb{P}[\Lambda > x]}{\log x} \leq -\alpha^*. \tag{20}$$

□

3.1.2 Lower Bound

Similarly as in the proof of the upper bound, we use the following lemmas. The following three easy results, specifically Corollary 1, allow us to obtain a lower bound for Λ while, maybe somewhat counterintuitively, increasing the lower barrier l.

Lemma 8. *For any* x_1,x_2,y_1,y_2,

$$\max(\, x_1 + x_2 \,,\, y_1 + y_2 \,) \leq \max(\, x_1 \,, y_1 \,) + \max(\, x_2 \,, y_2 \,).$$

Proof.

$$\begin{aligned}\max(\, x_1 + x_2 \,,\, y_1 + y_2 \,) &\leq \max(\, \max(x_1, y_1) + x_2 \,,\, \max(x_1, y_1) + y_2 \,) \\ &= \max(x_1, y_1) + \max(x_2, y_2).\end{aligned}$$

□

Lemma 9. *If* $\{\Lambda_n^{y_1}\}$ *and* $\{\Lambda_n^{y_2}\}$ *are defined on the same modulating sequence* $\{J_n\}_{n\geq 0}$ *and two i.i.d. sequences* $\left\{B_n^{i,1}(j)\right\}$, $\left\{B_n^{i,2}(j)\right\}$, *respectively, then,*

$$\Lambda_n^{y_1+y_2} \overset{d}{\leq} \Lambda_n^{y_1} + \Lambda_n^{y_2}.$$

Proof. We use induction to prove this lemma. Start with $n = 1$, and by Lemma 8, we obtain

$$\begin{aligned}\Lambda_1^{y_1+y_2} &= \max\left(\sum_{i=1}^{y_1+y_2} B_0^i(J_0),\ y_1 + y_2\right) = \max\left(\sum_{i=1}^{y_1} B_0^i(J_0) + \sum_{i=y_1+1}^{y_1+y_2} B_0^i(J_0),\ y_1 + y_2\right) \\ &\leq \max\left(\sum_{i=1}^{y_1} B_0^i(J_0),\ y_1\right) + \max\left(\sum_{i=y_1+1}^{y_1+y_2} B_0^i(J_0),\ y_2\right) \\ &\overset{d}{=} \Lambda_1^{y_1} + \Lambda_1^{y_2}.\end{aligned}$$

The proof is completed by induction in n,

$$\Lambda_{n+1}^{y_1+y_2} = \max\left(\sum_{i=1}^{\Lambda_n^{y_1+y_2}} B_1^i(J_n),\ y_1+y_2\right) \overset{d}{\leq} \max\left(\sum_{i=1}^{\Lambda_n^{y_1}+\Lambda_n^{y_2}} B_1^i(J_n),\ y_1+y_2\right)$$
$$\overset{d}{\leq} \Lambda_{n+1}^{y_1} + \Lambda_{n+1}^{y_2}.$$

□

Corollary 1. *If $\{\Lambda_{n,j}^1\}_{1\leq j\leq y}$ are conditionally i.i.d copies of Λ_n^1 given $\{J_i\}_{1\leq i\leq n}$, then,*

$$\Lambda_n^y \overset{d}{\leq} \sum_{j=1}^{y} \Lambda_{n,j}^1 .$$

Now, we basically establish that the supremum of Π_i occurs most likely for small indexes $i \leq h\log x$.

Lemma 10. *Assume that condition 1) of Theorem 3 is satisfied, then, for $0 \leq \varepsilon < 1$ and any $\beta > 0$, there exists $h > 0$ such that, when $x \to \infty$,*

$$\mathbb{P}\left[\sup_{i>h\log x} \Pi_i(1-\varepsilon)^i > x\right] = o\left(\frac{1}{x^\beta}\right).$$

Proof. Using condition 1) of Theorem 3, we can choose $0 < \alpha < \alpha^*$ with $n^{-1}\log\mathbb{E}[\Pi_n^\alpha] \to \Psi(\alpha) < 0$ and n_0 large enough, such that $\mathbb{E}[\Pi_n^\alpha] < \zeta^n$, $0 < \zeta < 1$, $n > n_0$. Thus, for $h = -\beta/\log\zeta > 0$ and $x > e^{n_0/h}$,

$$\mathbb{P}\left[\sup_{i>h\log x} \Pi_i(1-\varepsilon)^i > x\right] \leq \sum_{i>h\log x}^{\infty} \mathbb{P}[\Pi_i > x] \leq \sum_{i>h\log x}^{\infty} \frac{\mathbb{E}[\Pi_i^\alpha]}{x^\alpha} \leq \sum_{i>h\log x}^{\infty} \frac{\zeta^i}{x^\alpha} = o\left(\frac{1}{x^\beta}\right).$$

□

Finally, the last lemma shows that $\sum_{i=1}^{j} B_n^i(J_n)$ can not deviate by much from $j\mu(J_n)$ for large j.

Lemma 11. *For $1 > \delta, \varepsilon > 0$ and $\mathcal{C}_n^{l,\varepsilon} \triangleq \bigcup_{j\geq l}\{\ \sum_{i=1}^{j} B_n^i(J_n) < j\mu(J_n)(1-\varepsilon)\ \}$, we obtain, for any $\beta > 0$,*

$$\mathbb{P}\left[\mathcal{C}_0^{\lfloor x^\delta\rfloor,\varepsilon}\right] = o\left(\frac{1}{x^\beta}\right).$$

The **proof** of Lemma 11 is presented in Subsection 6.2. Next, we can prove the lower bound of Theorem 3.

Proof (of the lower bound of Theorem 3): First, using Corollary 1, we obtain, for any integer $y \geq 1$,

$$\mathbb{P}[\Lambda_n^l > x] \geq \mathbb{P}[\Lambda_n^1 > x] = \frac{y\mathbb{P}[\Lambda_n^1 > x]}{y} \geq \frac{\mathbb{P}[\sum_{j=1}^{y}\Lambda_{n,j}^1 > yx]}{y} \geq \frac{\mathbb{P}[\Lambda_n^y > yx]}{y}. \tag{21}$$

For $\Pi_n^i \triangleq \mu(J_i)\mu(J_{i+1})\cdots\mu(J_{n-1})$, $0<\varepsilon<1$ and $\mathcal{C}_n^{l,\varepsilon}$ defined in Lemma 11, we derive

$$\begin{aligned}
\mathbb{P}[\Lambda_n^y > yx] &\geq \mathbb{P}\left[\sup_{0\leq i\leq n-1} \Pi_n^i (1-\varepsilon)^{n-i} > x\right] - \mathbb{P}[\mathcal{C}_0^{y,\varepsilon}] - \cdots - \mathbb{P}[\mathcal{C}_{n-1}^{y,\varepsilon}] \\
&= \mathbb{P}\left[\sup_{1\leq i\leq n} \Pi_i (1-\varepsilon)^{i} > x\right] - n\mathbb{P}[\mathcal{C}_0^{y,\varepsilon}] \\
&\geq \mathbb{P}\left[\sup_{i\geq 1} \Pi_i (1-\varepsilon)^{i} > x\right] - \mathbb{P}\left[\sup_{i>n} \Pi_i (1-\varepsilon)^{i} > x\right] - n\mathbb{P}[\mathcal{C}_0^{y,\varepsilon}] \\
&\triangleq I_1 - I_2 - I_3. \qquad (22)
\end{aligned}$$

Next, similarly as in the proof of the upper bound, define a new process $\{\mu_\varepsilon(J_n) = \mu(J_n)(1-\varepsilon)\}_{n\geq 1}$ and let $\Pi_n^\varepsilon = \prod_{i=-n}^{-1}\mu_\varepsilon(J_i)$. Then, for ε small enough, we have

1) $n^{-1}\log\mathbb{E}(\Pi_n^\varepsilon)^\alpha \to \Psi^\varepsilon(\alpha) = \Psi(\alpha) + \alpha\log(1-\varepsilon)$ as $n\to\infty$ for $\mid \alpha-\alpha^* \mid < \varepsilon^*$,
2) $\Psi^\varepsilon(\alpha)$ is finite in a neighborhood of α_ε^*, $\alpha_\varepsilon^* > \alpha^*$ and differentiable at α_ε^* with $\Psi(\alpha_\varepsilon^*) + \alpha_\varepsilon^*\log(1-\varepsilon) = 0$, $\Psi'(\alpha_\varepsilon^*) > 0$, and
3) $\mathbb{E}\left[(\Pi_n^\varepsilon)^{\alpha_\varepsilon^*}\right] < \infty$ for $n\geq 1$.

Therefore, by Theorem 1, we obtain

$$\lim_{x\to\infty} \frac{\log\mathbb{P}[\sup_{i\geq 1}\Pi_i(1-\varepsilon)^i > x]}{\log x} = -\alpha_\varepsilon^*. \qquad (23)$$

Now, by setting $y = \lfloor x^\delta \rfloor, 0<\delta<1$, $n=\lfloor x \rfloor$ in (21), (22), and using Lemmas 10 and 11, it is easy to see that

$$I_2 + I_3 = o(I_1), \qquad (24)$$

which, by (22) and (24), further implies

$$\log\mathbb{P}[\Lambda > x] \geq \log\mathbb{P}[\Lambda_n^l > x] \geq \log(I_1 - I_2 - I_3) - \delta\log x = \log((1-o(1))I_1) - \delta\log x.$$

From the preceding inequality and (23), we obtain

$$\varliminf_{x\to\infty} \frac{\log\mathbb{P}[\Lambda > x]}{\log x} \geq -\alpha_\varepsilon^* - \delta. \qquad (25)$$

Since $\Psi^\varepsilon(\alpha)$ is continuous in a neighborhood of α^* in both α and ε, we have $\lim_{\varepsilon\to 0}\alpha_\varepsilon^* = \alpha^*$. Then, passing $\varepsilon,\delta\to 0$ in (25) completes the proof of the lower bound, which, in conjunction with (20), finishes the proof of Theorem 3. □

3.2 Proof of Theorem 4

Proof. Using the same arguments as in deriving (17) in the proof of the upper bound of Theorem 3, we obtain, for $l_x = \lfloor x \rfloor \geq l$ and $0<\varepsilon<1$,

$$\begin{aligned}
\mathbb{P}[\Lambda > x] &\leq \mathbb{P}\left[\Lambda_{\lfloor x\rfloor}^l > x\right] + \mathbb{P}\left[\sup_{j>\lfloor x\rfloor} Z_{-j}^l > x\right] \\
&\leq \mathbb{P}\left[\Lambda_{\lfloor x\rfloor}^{l_x} > x\right] + \sum_{j>\lfloor x\rfloor} \mathbb{P}\left[Z_j^l > x\right] \\
&\leq \mathbb{P}\left[\sup_{j\geq 1}\Pi_j(1+\varepsilon)^j > 1\right] + x\mathbb{P}\left[\mathcal{B}_0^{l_x,\varepsilon}\right] + \sum_{j>\lfloor x\rfloor}^{\infty} \mathbb{P}\left[Z_j^l > x\right] \\
&\triangleq I_1(x) + I_2(x) + I_3(x). \qquad (26)
\end{aligned}$$

Recalling $\Pi_j = \prod_{i=-1}^{-j} \mu(J_i)$ and noting $\sup_j \mu(j) < 1$, we can choose $\varepsilon > 0$ such that $\sup_j \mu(j)(1+\varepsilon) < 1$, which implies $I_1(x) = 0$. And, by using Lemma 6, we obtain $I_2(x) = o(e^{-\xi x})$ for some $\xi > 0$.

Next, using similar arguments as in deriving (14) in the proof of Lemma 4, we obtain, for $\varepsilon > 0$ and $j \geq 1$,

$$\mathbb{P}\left[Z_j^l > x\right] \leq \mathbb{E}\left[W_j e^{-\varepsilon j}\right] + \mathbb{P}\left[\Pi_j e^{\varepsilon j} > x\right],$$

which, by recalling $\sup_j \mu(j) < 1$ and choosing ε small enough such that $\mathbb{P}[\Pi_j e^{\varepsilon j} > x] = 0$ for $x > 1$, yields,

$$I_3(x) \leq \sum_{j > \lfloor x \rfloor}^{\infty} \mathbb{E}[W_j e^{-\varepsilon j}] = \sum_{j > \lfloor x \rfloor}^{\infty} e^{-\varepsilon j} = O\left(e^{-\varepsilon x}\right).$$

Finally, combining (26) and the bounds on $I_1(x)$, $I_2(x)$ and $I_3(x)$ finishes the proof. □

4 Exact Asymptotics

This section presents the exact asymptotic approximations of the RMPs and RMBPs in the following two subsections, respectively.

4.1 On the Exact Asymptotics of Reflected Multiplicative Processes

The following theorems are direct translations from the corresponding queueing theory results. Theorem 5 is based on the large deviation result that studies the situation when M is large, and Theorem 6 is derived from the heavy traffic approximation of a GI/GI/1 queue where we study the limiting behavior of a sequence of multiplicative processes with the multiplicative drift tending to one. These two theorems are basically corollaries of Theorem 5.2 in Chapter XIII and Theorem 7.1 in Chapter X of [5], respectively.

For a sequence of positive i.i.d. random variables $\{J_n\}_{n\geq 1}$, define G_+ to be the ladder height distribution of the random walk $\{S_n = \sum_{i=1}^n \log J_i\}_{n\geq 1}$ with $\|G_+\| = \mathbb{P}[S_n \leq 0 \text{ for all } n \geq 1]$.

Theorem 5. *If the sequence $\{\log J_n\}_{n\geq 1}$ is nonlattice, satisfying $\mathbb{E}[\log J_1] < 0$, $\mathbb{E}[J_1^{\alpha^*}] = 1$ and $\alpha^* > 0$, then*

$$\lim_{x\to\infty} \mathbb{P}[M > x] x^{\alpha^*} = \frac{1 - \|G_+\|}{\alpha^* \int_0^\infty x e^{\alpha^* x} G_+(dx)}.$$

Proof. The result is a direct consequence of Theorem 5.3 in Chapter XIII of [5]. □

Remark 8. If S_n is lattice valued, see Remark 5.4 of Chapter XIII on p. 366 of [5].

Now, we study the limiting behavior of a sequence of multiplicative processes indexed by an integer k where $J^{(k)}$, $S_n^{(k)}$ and $M^{(k)}$ are properly defined for all $k \geq 1$.

Theorem 6. *If $\left\{J^{(k)}, J_n^{(k)}\right\}_{n\geq 1}$ are positive and i.i.d. for each fixed k with $m_k \triangleq \mathbb{E}\left[\log J^{(k)}\right]$, $\sigma_k^2 \triangleq \mathbb{V}ar\left[\log J^{(k)}\right]$, the random walks $\left\{S_n^{(k)} = \sum_{i=1}^n \log J_i^{(k)}\right\}_{n\geq 1}$ satisfy $m_k < 0$, $\lim_{k\to\infty} m_k = 0$, $\underline{\lim}_{k\to\infty} \sigma_k^2 > 0$, and $\left(\log J^{(k)}\right)^2$ is uniformly integrable for all k, then, for $y \geq 1$,*

$$\lim_{k\to\infty} \mathbb{P}\left[\left(M^{(k)}\right)^{-m_k/\sigma_k^2} > y\right] = 1/y^2.$$

Proof. From Theorem 7.1 in Chapter X on p. 287 of [5], we have, for $z \geq 0$,

$$\lim_{k\to\infty} \mathbb{P}\left[-\frac{m_k}{\sigma_k^2}\log M^{(k)} > z\right] = e^{-2z},$$

which, by letting $z = \log y$, finishes the proof of Theorem 6. □

Remark 9. The preceding two theorems essentially provide a new *general* explanation of the measured double Pareto phenomenon (e.g., see [55, 65]) since they rely on two universal statistical laws: the first one being based on the large deviation theory and the latter being implied by the central limit theorem.

4.2 On the Exact Asymptotics of Reflected Branching Processes

Deriving the exact asymptotics for RMBPs is a difficult problem. However, in the scaling region when the boundary l grows as well, albeit slowly, one can derive an explicit asymptotic characterization. In this subsection, assume that $\{J, J_n\}_{n\geq 1}$ are i.i.d. and let G_+ be the ladder height distribution of the nonlattice random walk $\{S_n = \sum_{i=1}^n \log\mu(J_i)\}_{n\geq 1}$ with $\|G_+\| = \mathbb{P}[S_n \leq 0 \text{ for all } n \geq 1] < 1$.

Theorem 7. *If* $\mathbb{E}\left[e^{\theta \sup_k |B(k)-\mu(k)|}\right] < \infty$ *for some* $\theta > 0$*,* $\underline{\mu} \triangleq \inf_j \mu(j) > 0$*,* $\mathbb{E}[\log\mu(J)] < 0$*,* $\mathbb{E}[\mu(J)^{\alpha^*}] = 1$ *and* $\alpha^* > 0$*, then, for any* $\gamma > 0$*,*

$$\lim_{\substack{l_x \,\geq\, (\log x)^{3+\gamma} \\ x \,\to\, \infty}} \mathbb{P}[\Lambda^{l_x}/l_x > x]x^{\alpha^*} = \frac{1-\|G_+\|}{\alpha^* \int_0^\infty x e^{\alpha^* x} G_+(dx)}.$$

The **proof** of this theorem is presented in Subsection 6.3. Here, we illustrate the exact asymptotics of the reflected branching process with the following simulation example.

Example 5. Assume that $\{J_n\}_{n\geq 1}$ is a Bernoulli process with $\mathbb{P}[J_n = 1] = 0.4 = 1-\mathbb{P}[J_n = 0]$ and the i.i.d. random variables $\{B_n^i(1)\}_{i\geq 1}$, $\{B_n^i(0)\}_{i\geq 1}$ follow Poisson distributions with means 1.5 and 0.6, respectively. The simulation results of 10^7 samples, for $l = 1, 5, 13, 21$, are drawn in Figure 5. From the figure we can clearly see that $\mathbb{P}[\Lambda^l/l \geq x]$ approaches the limiting value very quickly, i.e., for $l = 13$ and $l = 21$, the plots of $\mathbb{P}[\Lambda^l/l \geq x]$ are basically indistinguishable.

5 Discussion of Related Models

Based on the study of reflected modulated branching processes, we address two related models: randomly stopped processes and modulated branching processes with absorbing barriers.

5.1 Randomly Stopped Processes

In this subsection we discuss randomly stopped multiplicative and branching processes, respectively.

5.1.1 Randomly Stopped Multiplicative Processes

The following two theorems show that randomly stopped multiplicative processes and reflected multiplicative processes are intimately related and, to a certain extend, basically equivalent under more restrictive conditions. By following the approach of Chapter VIII of [5], we study

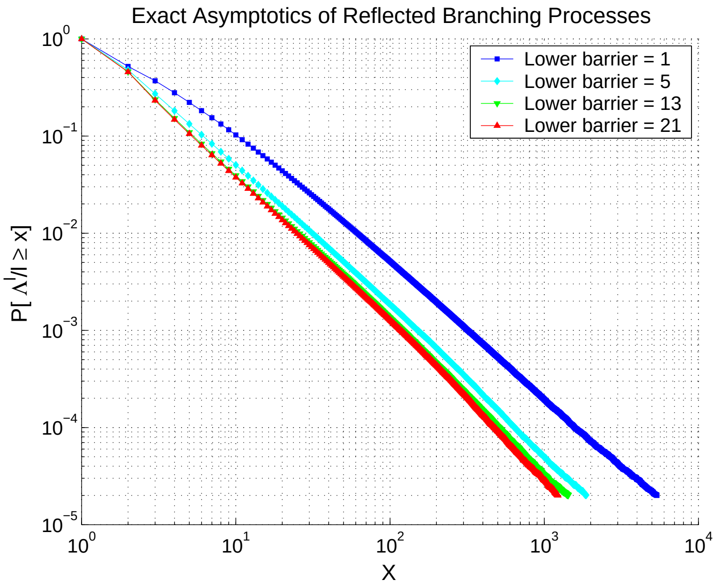

Figure 2: Simulation of $\mathbb{P}[\Lambda^l/l \geq x]$ versus x parameterized by l.

the ladder heights of a multiplicative process. For any RMP with i.i.d positive multiplicative increments, the random variable M, as defined in Lemma 3, can be represented in terms of the ladder heights. To this end, define $\Pi_n^0 \triangleq \prod_{i=0}^n J_i$ and the ladder height process $\{H_i\}_{i\geq 1}$ of $\{S_n = \sum_{i=1}^n \log J_i\}_{n\geq 1}$ with $\|G_+\| = \mathbb{P}[S_n \leq 0 \text{ for all } n \geq 1] < 1$ and $H_i^e \triangleq e^{H_i}$.

Theorem 8. *Suppose that $\{J, J_n\}_{n\geq 1}$ is a positive i.i.d. sequence with $\mathbb{E}[\log J] < 0$, then,*

$$M \stackrel{d}{=} \prod_{i=1}^{N} H_i^e, \tag{27}$$

where N is independent of $\{H_i^e\}_{i\geq 1}$ and follows a geometric distribution $\mathbb{P}[N > n] = \|G_+\|^n$.

Proof. Based on the well-known Pollaczek-Khinchin representation (see Chapter VIII of [5])

$$\log M \stackrel{d}{=} \sum_{i=1}^{N} H_i,$$

where N is independent of $\{H_i\}$ with $\mathbb{P}[N > n] = \|G_+\|^n$, it immediately follows that

$$\mathbb{P}[M > x] = \mathbb{P}\left[e^{\sum_{i=1}^N H_i} > x\right] = \mathbb{P}\left[\prod_{i=1}^{N} H_i^e > x\right].$$

□

Conversely, we can prove that if the observation time has exponential tail, the stopped process has a power law tail under quite general conditions. Note that here we do not require $\{J_n\}$ to be an i.i.d. sequence.

Theorem 9. *Let N be an integer random variable independent of $\{J_n\}$ with*

$$\lim_{x\to\infty} \frac{\log \mathbb{P}[N > x]}{x} = -\lambda < 0.$$

For a positive ergodic and stationary process $\{J_n\}_{n\geq 0}$, if $n^{-1}\log \mathbb{E}\left[\left(\Pi_n^0\right)^{\alpha}\right] \to \Psi(\alpha) < \infty$ as $n\to\infty$ in a neighborhood of $\alpha^* > 0$, $\Psi(\alpha)$ is differentiable at α^* with $\Psi(\alpha^*) = \lambda$, $\Psi'(\alpha^*) > 0$ and $\mathbb{E}\left[\left(\Pi_n^0\right)^{\alpha^*}\right] < \infty$ for $n \geq 1$, then,

$$\lim_{x\to\infty} \frac{\log \mathbb{P}\left[\Pi_N^0 > x\right]}{\log x} = -\alpha^*. \tag{28}$$

Remark 10. This theorem generalizes the previous results from [31, 62, 64], where only i.i.d. multiplicative increments are considered.

Proof. First, we prove the *upper bound*. For a fixed α that is in the neighborhood of α^* and $0 < \varepsilon < \lambda$, there exists n_ε such that $\mathbb{E}\left[\left(\Pi_n^0\right)^{\alpha}\right] < e^{(\Psi(\alpha)+\varepsilon)n}$ and $e^{-(\lambda-\varepsilon)n} > \mathbb{P}[N \geq n] > e^{-(\lambda+\varepsilon)n}$ for all $n \geq n_\varepsilon$. Since $\Psi(\alpha^*) = \lambda$ and $\Psi'(\alpha^*) > 0$, we can choose $\delta, \varepsilon > 0$ small enough such that $\Psi(\alpha^* - \delta) - \lambda + 2\varepsilon = -\xi < 0$. Thus, noting that N is independent of Π_n, we obtain

$$\begin{aligned}
\mathbb{P}\left[\Pi_N^0 > x\right] &= \sum_{n=1}^{\infty} \mathbb{P}[N = n]\mathbb{P}\left[\Pi_n^0 > x\right] \\
&\leq \sum_{n=1}^{n_\varepsilon} \mathbb{P}[N = n]\mathbb{P}\left[\Pi_n^0 > x\right] + \sum_{n=n_\varepsilon}^{\infty} \mathbb{P}[N \geq n]\mathbb{P}\left[\Pi_n^0 > x\right] \\
&\leq \sum_{n=1}^{n_\varepsilon} \mathbb{P}[N = n]\frac{\mathbb{E}\left[\left(\Pi_n^0\right)^{\alpha^*}\right]}{x^{\alpha^*}} + \sum_{n=n_\varepsilon}^{\infty} e^{-(\lambda-\varepsilon)n}\frac{\mathbb{E}[(\Pi_n^0)^{\alpha^*-\delta}]}{x^{\alpha^*-\delta}} \\
&\leq O\left(\frac{1}{x^{\alpha^*}}\right) + \frac{1}{x^{\alpha^*-\delta}}\sum_{n=n_\varepsilon}^{\infty} e^{-\xi n},
\end{aligned}$$

which implies

$$\overline{\lim_{x\to\infty}} \frac{\log \mathbb{P}\left[\Pi_N^0 > x\right]}{\log x} = -\alpha^* + \delta.$$

Passing $\delta \to 0$ in the preceding equality completes the proof of the upper bound.

The proof of the *lower bound* is presented in Subsection 6.4, which uses the standard exponential change of measure argument. □

Actually the following theorem shows that randomly stopped multiplicative processes and reflected multiplicative processes are basically equivalent under more restrictive conditions. This equivalence is established using classical results on $M/GI/1$ queue. In this regard, we assume that $\{J_n\}_{n\geq 1}$ is an i.i.d. process, Π_n^0 is the corresponding multiplicative process, N is a geometric random variable that is independent of Π_n^0 with $\mathbb{P}[N > n] = \rho^n, 0 < \rho < 1$, and $\bar{G}(t), t \geq 0$ is a complementary cumulative distribution function.

Theorem 10. *If for some $\alpha^* > 0$ and $\bar{G}(t)$, $\int_0^\infty e^{\alpha^* y}\bar{G}(y)dy = \rho^{-1}\int_0^\infty \bar{G}(y)dy$, $\int_0^\infty ye^{\alpha^* y}\bar{G}(y)dy < \infty$, and $\mathbb{P}[\log J_1 \leq x] = \int_0^x \bar{G}(y)dy / \int_0^\infty \bar{G}(y)dy, x \geq 0$, then, we can always construct a RMP such that $M \stackrel{d}{=} \Pi_N^0$, and, in particular,*

$$\lim_{x\to\infty} \mathbb{P}[M > x]\, x^{\alpha^*} = \lim_{x\to\infty} \mathbb{P}\left[\Pi_N^0 > x\right] x^{\alpha^*} = \frac{(1-\rho)\int_0^\infty \bar{G}(y)dy}{\alpha^*\rho\int_0^\infty ye^{\alpha^* y}\bar{G}(y)dy}.$$

Proof. We give a constructive proof based on the connection (duality) between the $M/GI/1$ queue and the geometrically stopped multiplicative process.

Consider a $M/GI/1$ queue with the service distribution $\mathbb{P}[S \geq t] = \bar{G}(t), t \geq 0$ and Poisson arrivals of rate $\lambda = \rho/\mathbb{E}[S], \mathbb{E}[S] < \infty$. Then, by the *Pollaczeck-Khinchine* formula (see, e.g., Theorem 5.7 on p. 237 of [5]), the stationary workload Q of this $M/GI/1$ queue is equal in distribution to $\sum_{i=1}^{N} H_i$, where $N, \{H_i\}_{i\geq 1}$ are independent with $\mathbb{P}[N > n] = \rho^n, n \geq 0$ and

$$\mathbb{P}[H_i \leq x] = \frac{\int_0^x \mathbb{P}[S \geq s]ds}{\mathbb{E}[S]} = \frac{\int_0^x \bar{G}(s)ds}{\int_0^\infty \bar{G}(s)ds} = \mathbb{P}[\log J_i \leq x], \quad x \geq 0,$$

which implies

$$\mathbb{P}[Q > \log x] = \mathbb{P}\left[\sum_{i=1}^{N} H_i > \log x\right] = \mathbb{P}\left[\sum_{i=1}^{N} \log J_i > \log x\right] = \mathbb{P}\left[\Pi_N^0 > x\right]. \tag{29}$$

By applying Cramér-Lundberg theory for the $M/GI/1$ queue (e.g., see Theorem 5.2 in Chapter XIII of [5]), we obtain

$$\lim_{x\to\infty} \mathbb{P}[Q > \log x]x^{\alpha^*} = \frac{(1-\rho)\int_0^\infty \bar{G}(y)dy}{\alpha^*\rho\int_0^\infty ye^{\alpha^* y}\bar{G}(y)dy},$$

which, by (29), completes the proof. □

5.1.2 Randomly Stopped Branching Processes

In the following theorem, we extend Theorem 9 of the preceding subsection to the context of randomly stopped branching processes. Define $\Pi_n^0 \triangleq \prod_{i=0}^{n} \mu(J_i)$.

Theorem 11. *Suppose that N is independent of $B_n^i(j) \geq 1$ for all n, i, j. Then, under the same conditions as in Theorem 9 with $\mathbb{E}\left[\left(\Pi_n^0\right)^\alpha\right] < \infty$ for $n \geq 1$ and $\Psi(\alpha)$ being differentiable in a neighborhood of $\alpha^* > 0$, we obtain, for $\{Z_n\}_{n\geq 0}$ defined in (1) with a bounded initial value $Z_0 < z_0 < \infty$,*

$$\lim_{x\to\infty} \frac{\log \mathbb{P}[Z_N > x]}{\log x} = \lim_{x\to\infty} \frac{\log \mathbb{P}[\Pi_N^0 > x]}{\log x} = -\alpha^*.$$

In order to avoid repetitions of similar arguments as in the proof of Theorem 3, we defer the **proof** of this theorem to Subsection 6.5.

5.2 Branching Processes with Absorbing Barriers

For many dynamic processes, e.g., city sizes, quite often when the sizes of the objects fall below a threshold, the whole object disappears, e.g., urban decay. Therefore, it is natural to study branching processes with absorbing barriers. As already discussed in Subsection 2.2, we know that a single object with an absorbing barrier can result in power law distributions based on the duality with the queueing cycle maximum. In this subsection, we study a more complicated situation where the newly generated objects can join the system and evolve together. This naturally models the arrivals to popular Web sites (hotspots), since information (news) is distributed according to a branching process, e.g., user A passes the information to B and C; further B may inform D, etc. Empirical examination shows that Web requests follow power law distributions, e.g., see [32, 1].

For a lower barrier $l > 0$ and the modulated branching process $\{Z_n^l\}_{n\geq 1}$ with $Z_0^l = l$ specified in Definition 1, define stopping time $P \triangleq \inf\{n > 0 : Z_n^l \leq l\}$, where the modulating process $\{J_n\}$ is a sequence of i.i.d. random variables. This branching process, denoted by Z_P, vanishes completely after P; it is easy to prove that $\mathbb{E}[P] < \infty$ when $\mathbb{E}[\log \mu(J_0)] < 0$.

Let the arrivals $\{A_n\}_{n>-\infty}$ be a sequence of i.i.d. Poisson random variables with $\mathbb{E}[A_n] = q > 0$ that is independent of other random variables. At time n, A_n objects are generated and join the system, each evolving according to an i.i.d. copy of the modulated branching process Z_P. Suppose that the system has reached its stationarity with N_n objects being in the system at time n, and then, by Little's Law, $\mathbb{E}[N_n] = q\mathbb{E}[P]$. Furthermore, assume that object j observed at time $n = 0$, if any, is generated at time $(-P_j^r)$ with a size $Z^l_{-P_j^r}$, where the random variables $\{P_j^r\}$ are i.i.d. and follow the equilibrium distribution of P. Then, the total size of all objects Z_s observed at time $n = 0$ in stationarity can be represented as

$$Z_s = \sum_{j=1}^{N_0} Z^l_{-P_j^r}.$$

Next, we show that Z_s follows a power law. The proof of the following theorem is essentially a corollary of Theorem 3. Recall $\bar{B} = \sup_k B(k)$.

Theorem 12. *Under the conditions described in this subsection, if $\{\mu(J_n)\}$ satisfies $\inf_j \mu(j) > 0$, $\mathbb{E}[\log \mu(J_1)] < 0$, $\mathbb{E}[\mu(J_1)^{\alpha^*}] = 1, \alpha^* > 0$ and $\mathbb{E}[e^{\theta \bar{B}}] < \infty$ for some $\theta > 0$, then,*

$$\lim_{x\to\infty} \frac{\log \mathbb{P}[Z_s > x]}{\log x} = -\alpha^*.$$

Proof. We begin with the *upper bound.* Notice that when the system reaches stationarity, N_n follows the Poisson distribution, and therefore, there exists $H > 0$ such that

$$\mathbb{P}[N_0 > \lfloor H \log x \rfloor] = o\left(\frac{1}{x^{\alpha^*}}\right). \tag{30}$$

Denoting by $\{\Lambda_i\}_{i\geq 1}$ the i.i.d. copies of the random variable Λ defined in Lemma 2, we obtain

$$\begin{aligned}
\mathbb{P}[Z_s > x] &\leq \mathbb{P}\left[\sum_{i=1}^{N_0} \Lambda_i > x\right] \\
&\leq \mathbb{P}\left[\sum_{i=1}^{\lfloor H\log x\rfloor} \Lambda_i > x\right] + \mathbb{P}[N_0 > \lfloor H \log x \rfloor] \\
&\leq H \log x \, \mathbb{P}\left[\Lambda > \frac{x}{H \log x}\right] + \mathbb{P}[N_0 > \lfloor H \log x \rfloor],
\end{aligned} \tag{31}$$

which, in conjunction with Theorem 3 and equation (30), yields

$$\overline{\lim_{x\to\infty}} \frac{\log \mathbb{P}[Z_s > x]}{\log x} \leq -\alpha^*. \tag{32}$$

Next, we proceed with the *lower bound.* Construct a new process that has the same arrivals $\{A_n\}$ as described before but only allows at most one object to exist in the system. The construction goes as follows: all the new arrivals will be dropped if there is an object present in the system; similarly, when newly generated objects arrive to the empty system, only one object will be accepted while others will be dropped; the object, if any, evolves according to an i.i.d. copy of the modulated branching process Z_P. Denote the total size of the object in the new system at time n by $\underline{Z}_n$, and observe that $\underline{Z}_n$ forms a renewal process. Then, taking out all the empty (idle) periods of the new system and concatenating the remaining periods sequentially yields a process equal in distribution to a reflected modulated branching process $\{\Lambda_n\}$, as defined in (2). Therefore, when the new system is in stationarity, we obtain, by the independence of $\{A_n\}$ and Z_P,

$$\mathbb{P}[Z_s > x] \geq \mathbb{P}[\underline{Z}_0 > x] = \mathbb{P}[\underline{Z}_0 > 0]\mathbb{P}[\Lambda > x],$$

which, by Theorem 3, yields

$$\varliminf_{x\to\infty} \frac{\log \mathbb{P}[Z_s > x]}{\log x} \geq -\alpha^*. \tag{33}$$

Finally, combining (32) and (33) finishes the proof. □

5.3 Truncated Power Laws

Truncated power laws have been observed empirically in many practical situations where the studied objects have natural upper boundaries. Here, we want to point out that by using the duality between the modulated branching processes and the queueing theory, one easily obtains truncated power laws when adding both a lower and an upper barrier to the modulated branching process. To illustrate this point, recall that $M/M/1/b$ queue with a finite buffer b results in a truncated geometric distribution for the number of customers in the queue, and by the duality, it essentially follows that in a proportional growth world with both a lower and an upper barrier, truncated power laws can naturally arise, playing a similar role as truncated exponential/geometric distributions do in an additive world.

6 Proofs

6.1 Proof of Lemma 6

The following easy lemma, which is part of Lemma 1.4 of [56], is used in the proof of Lemma 6.

Lemma 12. *Let X be a random variable satisfying*

$$\mathbb{P}[X > b] = 0, b > 0 \ \text{ and } \ \mathbb{E}[X^2] < \infty,$$

then, for $0 < h \leq 1/b$,

$$\log \mathbb{E}[e^{hX}] \leq h\mathbb{E}[X] + e\mathbb{E}[X^2]h^2/2.$$

Proof. It is easy to prove that, for all $y \leq 1$,

$$e^y \leq 1 + y + ey^2/2,$$

which, by noting that $hX \leq 1$ a.e., yields

$$\log \mathbb{E}[e^{hX}] \leq \log \mathbb{E}[1 + hX + e(hX)^2/2] \leq h\mathbb{E}[X] + e\mathbb{E}[X^2]h^2/2.$$

□

Now, we present the proof of Lemma 6.

Proof. First, by recalling $\bar{B} \overset{d}{=} \bar{B}_n^i \triangleq \sup_k B_n^i(k)$, we can bound the probability of the subset $\left\{\sum_{i=1}^n B_0^i(J_0) > \mu(J_0)(1+\varepsilon)n\right\}$ of $\mathcal{B}_0^{l_x,\varepsilon}$ by the following inequalities

$$\begin{aligned}
\bar{P}(n) &\triangleq \mathbb{P}\left[\sum_{i=1}^n B_0^i(J_0) > \mu(J_0)(1+\varepsilon)n\right] \\
&\le \mathbb{P}\left[\sum_{i=1}^n B_0^i(J_0)\mathbf{1}\left(B_0^i(J_0)\le k\right) + \sum_{i=1}^n \bar{B}_0^i\mathbf{1}\left(\bar{B}_0^i > k\right) > \mu(J_0)(1+\varepsilon)n\right] \\
&\le \mathbb{P}\left[\sum_{i=1}^n B_0^i(J_0)\mathbf{1}\left(B_0^i(J_0)\le k\right) > \mu(J_0)\left(1+\frac{\varepsilon}{2}\right)n\right] + \mathbb{P}\left[\sum_{i=1}^n \bar{B}_0^i\mathbf{1}\left(\bar{B}_0^i > k\right) > \frac{\underline{\mu}\varepsilon n}{2}\right] \\
&\triangleq I_1(n) + I_2(n). \qquad (34)
\end{aligned}$$

Next, for any $\theta > 0$ such that $\mathbb{E}\left[e^{\theta\bar{B}}\right] < \infty$, we can choose k large enough such that $\mathbb{E}\left[e^{\theta\bar{B}\mathbf{1}(\bar{B}>k)}\right]\Big/e^{\theta\underline{\mu}\varepsilon/2} < e^{-\theta\underline{\mu}\varepsilon/4} < 1$, implying

$$I_2(n) \le \left(\frac{\mathbb{E}\left[e^{\theta\bar{B}\mathbf{1}(\bar{B}>k)}\right]}{e^{\theta\underline{\mu}\varepsilon/2}}\right)^n < e^{-\theta\underline{\mu}\varepsilon n/4}. \qquad (35)$$

Regarding $I_1(n)$, note that $X_i(j) \triangleq B_0^i(j)\mathbf{1}\left(B_0^i(j)\le k\right) - \mathbb{E}[B_0^i(j)\mathbf{1}\left(B_0^i(j)\le k\right)]$ satisfies

$$\mathbb{E}[X_i(J_0)|J_0] = 0 \quad\text{and}\quad -k \le X_i(J_0) \le k,$$

which, by using Lemma 12 and denoting $\sigma_{J_0}^2 \triangleq \mathbb{E}[(X_i(J_0))^2 \mid J_0] \le k^2$, yields, for $0 < \theta \le 1/k$,

$$\mathbb{E}\left[e^{\theta X_i(J_0)}\middle|J_0\right] \le e^{e\sigma_{J_0}^2\theta^2/2} \le e^{ek^2\theta^2/2}.$$

Thus, by recalling $\underline{\mu} = \inf_j \mu(j) > 0$, we obtain

$$\begin{aligned}
I_1(n) &\le \mathbb{P}\left[\sum_{i=1}^n X_i(J_0) > \frac{\underline{\mu}\varepsilon n}{2}\right] \le e^{-\frac{\theta\underline{\mu}\varepsilon n}{2}}\mathbb{E}\left[\left(\mathbb{E}\left[e^{\theta X_i(J_0)}\,\middle|\, J_0\right]\right)^n\right] \\
&\le \exp\left(-\frac{\theta\underline{\mu}\varepsilon n}{2} + \frac{ek^2\theta^2 n}{2}\right),
\end{aligned}$$

which, by setting $\theta = \underline{\mu}\varepsilon e k^{-2}/2$ and choosing k large enough to ensure $0 < \theta \le 1/k$, yields

$$I_1(n) \le \exp\left(-\frac{(\underline{\mu}\varepsilon)^2 n}{8ek^2}\right). \qquad (36)$$

Therefore, from (34), (35) and (36), we obtain, for $l_x = \lfloor x^\delta \rfloor, 0 < \delta < 1$, some $\xi > 0$ and any $\beta > 0$, as $x \to \infty$,

$$\mathbb{P}\left[\mathcal{B}_0^{l_x,\varepsilon}\right] \le \sum_{i=l_x}^\infty \bar{P}(i) \le \sum_{i=\lfloor x^\delta\rfloor}^\infty \left(e^{-\frac{(\underline{\mu}\varepsilon)^2 n}{8ek^2}} + e^{-\frac{\theta\underline{\mu}\varepsilon n}{4}}\right) = O\left(e^{-\xi x^\delta}\right) = o\left(\frac{1}{x^\beta}\right).$$

□

6.2 Proof of Lemma 11

For $k_\varepsilon > 0, i \geq 1$, we define $X_i^\varepsilon(j) \triangleq \mathbb{E}\left[B_0^i(j)\mathbf{1}\left(B_0^i(j) \leq k_\varepsilon\right)\right] - B_0^i(j)\mathbf{1}\left(B_0^i(j) \leq k_\varepsilon\right)$ and obtain

$$\mathbb{E}[X_i^\varepsilon(J_0)|J_0] = 0 \text{ and } -k_\varepsilon \leq X_i^\varepsilon(J_0) \leq k_\varepsilon,$$

which, by using Lemma 12 and noting that $\mathbb{E}[(X_i^\varepsilon(J_0))^2 \mid J_0] \leq k_\varepsilon^2$, yields, for $0 < \theta \leq 1/k_\varepsilon$,

$$\mathbb{E}\left[e^{\theta X_i^\varepsilon(J_0)}\middle| J_0\right] \leq e^{ek_\varepsilon^2\theta^2/2}.$$

Next, the condition $\mathbb{E}[e^{\theta \bar{B}}] < \infty, \theta > 0$ implies that $\{B(j)\}$ is uniformly integrable with respect to j, and thus, for any $\varepsilon > 0$, there exists k_ε, such that

$$\sup_j \left(\mu(j) - \mathbb{E}\left[B(j)\mathbf{1}\left(B(j) \leq k_\varepsilon\right)\right]\right) \leq \frac{\varepsilon\underline{\mu}}{2},$$

where $\underline{\mu} \triangleq \inf_j \mu(j) > 0$. Therefore, for $\theta > 0$,

$$\begin{aligned}
\bar{P}(n) &\triangleq \mathbb{P}\left[\sum_{i=1}^n B_0^i(J_0) < \mu(J_0)(1-\varepsilon)n\right] \leq \mathbb{P}\left[\sum_{i=1}^n B_0^i(J_0) < (\mu(J_0) - \varepsilon\underline{\mu})n\right] \\
&= \mathbb{P}\left[\sum_{i=1}^n (-X_i^\varepsilon(J_0)) < \left(\mu(J_0) - \mathbb{E}\left[B(j)\mathbf{1}\left(B(j) \leq k_\varepsilon\right)\right]\right)n - \varepsilon\underline{\mu}n\right] \\
&\leq \mathbb{P}\left[\sum_{i=1}^n X_i^\varepsilon(J_0) > \frac{\varepsilon\underline{\mu}n}{2}\right] \leq \exp\left(-\frac{\theta\varepsilon\underline{\mu}n}{2}\right)\mathbb{E}\left[\left(\mathbb{E}\left[e^{\theta X_i^\varepsilon(J_0)} \,\middle|\, J_0\right]\right)^n\right] \\
&\leq \exp\left(-\frac{\theta\varepsilon\underline{\mu}n}{2} + \frac{k_\varepsilon^2 e\theta^2 n}{2}\right).
\end{aligned}$$

Then, by choosing k_ε large enough such that $\theta = \varepsilon\underline{\mu}ek_\varepsilon^{-2}/2 \leq 1/k_\varepsilon$, we obtain

$$\bar{P}(n) \leq \exp\left(-\frac{\left(\varepsilon\underline{\mu}\right)^2 n}{8ek_\varepsilon^2}\right),$$

which implies, for any $\beta > 0$ and $0 < \delta < 1$,

$$\mathbb{P}[\mathcal{C}_0^{y_x,\varepsilon}] \leq \sum_{n=\lfloor x^\delta \rfloor}^{\infty} \bar{P}(n) \leq \sum_{n=\lfloor x^\delta \rfloor}^{\infty} \exp\left(-\frac{\left(\varepsilon\underline{\mu}\right)^2 n}{8ek_\varepsilon^2}\right) = o\left(\frac{1}{x^\beta}\right).$$

□

6.3 Proof of Theorem 7

The proof of this theorem relies on the following lemmas; the first one is based on Theorem 3.7.1 of [22].

Lemma 13. *$\{X_i(j)\}_{i,j\in\mathbb{Z}}$ are zero mean independent random variables that are identically distributed for fixed j with $\bar{X}_i = \sup_j \mid X_i(j) \mid$. Fix a sequence $a_n \to 0$ such that $na_n \to \infty$ as $n \to \infty$. If $\mathbb{E}\left[e^{\theta \bar{X}_i}\right] < \infty$ for $\theta > 0$, then, there exist $n_0, h > 0$, such that for all $n > n_0$ and any random variable $J \in \mathbb{Z}$,*

$$\mathbb{P}\left[\sum_{i=1}^n X_i(J) > \sqrt{\frac{n}{a_n}}\right] \leq e^{-\frac{h}{a_n}}.$$

Proof. Define $\varphi_J(\omega) = \mathbb{E}\left[e^{\omega X_i(J)} \mid J\right]$ and use Taylor expansion to derive

$$\varphi_J(\omega) = \varphi_J(0) + \varphi_J'(0)\omega + \frac{\varphi_J''(\zeta)}{2!}\omega^2,\ 0 < \zeta < \omega \le \theta.$$

Noting that $\varphi_J(0) = 1$, $\varphi_J'(0) = 0$ and $\varphi_J''(\zeta) = \mathbb{E}\left[X_i^2(J)e^{\zeta X_i(J)} \mid J\right] \le \mathbb{E}\left[\bar{X}_i{}^2 e^{\theta \bar{X}_i}\right] \triangleq K_\theta$, we obtain, for $0 < \omega \le \theta$,

$$\varphi_J(\omega) \le 1 + K_\theta \omega^2,$$

which implies,

$$\mathbb{P}\left[\sum_{i=1}^n X_i(J) > \sqrt{\frac{n}{a_n}}\right] \le e^{-\omega\sqrt{\frac{n}{a_n}}}\mathbb{E}\left[(\varphi_J(\omega))^n\right] \le e^{-\omega\sqrt{\frac{n}{a_n}}}(1 + K_\theta\omega^2)^n \le e^{-\omega\sqrt{\frac{n}{a_n}}}e^{nK_\theta\omega^2}.$$

Since there exists n_0 such that $\theta > 1/(2K_\theta\sqrt{na_n})$ for all $n > n_0$, we can choose $\omega = 1/(2K_\theta\sqrt{na_n})$, which implies, for $n > n_0$,

$$\mathbb{P}\left[\sum_{i=1}^n X_i(J) > \sqrt{\frac{n}{a_n}}\right] \le e^{-\frac{1}{4K_\theta a_n}} = e^{-\frac{h}{a_n}},$$

where $h = 1/(4K_\theta) > 0$. □

Lemma 14. *For any $l \in \mathbb{N}$, define*

$$\mathscr{D}_n^l \triangleq \bigcup_{k \ge l}\left\{\sum_{i=1}^k B_n^i(J_n) > k\mu(J_n)\left(1 + \underline{\mu}^{-1}l^{-\frac{1}{3}}\right)\right\}$$

and

$$\mathscr{E}_n^l \triangleq \bigcup_{k \ge l}\left\{\sum_{i=1}^k B_n^i(J_n) < k\mu(J_n)\left(1 - \underline{\mu}^{-1}l^{-\frac{1}{3}}\right)\right\}.$$

If $l \ge (\log x)^{3+\gamma}$, $\gamma > 0$, then, under the conditions of Theorem 7, we obtain, for any $\beta > 0$,

$$\mathbb{P}\left[\mathscr{D}_n^l\right] = o\left(\frac{1}{x^\beta}\right) \text{ and } \mathbb{P}\left[\mathscr{E}_n^l\right] = o\left(\frac{1}{x^\beta}\right).$$

Proof. Defining $a_n = n^{-1/3}$ and observing that na_n is monotonically increasing in n, we obtain

$$\begin{aligned}
\mathbb{P}\left[\mathscr{D}_n^l\right] &\le \sum_{k=l}^\infty \mathbb{P}\left[\sum_{i=1}^k B_n^i(J_n) > k\mu(J_n)\left(1 + \frac{1}{\underline{\mu}}\sqrt{\frac{1}{la_l}}\right)\right] \\
&\le \sum_{k=l}^\infty \mathbb{P}\left[\sum_{i=1}^k B_n^i(J_n) > k\mu(J_n)\left(1 + \frac{1}{\underline{\mu}}\sqrt{\frac{1}{ka_k}}\right)\right] \\
&\le \sum_{k=l}^\infty \mathbb{P}\left[\sum_{i=1}^k \left(B_n^i(J_n) - \mu(J_n)\right) > \sqrt{\frac{k}{a_k}}\right],
\end{aligned}$$

which, by applying Lemma 13, yields

$$\mathbb{P}\left[\mathscr{D}_n^l\right] \le \sum_{k=l}^\infty e^{-\frac{h}{a_k}} \le \sum_{k \ge (\log x)^{3+\gamma}} e^{-hk^{1/3}} = o\left(\frac{1}{x^\beta}\right) \text{ as } x \to \infty.$$

By the same argument,

$$\begin{aligned}
\mathbb{P}\left[\mathscr{E}_n^l\right] &\le \sum_{k=l}^{\infty} \mathbb{P}\left[\sum_{i=1}^{k} B_n^i(J_n) < k\mu(J_n)\left(1-\frac{1}{\underline{\mu}}\sqrt{\frac{1}{l\cdot a_l}}\right)\right] \\
&\le \sum_{k=l}^{\infty} \mathbb{P}\left[\sum_{i=1}^{k} \left(\mu(J_n)-B_n^i(J_n)\right) > \sqrt{\frac{k}{a_k}}\right] \\
&= o\left(\frac{1}{x^{\beta}}\right) \quad \text{as } x\to\infty.
\end{aligned}$$

□

Following the proof of Lemma 4 with minor modifications, we can prove the following stronger result.

Lemma 15. *For any $\beta > 0$, there exists $h > 0$ such that the branching process defined in (1) satisfies*

$$\sum_{n\ge h\log x}^{\infty} \mathbb{P}\left[Z_n^l > x\right] = o\left(\frac{1}{x^{\beta}}\right) \quad \text{as } x\to\infty.$$

Now, we proceed with the proof of Theorem 7.

Proof (of Theorem 7): First, we establish the upper bound. Setting $\varepsilon = \underline{\mu}^{-1}l^{-\frac{1}{3}}$ in Lemma 5, we obtain

$$\mathbb{P}\left[\Lambda_n^l > lx\right] \le \mathbb{P}\left[\max_{1\le j\le n} \Pi_j\left(1+\underline{\mu}^{-1}l^{-\frac{1}{3}}\right)^j > x\right] + n\mathbb{P}\left[\mathscr{D}_0^l\right],$$

where $\mathscr{D}_0^l$ is defined in Lemma 14. For $l \ge (\log x)^{3+\gamma}$, we obtain

$$\begin{aligned}
\mathbb{P}\left[\Lambda^l > lx\right] = \mathbb{P}\left[\sup_{j\ge 1} Z_{-j} > lx\right] &\le \mathbb{P}\left[\Lambda_n^l > lx\right] + \mathbb{P}\left[\sup_{j>n} Z_{-j} > lx\right] \\
&\le \mathbb{P}\left[\sup_{1\le j\le n} \Pi_j\left(1+\underline{\mu}^{-1}l^{-\frac{1}{3}}\right)^j > x\right] + n\mathbb{P}\left[\mathscr{D}_0^l\right] + \sum_{j>n}^{\infty}\mathbb{P}\left[Z_j^l > lx\right],
\end{aligned}$$

which, by setting $n = \lfloor h\log x\rfloor$ with h being chosen as in Lemma 15 and applying Lemmas 14, 15, yields,

$$\begin{aligned}
\mathbb{P}\left[\Lambda^l > lx\right] &\le \mathbb{P}\left[\sup_{1\le j\le h\log x} \Pi_j\left(1+\underline{\mu}^{-1}l^{-\frac{1}{3}}\right)^j > x\right] + o\left(\frac{1}{x^{\alpha^*}}\right) \\
&\le \mathbb{P}\left[\sup_{j\ge 1} \Pi_j > x\left(1+\frac{1}{\underline{\mu}(\log x)^{1+\gamma/3}}\right)^{-h\log x}\right] + o\left(\frac{1}{x^{\alpha^*}}\right).
\end{aligned}$$

Finally, by using Theorem 5 and observing that $\lim_{x\to\infty}\left(1+\frac{1}{\underline{\mu}(\log x)^{1+\gamma/3}}\right)^{-h\log x} = 1$, we obtain

$$\overline{\lim_{\substack{l\,\ge\,(\log x)^{3+\gamma}\\ x\,\to\,\infty}}}\ \mathbb{P}\left[\Lambda^l/l > x\right]x^{\alpha^*} \le \frac{1-\|G_+\|}{\alpha^*\int_0^{\infty} xe^{\alpha^* x}G_+(dx)}.$$

Next, we prove the lower bound. Recall that $\Pi_n^i \triangleq \prod_{j=i}^{n-1} \mu(J_j)$ and $\Pi_i \triangleq \prod_{j=-1}^{-i} \mu(J_j)$. Then, for $l \geq (\log x)^{3+\gamma}$ and $n = \lfloor h \log x \rfloor$ where h is chosen as in Lemma 15, we have

$$\begin{aligned}
\mathbb{P}\left[\Lambda_n^l > lx\right] &\geq \mathbb{P}\left[\sup_{0\leq i\leq n-1} \Pi_n^i \left(1-\underline{\mu}^{-1}l^{-\frac{1}{3}}\right)^{n-i} > x\right] - \mathbb{P}\left[\mathscr{E}_0^l\right] - \cdots - \mathbb{P}\left[\mathscr{E}_{n-1}^l\right] \\
&= \mathbb{P}\left[\sup_{1\leq i\leq n} \Pi_i \left(1-\underline{\mu}^{-1}l^{-\frac{1}{3}}\right)^{i} > x\right] - n\mathbb{P}\left[\mathscr{E}_0^l\right] \\
&\geq \mathbb{P}\left[\sup_{i\geq 1} \Pi_i \left(1-\underline{\mu}^{-1}l^{-\frac{1}{3}}\right)^{h\log x} > x\right] - \mathbb{P}\left[\sup_{i>n} \Pi_i \left(1-\underline{\mu}^{-1}l^{-\frac{1}{3}}\right)^{i} > x\right] - n\mathbb{P}\left[\mathscr{E}_0^l\right] \\
&\triangleq I_1 - I_2 - I_3, \qquad (37)
\end{aligned}$$

where $\mathscr{E}_0^l$ is defined in Lemma 14. By Lemma 10, we obtain,

$$I_2 \leq \mathbb{P}\left[\sup_{i>h\log x} \Pi_i > x\right] = o\left(\frac{1}{x^{\alpha^*}}\right), \qquad (38)$$

and by Lemma 14,

$$I_3 = o\left(\frac{1}{x^{\alpha^*}}\right). \qquad (39)$$

Thus, combining (37), (38) and (39), we obtain

$$\mathbb{P}\left[\Lambda_n^l > lx\right] \geq \mathbb{P}\left[\sup_{j\geq 1} \Pi_j \left(1 - \frac{1}{\underline{\mu}\,(\log x)^{1+\gamma/3}}\right)^{h\log x} > x\right] - o\left(\frac{1}{x^{\alpha^*}}\right),$$

which, by using the same argument as in the proof of the upper bound, yields

$$\underset{\substack{l\,\geq\,(\log x)^{3+\gamma} \\ x\,\to\,\infty}}{\underline{\lim}} \mathbb{P}\left[\Lambda^l/l > x\right] x^{\alpha^*} \geq \frac{1-\|G_+\|}{\alpha^* \int_0^\infty x e^{\alpha^* x} G_+(dx)}.$$

□

6.4 Proof of the Lower Bound of Theorem 9

For $0 < 3\varepsilon < \lambda$, $\delta > 2\varepsilon/(\lambda - 3\varepsilon)$ and $\log x > n_\varepsilon$, recalling that $e^{-(\lambda-\varepsilon)n} > \mathbb{P}[N \geq n] > e^{-(\lambda+\varepsilon)n}$, we obtain, for large x,

$$\begin{aligned}
\mathbb{P}\left[\frac{(1+\delta)\log x}{\Psi'(\alpha^*)} \leq N \leq \frac{(1+2\delta)\log x}{\Psi'(\alpha^*)}\right] &\geq e^{-\frac{(\lambda+\varepsilon)(1+\delta)\log x}{\Psi'(\alpha^*)}} - e^{-\frac{(\lambda-\varepsilon)(1+2\delta)\log x}{\Psi'(\alpha^*)}} \\
&\geq (1-\varepsilon)e^{-\frac{(\lambda+\varepsilon)(1+\delta)\log x}{\Psi'(\alpha^*)}},
\end{aligned}$$

which implies that there exists $\delta \leq \zeta \leq 2\delta$ such that $n_x = \lceil (1+\zeta)(\log x)/\Psi'(\alpha^*) \rceil$ satisfies

$$\mathbb{P}[N = n_x] \geq \frac{(1-\varepsilon)\Psi'(\alpha^*)e^{-(\lambda+\varepsilon)(1+\delta)\log x/\Psi'(\alpha^*)}}{\delta \log x}. \qquad (40)$$

Therefore, using (40) and denoting $\log J_i$ by X_i, we obtain

$$\begin{aligned}
\mathbb{P}[\Pi_N^0 > x] &\geq \mathbb{P}[N = n_x]\mathbb{P}\left[\sum_{i=1}^{n_x} \log J_i > \log x\right] \\
&\geq \frac{(1-\varepsilon)\Psi'(\alpha^*)e^{-(\lambda+\varepsilon)(1+\delta)\log x/\Psi'(\alpha^*)}}{\delta \log x}\mathbb{P}\left[\sum_{i=1}^{n_x} X_i > \frac{\Psi'(\alpha^*)}{1+\delta} n_x\right]. \qquad (41)
\end{aligned}$$

Next, we perform an exponential change of measure for the probability on the right-hand side of (41). Let $\mathbb{P}_n^*$ be the probability measure on $\mathbb{R}^n$ defined by the probability measure $\mathbb{P}$ of the stationary and ergodic process $\{X_i\}_{i\geq 1}$

$$\mathbb{P}_n^*(dx_1, \cdots, dx_n) = e^{\alpha^* S_n - \Psi_n(\alpha^*)}\mathbb{P}(dx_1, \cdots, dx_n),$$

where $S_n = \sum_{i=1}^n X_i$ and $\Psi_n(\alpha) \triangleq \log \mathbb{E}[e^{\alpha S_n}]$ satisfying $n^{-1}\Psi_n(\alpha) \to \Psi(\alpha)$ in the neighborhood of α^*. Thus,

$$\begin{aligned}
\mathbb{P}\left[\sum_{i=1}^n X_i > \frac{\Psi'(\alpha^*)}{1+\delta}n\right] &= \mathbb{E}_n^*\left[e^{-\alpha^* S_n + \Psi_n(\alpha^*)}\mathbf{1}\left(S_n > \frac{\Psi'(\alpha^*)}{1+\delta}n\right)\right] \\
&\geq \mathbb{E}_n^*\left[e^{-\alpha^* S_n + \Psi_n(\alpha^*)}\mathbf{1}\left(\left|\frac{S_n}{n} - \Psi'(\alpha^*)\right| < \frac{\Psi'(\alpha^*)\delta}{1+\delta}\right)\right] \\
&\geq e^{-\alpha^* \frac{(1+2\delta)\Psi'(\alpha^*)}{1+\delta}n + \Psi_n(\alpha^*)}\mathbb{P}_n^*\left[\left|\frac{S_n}{n} - \Psi'(\alpha^*)\right| < \frac{\Psi'(\alpha^*)\delta}{1+\delta}\right]. \qquad (42)
\end{aligned}$$

Then, by Claim 1 on page 17 of [14], we know

$$\mathbb{P}_n^*\left[\left|\frac{S_n}{n} - \Psi'(\alpha^*)\right| < \frac{\Psi'(\alpha^*)\delta}{1+\delta}\right] \to 1 \text{ as } x \to \infty,$$

which, in conjunction with (41) and (42), implies

$$\varliminf_{x\to\infty} \frac{\log \mathbb{P}\left[\Pi_N^0 > x\right]}{\log x} \geq -\frac{(\lambda+\varepsilon)(1+\delta)}{\Psi'(\alpha^*)} - \frac{\alpha^*(1+2\delta)^2}{1+\delta} + \frac{(1+\delta)\Psi(\alpha^*)}{\Psi'(\alpha^*)}.$$

Finally, by passing $\varepsilon, \delta \to 0$ in the preceding equality and noting $\Psi(\alpha^*) = \lambda$, we prove the lower bound. □

6.5 Proof of Theorem 11

The second equality is implied by Theorem 9, and we only need to prove the first one. We begin with proving the *upper bound*. Recalling the definition of $\mathcal{B}_n^{l,\varepsilon}$ in Lemma 5 and, for $n \geq 1$, $0 < \varepsilon, \xi < 1$, choosing $x^\xi > z_0 > Z_0$, we obtain

$$\begin{aligned}
\mathbb{P}\left[Z_n^{Z_0} > x\right] &\leq \mathbb{P}\left[Z_n^{\lfloor x^\xi \rfloor} > x\right] \\
&\leq \mathbb{P}\left[Z_n^{\lfloor x^\xi \rfloor} > x, \bigcap_{i=0}^{n-1}\left(\mathcal{B}_i^{\lfloor x^\xi \rfloor, \varepsilon}\right)^C\right] + \mathbb{P}\left[\bigcup_{i=0}^{n-1}\mathcal{B}_i^{\lfloor x^\xi \rfloor, \varepsilon}\right] \\
&\leq \mathbb{P}\left[\Pi_n^0(1+\varepsilon)^n > x^{1-\xi}\right] + n\mathbb{P}\left[\mathcal{B}_0^{\lfloor x^\xi \rfloor, \varepsilon}\right],
\end{aligned}$$

which, by the independence of N and $\{B_n^i(j), J_n\}$, implies

$$\mathbb{P}[Z_N > x] \leq \mathbb{P}\left[\Pi_N^0(1+\varepsilon)^N > x^{1-\xi}\right] + \mathbb{E}[N]\mathbb{P}\left[\mathcal{B}_0^{\lfloor x^\xi \rfloor, \varepsilon}\right]. \qquad (43)$$

Next, define a new process $\{\Pi_n^\varepsilon = \Pi_n^0(1+\varepsilon)^n\}$. It is easy to see that, for ε small enough, the sequence $\{\Pi_n^\varepsilon\}$ satisfies $n^{-1}\log \mathbb{E}\left[(\Pi_n^\varepsilon)^\alpha\right] \to \Psi(\alpha) + \alpha\log(1+\varepsilon)$. Therefore, by Theorem 9, we obtain

$$\lim_{x\to\infty} \frac{\log \mathbb{P}\left[\Pi_N^0(1+\varepsilon)^N > x^{1-\xi}\right]}{\log x} = -(1-\xi)\alpha_\varepsilon^*, \qquad (44)$$

where α_ε^* satisfies $\Psi(\alpha_\varepsilon^*) + \alpha_\varepsilon^* \log(1+\varepsilon) = 0$. Combining (43), (44) and Lemma 6, we obtain

$$\overline{\lim_{x\to\infty}} \frac{\log \mathbb{P}[Z_N > x]}{\log x} \le -(1-\xi)\alpha_\varepsilon^*,$$

which, by passing $\varepsilon, \xi \to 0$, completes the proof of the upper bound.

Now, we prove the *lower bound*. For $0 < \xi < 1, n \ge 0$, noting

$$\mathbb{P}[Z_n > x] \ge \frac{\lfloor x^\xi \rfloor}{x^\xi} \mathbb{P}[Z_n^1 > x] \ge \frac{1}{x^\xi} \mathbb{P}\left[Z_n^{\lfloor x^\xi \rfloor} > x \lfloor x^\xi \rfloor \right],$$

and recalling the definition of $\mathcal{C}_n^{l,\varepsilon}$ in Lemma 11, we derive

$$\begin{aligned} \mathbb{P}[Z_n > x] &\ge \frac{1}{x^\xi} \mathbb{P}\left[Z_n^{\lfloor x^\xi \rfloor} > x \lfloor x^\xi \rfloor, \bigcap_{i=0}^{n-1} \left(\mathcal{C}_i^{\lfloor x^\xi \rfloor, \xi} \right)^C \right] \\ &\ge \frac{1}{x^\xi} \left(\mathbb{P}\left[\Pi_n^0 (1-\xi)^n > x \right] - n \mathbb{P}\left[\mathcal{C}_0^{\lfloor x^\xi \rfloor, \xi} \right] \right), \end{aligned}$$

which, by the independence of N and $\{B_n^i(j), J_n\}$, implies

$$\mathbb{P}[Z_N > x] \ge \frac{1}{x^\xi} \left(\mathbb{P}\left[\Pi_N^0 (1-\xi)^N > x \right] - \mathbb{E}[N] \mathbb{P}\left[\mathcal{C}_0^{\lfloor x^\xi \rfloor, \xi} \right] \right).$$

Then, by using the same approach as in the proof of the upper bound and Lemma 11, we can easily show that

$$\underline{\lim_{x\to\infty}} \frac{\log \mathbb{P}[Z_N > x]}{\log x} \ge -\alpha^*.$$

Finally, by combining the upper bound and the lower bound, we finish the proof. □

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