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Reduced-Load Equivalence and Induced Burstiness in GPS Queues with Long-Tailed Traffic Flows

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Abstract. We analyze the queueing behavior of long-tailed traffic flows under the Generalized Processor Sharing (GPS) discipline. We show a sharp dichotomy in qualitative behavior, depending on the relative values of the weight parameters. For certain weight combinations, an individual flow with long-tailed traffic characteristics is effectively served at a *constant* rate. The effective service rate may be interpreted as the maximum average traffic rate for the flow to be stable, which is only influenced by the traffic characteristics of the other flows through their average rates. In particular, the flow is essentially immune from excessive activity of flows with 'heavier'-tailed traffic characteristics. In many situations, the effective service rate is simply the link rate reduced by the aggregate average rate of the other flows. This confirms that GPS-based scheduling algorithms provide a potential mechanism for extracting significant multiplexing gains, while isolating individual flows. For other weight combinations however, a flow may be strongly affected by the activity of 'heavier'-tailed flows, and may inherit their traffic characteristics, causing induced burstiness. The stark contrast in qualitative behavior illustrates the crucial importance of the weight parameters.

Keywords: Generalized Processor Sharing, induced burstiness, long-tailed traffic, reduced-load equivalence, Weighted Fair Queueing, workload asymptotics

1. Introduction

Measurements indicate that traffic in high-speed networks exhibits burstiness on a wide range of time scales, manifesting itself in long-range dependence and self-similarity, see, for instance, [32,39]. The occurrence of these phenomena is commonly attributed to extreme variability and long-tailed characteristics in the underlying activity patterns (connection times, file sizes, scene lengths), see, for instance, [6,25,43]. This has triggered a strong interest in queueing models with long-tailed traffic characteristics.

Although the presence of long-tailed traffic characteristics is widely acknowledged, the practical implications for network performance and traffic engineering remain to be fully resolved. For small buffer sizes and a large number of sources, the impact of long-tailed traffic characteristics may not be as pronounced as found in studies for infinite buffers, see [28,29,33,40]. For larger buffer sizes, flow control mechanisms play a critical role in preventing long-tailed activity patterns from overwhelming the buffer contents, see [4], although the end-to-end delay may still be affected.

Scheduling and priority mechanisms also play a major role in controlling the effect of long-tailed traffic characteristics on network performance. The present paper specifically examines the effectiveness of Generalized Processor Sharing (GPS) in isolating long-tailed traffic flows. As a design paradigm, GPS is at the heart of commonly-used scheduling algorithms for high-speed switches, such as Weighted Fair Queueing, see, for instance, [37,38].

A basic approach in the analysis of long-tailed traffic phenomena is the use of fluid models with long-tailed arrival processes (e.g., on/off sources with long-tailed on-periods). Fluid models are closely related to the ordinary single-server queue, thus bringing within reach the classical results on regularly-varying [24] or subexponential [36,42] behavior of the service and waiting-time distribution in the $GI/G/1$ queue. Those results are immediately applicable in the case of a single long-tailed arrival stream, see [16,22]. They are also of use when a single long-tailed stream is multiplexed with exponential streams, see [17,31]. We refer to [20] for a comprehensive survey on fluid queues with long-tailed arrival processes. See also [30] for an extensive list of references on subexponential queueing models.

The impact of priority and scheduling mechanisms on long-tailed traffic phenomena has received relatively little attention. Some recent studies have investigated the effect of the scheduling discipline on the waiting-time distribution in the classical $M/G/1$ queue, see, for instance, [3]. For FCFS, it is well known [24] that the waiting-time tail is regularly varying of index $1 - \nu$ iff the service time tail is regularly varying of index $-\nu$. For LCFS preemptive resume as well as for processor sharing, the waiting-time tail turns out to be regularly varying of the *same* index as the service time tail, see [18,49], although with different pre-factors. In the case of processor sharing with several customer classes, Zwart [47] showed that the sojourn time distribution of a class-i customer is regularly varying of index $-\nu_i$ iff the service time distribution of that class is regularly varying of index $-\nu_i$, *regardless* of the service time distributions of the other classes. In contrast, for two customer classes with ordinary non-preemptive priority, the tail behavior of the waiting and sojourn time distributions is determined by the *heaviest* of the (regularly-varying) service time distributions, see [1,19].

In the present paper, we consider the Generalized Processor Sharing (GPS) discipline. GPS-based scheduling algorithms, such as weighted fair queueing, play a major role in achieving differentiated quality-of-service in integrated-services networks. The queueing analysis of GPS is extremely difficult. One of the earliest studies is [27]. Interesting partial results for exponential traffic models were obtained in [7,26,34,44,45].

Here, we focus on non-exponential traffic models, extending the results of [9,10,12]. We show that, for certain weight combinations, an individual flow with long-tailed traffic characteristics is effectively served at a *constant* rate. The effective service rate may be interpreted as the maximum average traffic rate for the flow to be stable, which is only influenced by the traffic characteristics of the other flows through their average rates. In particular, the flow is essentially immune from excessive activity of flows with 'heavier'-tailed traffic characteristics. In many situations, the effective service rate is simply the link rate reduced by the aggregate average rate of the other flows. This is strongly reminiscent of the reduced-load equivalence established by Agrawal et al. [2]. For other weight combinations however, a flow may be strongly affected by the activity of 'heavier'-tailed flows, and may inherit their traffic characteristics, causing induced burstiness. In [11], qualitatively similar results were obtained for a closely related model of two coupled processors. For that model however, transform techniques were used, whereas in the present paper we will develop probabilistic lower and upper bounds and prove that these asymptotically coincide.

The remainder of the paper is organized as follows. In section 2, we present a detailed model description. In section 3, we consider a scenario where the traffic intensity of each flow is smaller than its weight in the GPS scheme. We start by deriving lower and upper bounds for the workload distribution of an individual flow. We show that the bounds, although quite crude by themselves, agree in terms of tail behavior, resulting in the exact workload asymptotics. Next, we consider the general situation where the traffic intensity of a flow may be larger than its GPS weight. We start by discussing some stability issues, and then introduce a stability-related notion which plays a crucial role in the analysis. We distinguish between two cases, depending on whether a flow may be affected by other flows or not. These cases are examined in sections 4 and 5, respectively. In both cases, we establish bounds for the workload distribution of an individual flow, which are now more complicated and rely on more refined GPS properties. As before though, the bounds coincide as far as tail behavior is concerned, thus yielding exact asymptotic results. In section 6, we make some concluding remarks.

2. Model description

Consider N traffic flows sharing a link of unit rate. Traffic from the flows is served in accordance with the Generalized Processor Sharing (GPS) discipline, which operates as follows. Flow i is assigned a weight ϕ_i, with $\sum_{i=1}^{N} \phi_i = 1$. If all the flows are backlogged at time t, then flow i is served at rate ϕ_i. If some of the flows are not backlogged, however, then the excess capacity is redistributed among the backlogged flows in proportion to their respective weights. We refer to [26] for a formal description of the evolution of the backlog process.

Denote by $A_i(s, t)$ the amount of traffic generated by flow i during the time interval $(s, t]$. We assume that the process $A_i(s, t)$ is stationary. Denote by ρ_i the traffic intensity of flow i as will be defined below in detail for the two traffic scenarios that we consider. Denote by $V_i(t)$ the backlog (workload) of flow i at time t. Let $\mathbf{V}_i$ be a stochastic variable having the limiting distribution of $V_i(t)$ for $t \to \infty$ (assuming it exists). In the cases that we consider, the limiting distribution when it exists does not depend on the initial state of the system. As we are primarily interested in studying $\mathbf{V}_i$, we may thus assume without loss of generality that the system is initially empty, i.e., $V_j(0) = 0$ for all $j = 1, \ldots, N$.

Define $B_i(s,t)$ as the amount of service received by flow i during the time interval $(s,t]$. Then the following identity relation holds for all $0 \leqslant s \leqslant t$:

$$V_i(t) = V_i(s) + A_i(s,t) - B_i(s,t). \tag{1}$$

Denote by $A(s,t) := \sum_{i=1}^N A_i(s,t)$ the total amount of traffic generated during $(s,t]$. Define $\rho := \sum_{i=1}^N \rho_i$ as the total traffic intensity. Define $V(t) := \sum_{i=1}^N V_i(t)$ as the total workload at time t.

For any $c \geqslant 0$, denote by $V_i^c(t) := \sup_{0 \leqslant s \leqslant t}\{A_i(s,t) - c(t-s)\}$ the workload at time t in a queue with constant service rate c fed by flow i only (assuming $V_i^c(0) = 0$), see, for instance, [5]. For $c > \rho_i$, let $\mathbf{V}_i^c$ be a stochastic variable with as distribution the limiting distribution of $V_i^c(t)$ for $t \to \infty$. Define $B_i^c(s,t)$ as the amount of service received by flow i during $(s,t]$ in a queue with service rate c. Similarly to the identity relation above, for all $0 \leqslant s \leqslant t$:

$$V_i^c(t) = V_i^c(s) + A_i(s,t) - B_i^c(s,t). \tag{2}$$

Denote by $\mathbf{P}_i^c$ the duration of the busy period associated with the workload process $\mathbf{V}_i^c$. We occasionally use the short-hand notation $\mathbf{P}_i$ when the service capacity c is clear from the context.

Before describing the traffic model, we first introduce some additional notation.

For any two real functions $g(\cdot)$ and $h(\cdot)$, we use the notational convention $g(x) \sim h(x)$ to denote $\lim_{x\to\infty} g(x)/h(x) = 1$, or equivalently, $g(x) = h(x)(1 + \mathrm{o}(1))$ as $x \to \infty$.

For any stochastic variable $\mathbf{X}$ with distribution function $F(\cdot)$, $\mathbb{E}\mathbf{X} < \infty$, denote by $F^r(\cdot)$ the distribution function of the residual lifetime of $\mathbf{X}$, i.e., $F^r(x) = (1/\mathbb{E}\mathbf{X}) \int_0^x (1 - F(y))\,\mathrm{d}y$, and by $\mathbf{X}^r$ a stochastic variable with distribution $F^r(\cdot)$.

The classes of *long-tailed*, *subexponential*, *regularly varying*, and *intermediately regularly varying* distributions are denoted with the symbols $\mathcal{L}$, $\mathcal{S}$, $\mathcal{R}$, and $\mathcal{IR}$, respectively. The definitions of these classes may be found in [8].

We now describe the two traffic scenarios that we consider.

2.1. *Instantaneous input*

Here, a flow generates instantaneous traffic bursts according to a renewal process. The interarrival times between bursts of flow i have distribution function $U_i(\cdot)$ with mean $1/\lambda_i$. The burst sizes of flow i have distribution $S_i(\cdot)$ with mean $\sigma_i < \infty$. Thus, the traffic intensity of flow i is $\rho_i = \lambda_i \sigma_i$.

We now state some results which will play a crucial role in the analysis.

Theorem 2.1 (Pakes [36]). If $S_i^r(\cdot) \in \mathcal{S}$, and $\rho_i < c$, then

$$\mathbb{P}\{\mathbf{V}_i^c > x\} \sim \frac{\rho_i}{c - \rho_i}\mathbb{P}\{\mathbf{S}_i^r > x\}.$$

Theorem 2.2 (Zwart [48]). If $U_i(\cdot)$ is an exponential distribution, i.e., the arrival process is Poisson, $S_i(\cdot) \in \mathcal{IR}$, and $\rho_i < c$, then

$$\mathbb{P}\{\mathbf{P}_i > x\} \sim \frac{c}{c - \rho_i}\mathbb{P}\{\mathbf{S}_i > x(c - \rho_i)\}.$$

In fact, the preceding theorem can be extended to non-Poisson arrival processes, see [47]. In the analysis we will need a slight modification:

Theorem 2.3. If $U_i(\cdot)$ is an exponential distribution, $S_i^r(\cdot) \in \mathcal{IR}$, and $\rho_i < c$, then

$$\mathbb{P}\{\mathbf{P}_i^r > x\} \sim \frac{c}{c - \rho_i}\mathbb{P}\{\mathbf{S}_i^r > x(c - \rho_i)\}.$$

Remark 2.1. Although theorem 2.3 is only a minor extension of theorem 2.2, the proof (see [13]) is new and might be of independent interest. It directly uses theorem 2.1 to derive the asymptotic behavior of the residual busy period. Note that if $S_i(\cdot) \in \mathcal{IR}$, then theorem 2.2 implies theorem 2.3. However, if we only assume $S_i^r(\cdot) \in \mathcal{IR}$, then we cannot directly use theorem 2.2, since $S_i^r(\cdot) \in \mathcal{IR}$ does not necessarily imply $S_i(\cdot) \in \mathcal{IR}$.

2.2. *Fluid input*

Here, a flow generates traffic according to an on–off process, alternating between on- and off-periods. The off-periods of flow i have distribution function $U_i(\cdot)$ with mean $1/\lambda_i$. The on-periods of flow i have distribution $S_i(\cdot)$ with mean $\sigma_i < \infty$. While on, flow i produces traffic at a constant rate r_i, so the mean burst size is $\sigma_i r_i$. The fraction of time that flow i is off is

$$p_i = \frac{1/\lambda_i}{1/\lambda_i + \sigma_i} = \frac{1}{1 + \lambda_i \sigma_i}.$$

The traffic intensity of flow i is

$$\rho_i = (1 - p_i) r_i = \frac{\lambda_i \sigma_i r_i}{1 + \lambda_i \sigma_i}.$$

We now state the analogues of theorems 2.1–2.3 in the case of on–off processes.

Theorem 2.4 (Jelenković and Lazar [31]). If $S_i^r(\cdot) \in \mathcal{S}$, and $\rho_i < c < r_i$, then

$$\mathbb{P}\{\mathbf{V}_i^c > x\} \sim p_i \frac{\rho_i}{c - \rho_i}\mathbb{P}\left\{\mathbf{S}_i^r > \frac{x}{r_i - c}\right\}.$$

Theorem 2.5 (Boxma and Dumas [21], Zwart [48]). If $U_i(\cdot)$ is an exponential distribution, i.e., the off-periods are exponentially distributed, $S_i(\cdot) \in \mathcal{IR}$, and $\rho_i < c < r_i$, then

$$\mathbb{P}\{\mathbf{P}_i > x\} \sim p_i \frac{c}{c-\rho_i} \mathbb{P}\left\{\mathbf{S}_i > \frac{x(c-\rho_i)}{r_i-\rho_i}\right\}.$$

In addition, the following minor extension of the preceding theorem holds:

Theorem 2.6. If $U_i(\cdot)$ is an exponential distribution, $S_i^r(\cdot) \in \mathcal{IR}$, and $\rho_i < c < r_i$, then

$$\mathbb{P}\left\{\mathbf{P}_i^r > x\right\} \sim p_i \frac{c}{c-\rho_i} \mathbb{P}\left\{\mathbf{S}_i^r > \frac{x(c-\rho_i)}{r_i-\rho_i}\right\}.$$

Remark 2.2. Theorems 2.5 and 2.6 follow directly from theorems 2.2 and 2.3 because of a useful equivalence relation observed by Boxma and Dumas [21] and Zwart [46]. The busy period in a fluid queue is equal in distribution to the busy period in a corresponding $G/G/1$ queue scaled by a factor $r_i/(r_i - c_i)$. The interarrival times in the $G/G/1$ queue are exactly the off-periods in the fluid queue, and the service times correspond to the *net input* during the on-periods. Thus, with some minor abuse of notation, $\mathbb{P}\{\mathbf{P}_i > x\} = \mathbb{P}\{\mathbf{P}_i^{G/G/1} > x(r_i - c)/r_i\}$ for all values of x, with $U_i^{G/G/1}(\cdot) = U_i(\cdot)$ and $\mathbf{S}_i^{G/G/1} := (r_i - c)\mathbf{S}_i$.

From theorem 2.2, noting that $c - \rho_i^{M/G/1} = (c - \rho_i)/p_i$ and $p_i r_i = r_i - \rho_i$,

$$\begin{aligned}\mathbb{P}\left\{\mathbf{P}_i^{M/G/1} > \frac{x(r_i-c)}{r_i}\right\} &\sim \frac{c}{c-\rho_i^{M/G/1}} \mathbb{P}\left\{\mathbf{S}_i^{M/G/1} > x\left(c-\rho_i^{M/G/1}\right)\frac{r_i-c}{r_i}\right\} \\ &= p_i \frac{c}{c-\rho_i} \mathbb{P}\left\{\mathbf{S}_i > \frac{x(c-\rho_i)}{r_i-\rho_i}\right\},\end{aligned}$$

yielding theorem 2.5.

In [21] theorem 2.6 was essentially obtained in this manner from a weaker version of [35, theorem 2.2] for the case $S_i(\cdot) \in \mathcal{R}$. Similarly, theorem 2.6 for the residual busy period can be directly obtained from theorem 2.3.

Alternatively, theorem 2.6 can be proved by mimicking the proof of theorem 2.3 as provided in [13].

3. Reduced-load equivalence

3.1. Bounds

We first derive bounds for the workload distribution which we will use in the next subsection to analyze the tail behavior. We focus on a particular yet arbitrary flow i. The bounds do not involve any specific assumptions regarding the traffic model. In particular, the bounds apply for the two traffic scenarios described in sections 2.1 and 2.2.

The bounds rely on the following two simple properties of the GPS discipline:

(i) it is work-conserving, i.e., it serves at the full link rate whenever any of the flows is backlogged;

(ii) it guarantees minimum rates $\phi_1, \dots, \phi_N$, i.e., it serves flow i at least at rate ϕ_i whenever flow i is backlogged.

From property (i),

$$V(t) = \sup_{0 \leqslant s \leqslant t} \bigl\{A(s,t) - (t-s)\bigr\} \quad \text{for all } t \geqslant 0. \tag{3}$$

From property (ii),

$$V_i(t) \leqslant V_i^{\phi_i}(t) \quad \text{for all } t \geqslant 0. \tag{4}$$

In the remainder of the section, we make the following crucial assumption.

Assumption 3.1. The traffic intensities and weights satisfy $\rho_i < \phi_i$ for all $i = 1, \dots, N$.

Note that the above assumption ensures stability of the flows. For a formal stability proof, we refer to [26].

We first present a lower bound for the workload distribution of flow i. For compactness, define $A_{-i}(s,t) := A(s,t) - A_i(s,t) = \sum_{j \neq i} A_j(s,t)$ as the aggregate amount of traffic generated by all flows other than i during the time interval $(s,t]$. Also, denote by $\rho_{-i} := \rho - \rho_i = \sum_{j \neq i} \rho_j$ the aggregate traffic intensity of these flows. For any $c \geqslant 0$, define $Z_{-i}^c(t) := \sup_{0 \leqslant s \leqslant t}\{c(t-s) - A_{-i}(s,t)\}$. For $c < \rho_{-i}$, let $\mathbf{Z}_{-i}^c$ be a stochastic variable having the limiting distribution of $Z_{-i}^c(t)$ for $t \to \infty$.

Lemma 3.1 (Lower bound). For any $\delta > 0$,

$$\mathbb{P}\{\mathbf{V}_i > x\} \geqslant \mathbb{P}\Biggl\{\mathbf{V}_i^{1-\rho_{-i}+\delta} - \mathbf{Z}_{-i}^{\rho_{-i}-\delta} - \sum_{j \neq i} \mathbf{V}_j^{\phi_j} > x\Biggr\}. \tag{5}$$

Proof. Using properties (3), (4) we obtain, for any θ,

$$\begin{aligned}
V_i(t) &= V(t) - \sum_{j \neq i} V_j(t) \\
&\geqslant V(t) - \sum_{j \neq i} V_j^{\phi_j}(t) \\
&= \sup_{0 \leqslant s \leqslant t} \bigl\{A(s,t) - (t-s)\bigr\} - \sum_{j \neq i} V_j^{\phi_j}(t) \\
&= \sup_{0 \leqslant s \leqslant t} \bigl\{A_i(s,t) - (1-\theta)(t-s) + A_{-i}(s,t) - \theta(t-s)\bigr\} - \sum_{j \neq i} V_j^{\phi_j}(t)
\end{aligned}$$

$$\begin{aligned}
&\geqslant \sup_{0\leqslant s\leqslant t}\bigl\{A_i(s,t)-(1-\theta)(t-s)\bigr\}+\inf_{0\leqslant s\leqslant t}\bigl\{A_{-i}(s,t)-\theta(t-s)\bigr\}-\sum_{j\neq i}V_j^{\phi_j}(t)\\
&=\sup_{0\leqslant s\leqslant t}\bigl\{A_i(s,t)-(1-\theta)(t-s)\bigr\}-\sup_{0\leqslant s\leqslant t}\bigl\{\theta(t-s)-A_{-i}(s,t)\bigr\}-\sum_{j\neq i}V_j^{\phi_j}(t)\\
&=V_i^{1-\theta}(t)-Z_{-i}^{\theta}(t)-\sum_{j\neq i}V_j^{\phi_j}(t)\quad\text{for all } t\geqslant 0.
\end{aligned}$$

In particular, taking $\theta=\rho_{-i}-\delta$, we have

$$V_i(t)\geqslant V_i^{1-\rho_{-i}+\delta}(t)-Z_{-i}^{\rho_{-i}-\delta}(t)-\sum_{j\neq i}V_j^{\phi_j}(t)\quad\text{for all } t\geqslant 0.$$

Thus, in the stationary regime (5) holds. □

We now provide an upper bound for the workload distribution of flow i. For any $c\geqslant 0$, define $V_{-i}^c(t):=\sup_{0\leqslant s\leqslant t}\{A_{-i}(s,t)-c(t-s)\}$ as the workload at time t in a queue with constant service rate c fed by all flows other than i. For $c>\rho_{-i}$, let $\mathbf{V}_{-i}^c$ be a stochastic variable having the limiting distribution of $V_{-i}^c(t)$ for $t\to\infty$.

Lemma 3.2 (Upper bound). For any $\delta>0$

$$\mathbb{P}\{\mathbf{V}_i>x\}\leqslant\mathbb{P}\bigl\{\mathbf{V}_i^{\phi_i}>x,\mathbf{V}_i^{1-\rho_{-i}-\delta}+\mathbf{V}_{-i}^{\rho_{-i}+\delta}>x\bigr\}.\tag{6}$$

Proof. Using property (3), we have, for any θ,

$$\begin{aligned}
V_i(t)&\leqslant V(t)\\
&=\sup_{0\leqslant s\leqslant t}\bigl\{A(s,t)-(t-s)\bigr\}\\
&=\sup_{0\leqslant s\leqslant t}\bigl\{A_i(s,t)-(1-\theta)(t-s)+A_{-i}(s,t)-\theta(t-s)\bigr\}\\
&\leqslant\sup_{0\leqslant s\leqslant t}\bigl\{A_i(s,t)-(1-\theta)(t-s)\bigr\}+\sup_{0\leqslant s\leqslant t}\bigl\{A_{-i}(s,t)-\theta(t-s)\bigr\}\\
&=V_i^{1-\theta}(t)+V_{-i}^{\theta}(t)\quad\text{for all } t\geqslant 0.
\end{aligned}$$

Invoking property (4), and taking $\theta=\rho_{-i}+\delta$, we obtain

$$V_i(t)\leqslant\min\bigl\{V_i^{\phi_i}(t),V_i^{1-\rho_{-i}-\delta}(t)+V_{-i}^{\rho_{-i}+\delta}(t)\bigr\}\quad\text{for all } t\geqslant 0.$$

Thus, in the stationary regime (6) holds. □

3.2. Asymptotic behavior

We now use the bounds from the previous subsection to determine the tail distribution of the workload. Denote by $c_i:=1-\rho_{-i}=1-\sum_{j\neq i}\rho_j$ the link rate reduced by the aggregate average rate of all flows other than i. We consider a specific flow i which satisfies the following three properties for $c=c_i$.

Property 3.1. $\mathbb{P}\{\mathbf{V}_i^c > x\} \in \mathcal{L}$, i.e.,

$$\lim_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_i^c > x - y\}}{\mathbb{P}\{\mathbf{V}_i^c > x\}} = 1, \quad \text{for all real } y.$$

Property 3.2. For any $\theta > 0$,

$$\liminf_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_i^{c+\theta} > x\}}{\mathbb{P}\{\mathbf{V}_i^c > x\}} = F_i^c(\theta),$$

with $\lim_{\theta\downarrow 0} F_i^c(\theta) = 1$.

Property 3.3. For any $0 < \theta < c - \rho_i$,

$$\limsup_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_i^{c-\theta} > x\}}{\mathbb{P}\{\mathbf{V}_i^c > x\}} = G_i^c(\theta) < \infty,$$

with $\lim_{\theta\downarrow 0} G_i^c(\theta) = 1$.

According to theorem 2.1, in case of instantaneous input, flow i satisfies properties 3.1–3.3 for any $c > \rho_i$ if $S_i^r(\cdot) \in \mathcal{S}$.

According to theorem 2.4, in case of fluid input, flow i satisfies property 3.1 for any $r_i > c > \rho_i$ if $S_i^r(\cdot) \in \mathcal{S}$, and properties 3.2 and 3.3 if $S_i^r(\cdot) \in \mathcal{IR}$.

We now give the main result of this section.

Theorem 3.1. Consider a flow i which satisfies properties 3.1–3.3 for $c = c_i$. If assumption 3.1 holds, then

$$\mathbb{P}\{\mathbf{V}_i > x\} \sim \mathbb{P}\big\{\mathbf{V}_i^{c_i} > x\big\}.$$

Proof (Lower bound). From lemma 3.1, using independence, for any $\delta > 0$ and y,

$$\begin{aligned}\mathbb{P}\{\mathbf{V}_i > x\} &\geqslant \mathbb{P}\Bigg\{\mathbf{V}_i^{c_i+\delta} > x + y, \mathbf{Z}_{-i}^{\rho_{-i}-\delta} + \sum_{j\neq i} \mathbf{V}_j^{\phi_j} \leqslant y\Bigg\} \\ &= \mathbb{P}\big\{\mathbf{V}_i^{c_i+\delta} > x + y\big\}\mathbb{P}\Bigg\{\mathbf{Z}_{-i}^{\rho_{-i}-\delta} + \sum_{j\neq i} \mathbf{V}_j^{\phi_j} \leqslant y\Bigg\}.\end{aligned}$$

Thus

$$\frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{V}_i^{c_i} > x\}} \geqslant \frac{\mathbb{P}\{\mathbf{V}_i^{c_i+\delta} > x + y\}}{\mathbb{P}\{\mathbf{V}_i^{c_i} > x + y\}} \frac{\mathbb{P}\{\mathbf{V}_i^{c_i} > x + y\}}{\mathbb{P}\{\mathbf{V}_i^{c_i} > x\}} \mathbb{P}\Bigg\{\mathbf{Z}_{-i}^{\rho_{-i}-\delta} + \sum_{j\neq i} \mathbf{V}_j^{\phi_j} \leqslant y\Bigg\}.$$

Using the fact that $\mathbb{P}\{\mathbf{V}_i^{c_i} > x\}$ satisfies properties 3.1 and 3.2,

$$\liminf_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{V}_i^{c_i} > x\}} \geqslant F_i^{c_i}(\delta)\mathbb{P}\Bigg\{\mathbf{Z}_{-i}^{\rho_{-i}-\delta} + \sum_{j\neq i} \mathbf{V}_j^{\phi_j} \leqslant y\Bigg\}.$$

Letting $y \to \infty$, and then $\delta \downarrow 0$, we have

$$\liminf_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{V}_i^{c_i} > x\}} \geqslant 1.$$

(*Upper bound*). From lemma 3.2, using independence, for any $0 < \delta < 1 - \rho$ and y,

$$\begin{aligned}\mathbb{P}\{\mathbf{V}_i > x\} &\leqslant \mathbb{P}\big\{\mathbf{V}_i^{\phi_i} > x, \mathbf{V}_i^{c_i-\delta} > x - y \text{ or } \mathbf{V}_{-i}^{\rho_{-i}+\delta} > y\big\} \\ &\leqslant \mathbb{P}\big\{\mathbf{V}_i^{c_i-\delta} > x - y\big\} + \mathbb{P}\big\{\mathbf{V}_i^{\phi_i} > x, \mathbf{V}_{-i}^{\rho_{-i}+\delta} > y\big\} \\ &= \mathbb{P}\big\{\mathbf{V}_i^{c_i-\delta} > x - y\big\} + \mathbb{P}\big\{\mathbf{V}_i^{\phi_i} > x\big\}\mathbb{P}\big\{\mathbf{V}_{-i}^{\rho_{-i}+\delta} > y\big\}.\end{aligned}$$

Thus

$$\frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{V}_i^{c_i} > x\}} \leqslant \frac{\mathbb{P}\{\mathbf{V}_i^{c_i-\delta} > x - y\}}{\mathbb{P}\{\mathbf{V}_i^{c_i} > x - y\}} \frac{\mathbb{P}\{\mathbf{V}_i^{c_i} > x - y\}}{\mathbb{P}\{\mathbf{V}_i^{c_i} > x\}} + \frac{\mathbb{P}\{\mathbf{V}_i^{\phi_i} > x\}}{\mathbb{P}\{\mathbf{V}_i^{c_i} > x\}} \mathbb{P}\big\{\mathbf{V}_{-i}^{\rho_{-i}+\delta} > y\big\}.$$

Using the fact that $\mathbb{P}\{\mathbf{V}_i^{c_i} > x\}$ satisfies properties 3.1 and 3.3,

$$\limsup_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{V}_i^{c_i} > x\}} \leqslant G_i^{c_i}(\delta) + G_i^{c_i}(c_i - \phi_i)\mathbb{P}\big\{\mathbf{V}_{-i}^{\rho_{-i}+\delta} > y\big\}.$$

Letting $y \to \infty$, and then $\delta \downarrow 0$, we obtain

$$\limsup_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{V}_i^{c_i} > x\}} \leqslant 1. \qquad \square$$

Theorem 3.1 states that the workload of an individual flow i (with long-tailed traffic characteristics) is asymptotically equivalent to that in an isolated system. In the isolated system, flow i is served at a *constant* rate, which is equal to the link rate reduced by the aggregate average rate of all other flows. The result suggests that the most likely way for flow i to build a large queue is that the flow itself generates a large burst, or experiences a long on-period, while all other flows show roughly average behavior, each flow j consuming a fraction ρ_j of the link rate. During that period, flow i then receives service approximately at rate $c_i = 1 - \sum_{j\neq i} \rho_j$.

Thus, asymptotically, the workload of flow i is only affected by the traffic characteristics of the other flows through their aggregate average rate. In particular, flow i is essentially immune from excessive activity of other flows, even when those have 'heavier'-tailed traffic characteristics.

The result is reminiscent of the 'reduced-load equivalence' established by Agrawal et al. [2] and a result derived in [31] for multiplexing exponential with subexponential flows. In these scenarios, the total workload is asymptotically equivalent to that in a reduced system. The reduced system consists of a single dominant flow i served at the link rate reduced by the aggregate average rate of all other flows. However, these results do require bounding conditions on the variability of the other flows. Here, such conditions are not needed because of the properties of the GPS discipline. In fact, we have only used the following two properties of the GPS discipline in establishing theorem 3.1:

(i) it is work-conserving; (ii) it guarantees minimum rates $\phi_1, \ldots, \phi_N$. Thus, the result does not rely on the specific way in which excess capacity is redistributed in GPS, but holds for any rate sharing algorithm with the above two properties. Also, the workload is not significantly influenced by the exact values of the GPS weights (as long as they are larger than the average flow rates as stipulated in assumption 3.1).

Now suppose each of the flows were served in isolation. Then the required service capacity to achieve similar tail behavior is

$$\sum_{i=1}^{N} c_i = \sum_{i=1}^{N}\left(1 - \sum_{j \neq i} \rho_j\right) = \sum_{i=1}^{N}(1 - \rho + \rho_i) = 1 + (N - 1)(1 - \rho).$$

The latter quantity may typically be expected to be substantially larger than 1. This confirms that GPS-based scheduling algorithms provide an effective mechanism for extracting significant multiplexing gains, while isolating indvidual flows.

To conclude the section, we briefly discuss the significance of assumption 3.1. The assumption that $\rho_i < \phi_i$ for all $i = 1, \ldots, N$ implies two crucial properties:

(i) flow i is always guaranteed to receive service at a stable rate, even when other flows generate large bursts or experience long on-periods;

(ii) when flow i generates a large burst, or experiences a long on-period, and thus builds up a large queue, all other flows j continue to be served at a stable rate, demanding a fraction ρ_j of the link rate.

If assumption 3.1 is relaxed, then two complicating situations may arise: (i) when other flows generate large bursts or experience long on-periods, flow i may not receive service at a stable rate, and thus build up a large queue; (ii) when flow i generates a large burst, or experiences a long on-period, not all other flows j may continue to be served at a stable rate, so some may consume *less* than a fraction ρ_j of the link rate.

In scenario (i), the tail behavior of flow i may potentially be affected by other flows with 'heavier'-tailed traffic characteristics, which drastically complicates the analysis. In case (ii), precisely what rate the other flows will get, depends on the exact values of the GPS weights (or the detailed mechanics of the rate sharing algorithm in general). We will examine these scenarios in detail in the next sections when we relax assumption 3.1.

4. Generalized reduced-load equivalence

4.1. Stability issues

We now relax the assumption that the traffic intensities and weights satisfy $\rho_i < \phi_i$ for all $i = 1, \ldots, N$, so that stability of the flows is not automatically ensured. In case $\sum_{i=1}^{N} \rho_i < 1$, all flows will remain stable, because the GPS discipline is work-conserving. However, the scenario $\sum_{i=1}^{N} \rho_i > 1$ may occur as well now. In that case, at least one of the flows will be unstable, while others may still be stable.

We now identify which flows are stable and which ones are unstable. To avoid technical subtleties, flow i is considered ‘stable’ if its mean service rate is ρ_i, see also remark 4.1 below. For ease of presentation, we assume the flows are indexed such that

$$\frac{\rho_1}{\phi_1} \leqslant \cdots \leqslant \frac{\rho_N}{\phi_N}.$$

Lemma 4.1. With the above ordering, the set of stable flows is $S^* = \{1, \ldots, K^*\}$, with

$$K^* = \max_{k=1,\ldots,N}\left\{k\colon \frac{\rho_k}{\phi_k} \leqslant \frac{1-\sum_{j=1}^{k-1}\rho_j}{\sum_{j=k}^{N}\phi_j}\right\}.$$

Proof. See appendix A. □

It may be verified that $K^* = N$ (i.e. all the flows receive a stable service rate) iff $\sum_{i=1}^{N}\rho_i \leqslant 1$. By definition, each of the stable flows $i \in S^*$ receives a mean service rate ρ_i. Each of the unstable flows $i \notin S^*$ receives a mean service rate $\phi_i R < \rho_i$, with

$$R = \frac{1}{\sum_{j\notin S^*}\phi_j}\left(1 - \sum_{j\in S^*}\rho_j\right).$$

To understand the above formula, notice that the stable flows consume an average aggregate rate $\sum_{j\in S^*}\rho_j$, leaving an average rate $1-\sum_{j\in S^*}\rho_j$ for the unstable flows, which is shared in proportion to the weights ϕ_i.

We now introduce a stability-related notion which will play a fundamental role in the analysis. Define γ_{iE} as the mean rate at which flow i would receive service if the flows $j \in E$ were to continuously claim their full share of the link rate according to the assigned weights ϕ_j (while the remaining flows $j \notin E$ still acted ‘normally’). (With minor abuse of notation we write γ_{ij} for $\gamma_{i\{j\}}$ and abbreviate γ_{ii} to γ_i.) Now observe that the flows $j \in E$ would in fact show such greedy behavior if they were unstable (which they need not be in reality). So we may determine γ_{iE} by forcing the flows $j \in E$ into the set of unstable flows, and then apply lemma 4.1. The set of flows which would receive a stable service rate if the flows $j \in E$ were to continuously claim their full share of the link rate, is then $S_E := \{1, \ldots, K_E^*\} \setminus E$, with

$$K_E^* = \max_{k=1,\ldots,N}\left\{k : \frac{\rho_k}{\phi_k} \leqslant \frac{1-\sum_{j=1}^{k-1}\rho_j \mathrm{I}_{\{j\notin E\}}}{\sum_{j=k}^{N}\phi_j \mathrm{I}_{\{j\notin E\}} + \sum_{j\in E}\phi_j}\right\}.$$

Thus, $\gamma_{iE} = \rho_i$ for all $i \in S_E$, and $\gamma_{iE} = \phi_i R_E < \rho_i$ for all $i \notin S_E$, with

$$R_E = \frac{1}{\sum_{j\notin S_E}\phi_j}\left(1 - \sum_{j\in S_E}\rho_j\right).$$

To explain the above formula, observe that the flows $j \in S_E$ by definition receive an average aggregate rate $\sum_{j\in S_E}\rho_j$, leaving an average rate $1-\sum_{j\in S_E}\rho_j$ for the flows $j \notin S_E$, which is shared in proportion to the weights ϕ_i.

Remark 4.1. For later purposes, we find it convenient to label flow i as 'stable' if the mean service rate is ρ_i. In fact, the latter condition is necessary for stability in the usual sense, but not entirely sufficient. A sufficient condition is $\rho_i < \gamma_i$. Indeed, if the queue of flow i never emptied, then it would receive a mean service rate γ_i, so that γ_i is the critical mean rate for stability.

4.2. Bounds

We first derive bounds for the workload distribution which we will use in the next subsection to analyze the tail behavior. We focus on a particular yet arbitrary flow i for which we assume $\rho_i < \gamma_i$ to ensure stability.

We first introduce some additional notation. For any subset $E \subseteq \{1, \ldots, N\}$, define

$$\gamma_{iE}(\delta) = (1-\delta)\gamma_{iE} = (1-\delta)\rho_i \quad \text{for all } i \in S_E,$$

and

$$\gamma_{iE}(\delta) = \phi_i R_E(\delta) \quad \text{for all } i \notin S_E,$$

with

$$R_E(\delta) = \frac{1}{\sum_{j \notin S_E} \phi_j}\left(1 - \sum_{j \in S_E} \gamma_{jE}(\delta)\right) = \frac{1}{\sum_{j \notin S_E} \phi_j}\left(1 - (1-\delta)\sum_{j \in S_E} \rho_j\right).$$

Note that $\sum_{i=1}^N \gamma_{iE}(\delta) = 1$ for all values of δ (unless $E = \emptyset$).

We now state some preliminary results which will play a crucial role in deriving the bounds.

Lemma 4.2. Let $E, S, T \subseteq \{1, \ldots, N\}$ be sets with $S_E \subseteq S$, $S \cap T = \emptyset$.

Then

$$\sum_{j \in S} B_j(r,t) \geqslant \sum_{j \in S} \inf_{r \leqslant s \leqslant t}\left\{A_j(r,s) + \frac{\gamma_{jE}(\delta)}{\sum_{k \notin T} \gamma_{kE}(\delta)}\left[t - s - \sum_{k \in T} B_k(s,t)\right]\right\},$$

for all $\delta \geqslant \delta_0$ for some $\delta_0 < 0$.

The proof of the above lemma may be found in [13].

Lemma 4.3. Let $E, S \subseteq \{1, \ldots, N\}$ be sets with $E \neq \emptyset$, $S_E \subseteq S$.

Then

$$\sum_{j \in S} B_j(r,t) \geqslant \sum_{j \in S} \inf_{r \leqslant s \leqslant t}\left\{A_j(r,s) + \gamma_{jE}(\delta)(t-s)\right\},$$

for all $\delta \geqslant \delta_0$ for some $\delta_0 < 0$.

Proof. The statement follows immediately from lemma 4.2 when taking $T = \emptyset$ so that $\sum_{k \in T} B_k(s,t) = 0$ and $\sum_{k \notin T} \gamma_{kE}(\delta) = \sum_{k=1}^N \gamma_{kE}(\delta) = 1$. □

Lemma 4.4. Let $E, S \subseteq \{1, \ldots, N\}$ be sets with $E \neq \emptyset$, $S_E \subseteq S$.
Then

$$\sum_{j \in S} V_j(t) \leqslant \sum_{j \in S} V_j^{\gamma_{jE}(\delta)}(t),$$

for all $\delta \geqslant \delta_0$ for some $\delta_0 < 0$.

Proof. Using the identity relation (1), lemma 4.3, and the assumption that $V_j(0) = 0$ for all $j = 1, \ldots, N$,

$$\begin{aligned} \sum_{j \in S} V_j(t) &= \sum_{j \in S} \big[A_j(0, t) - B_j(0, t)\big] \\ &\leqslant \sum_{j \in S} \big[A_j(0, t) - \inf_{0 \leqslant s \leqslant t} \big\{A_j(0, s) + \gamma_{jE}(\delta)(t - s)\big\}\big] \\ &= \sum_{j \in S} \sup_{0 \leqslant s \leqslant t} \big\{A_j(s, t) - \gamma_{jE}(\delta)(t - s)\big\} = \sum_{j \in S} V_j^{\gamma_{jE}(\delta)}(t). \end{aligned}$$ □

We first present a lower bound for the workload distribution of flow i. For any $c \geqslant 0$, define $Z_j^c(r) := \sup_{s \geqslant r} \{c(s - r) - A_j(r, s)\}$. For $c < \rho_j$, let $\mathbf{Z}_j^c$ be a stochastic variable with as distribution the distribution of $Z_j^c(r)$ (which in fact does not depend on r because the process $A_j(s, t)$ is stationary).

Lemma 4.5 (Lower bound). For $\delta > 0$ sufficiently small,

$$\mathbb{P}\{\mathbf{V}_i > x\} \geqslant \mathbb{P}\Bigg\{\mathbf{V}_i^{\gamma_i(\delta)} - \sum_{j \neq i} \mathbf{Z}_j^{\rho_j(1-\delta)} > x\Bigg\}. \tag{7}$$

Proof. See appendix B. □

We now provide an upper bound for the workload distribution of flow i. For any subset $E \subseteq \{1, \ldots, N\}$, define $\psi_{iE} = \phi_i / \sum_{j \notin S_E} \phi_j$.

Lemma 4.6 (Upper bound). For $\delta > 0$ sufficiently small,

$$\mathbb{P}\{\mathbf{V}_i > x\} \leqslant \mathbb{P}\Bigg\{\mathbf{V}_i^{\gamma_{iE}(-\delta)} + \psi_{iE} \sum_{j \in S_E} \mathbf{V}_j^{\rho_j(1+\delta)} > x \text{ for all sets } E \ni i \text{ with } \gamma_{iE} > \rho_i\Bigg\}. \tag{8}$$

Proof. See appendix C. □

4.3. *Asymptotic behavior*

We now use the bounds from the previous subsection to determine the tail distribution of the workload. As before, we consider a specific flow i which satisfies properties 3.1–3.3, but now for $c = \gamma_i$.

We make the following assumption.

Assumption 4.1. At least one of the following two conditions holds:

(i) $\rho_i < \phi_i$;

(ii) for all sets $E \not\ni i$ with $\gamma_{iE\cup\{i\}} \leqslant \rho_i$, for any $\delta > 0$,

$$\prod_{j \in E} \mathbb{P}\{\mathbf{V}_j^{\rho_j(1+\delta)} > x\} = \mathrm{o}(\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}) \quad \text{as } x \to \infty.$$

In addition, flow i satisfies the following property for $c = \gamma_i$.

Property 4.1. $\mathbb{P}\{\mathbf{V}_i^c > x\}$ is *dominatedly varying* (see [23]), i.e.,

$$\limsup_{x \to \infty} \frac{\mathbb{P}\{\mathbf{V}_i^c > \eta x\}}{\mathbb{P}\{\mathbf{V}_i^c > x\}} = H_i^c(\eta) < \infty, \quad \text{for some real } \eta \in (0, 1)$$

(which implies the property holds for all $\eta > 0$).

As will be formally shown below, the above assumption ensures that flow i is not significantly affected by flows with 'heavier'-tailed traffic characteristics. Specifically, the assumption implies that temporary instability caused by activity of other flows does not substantially influence the workload of flow i compared to the contribution of flow i itself. Condition (i), in fact, guarantees unconditional stability of flow i, regardless of the activity of the other flows. Note that the inequality $\gamma_{iE\cup\{i\}} \leqslant \rho_i$ implies that flow i would be pushed into instability if the flows $j \in E$ continuously claimed their full share of the link rate according to the assigned weights ϕ_j. Thus, condition (ii) guarantees that only sets of flows with 'combined lighter tails', could potentially drive flow i into instability. Or equivalently, sets of flows with 'combined heavier tails' *cannot* drive flow i into instability.

According to theorems 2.1 and 2.4, if $S_j^r(\cdot) \in \mathcal{IR}$, then for $c > \rho_j$, $\mathbb{P}\{\mathbf{V}_j^c > x\} \sim K_j^c \mathbb{P}\{\mathbf{S}_j^r > x\}$ for some constant $0 < K_j^c < \infty$. If $S_j(\cdot)$ is light-tailed, i.e., $\mathbb{P}\{\mathbf{S}_j > x\} = \mathrm{o}(\mathrm{e}^{-\kappa_1 x})$ for some $\kappa_1 > 0$, then for $c > \rho_j$, $\mathbb{P}\{\mathbf{V}_j^c > x\} = \mathrm{o}(\mathrm{e}^{-\kappa_2 x})$ for some $\kappa_2 > 0$. Thus, a sufficient requirement for condition (ii) of assumption 4.1 to hold is $S_i^r(\cdot) \in \mathcal{IR}$, and for all sets $E \subseteq \{1, \ldots, N\}$ with $\gamma_{iE\cup\{i\}} \leqslant \rho_i$, either $S_j(\cdot)$ is light-tailed for some $j \in E$, or $S_j^r(\cdot) \in \mathcal{IR}$ for all $j \in E$ and $\prod_{j \in E} \mathbb{P}\{\mathbf{S}_j^r > x\} = \mathrm{o}(\mathbb{P}\{\mathbf{S}_i^r > x\})$ as $x \to \infty$.

Now consider the special case where the flows $j \in R$ have regularly varying tails of index $-\nu_j$, whereas the flows $j \notin R$ have exponential tails. In that case, for flows $i \in R$, the sufficient condition indicated above may be expressed as follows: for all sets $E \subseteq R$ with $\gamma_{iE\cup\{i\}} \leqslant \rho_i$, $\sum_{j \in E}(\nu_j - 1) > \nu_i - 1$.

We now give the main result of this section.

Theorem 4.1. Consider a flow i which satisfies properties 3.1–3.3 for $c = \gamma_i$. If assumption 4.1 holds, then

$$\mathbb{P}\{\mathbf{V}_i > x\} \sim \mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}.$$

Before giving the formal proof of theorem 4.1, we first provide an intuitive explanation. Theorem 4.1 states that the workload of an individual flow i (with long-tailed traffic characteristics) is asymptotically equivalent to that in an isolated system. In the isolated system, flow i is served at a *constant* rate γ_i, which is equal to the average rate that flow i would receive if it continuously claimed its full share of the link rate. The result suggests that the most likely way for flow i to build a large queue is that the flow itself generates a large burst, or experiences a long on-period, while all other flows show roughly average behavior. During that period, flow i then receives service approximately at rate γ_i.

Thus, asymptotically, the workload of flow i is only affected by the traffic characteristics of the other flows through their average rates. In particular, flow i is largely insensitive to extreme activity of other flows, even when those have 'heavier'-tailed traffic characteristics.

We now briefly discuss the significance of assumption 4.1. As mentioned earlier, the assumption ensures that flow i is not significantly affected by flows with 'heavier'-tailed traffic characteristics. If assumption 4.1 does *not* hold, then there exists some set E with heavier combined tails than flow i and $\gamma_{iE\cup\{i\}} \leqslant \rho_i$. We conjecture that the tail distribution of $\mathbf{V}_i$ in that case is determined by the set E^* with the 'heaviest' tails, i.e.,

$$\prod_{j\in E} \mathbb{P}\{\mathbf{S}_j^r > x\} = \mathrm{o}\Bigg(\prod_{j\in E^*} \mathbb{P}\{\mathbf{S}_j^r > x\}\Bigg) \quad \text{for all } E \neq E^* \text{ with } \gamma_{iE\cup\{i\}} \leqslant \rho_i.$$

The tail distribution of $\mathbf{V}_i$ is then *heavier* than when flow i were served in isolation at a stable rate. The most likely way for flow i to build a large queue is that the flows $j \in E^*$ generate large bursts, or experience long on-periods, while the other flows, including flow i itself, show roughly average behavior. Flow i then receives service approximately at rate $\gamma_{iE^*} \leqslant \rho_i$, so that the queue will roughly grow at rate $\rho_i - \gamma_{iE^*}$ for a substantial period of time. In the next section we investigate this scenario in detail for the case where the 'dominant' set E^* consists of just a single flow k^*.

For theorem 4.1 to hold in case of fluid input, we need besides stability, i.e., $\rho_i < \gamma_i$, also $r_i > \gamma_i$ as implicitly required in properties 3.1–3.3. If assumption 4.1 does *not* hold, then we expect the tail behavior of $\mathbf{V}_i$ in case $r_i < \gamma_i$ is still determined by the set E^* as described above. We will prove this in the next section for the case where the 'dominant' set E^* consists of just a single flow. If assumption 4.1 *does hold*, however, then we conjecture that, possibly under some additional conditions, the tail behavior is determined by the set E^* with the heaviest tails for which either (i) $\gamma_{iE^*} < \rho_i$, if $i \notin E^*$ or (ii) $\gamma_{iE^*} < r_i$, if $i \in E^*$. The tail distribution of $\mathbf{V}_i$ is then *lighter* than when flow i were served in isolation. The most likely way for flow i to build a large queue is still that the flows $j \in E^*$ generate large bursts or experience long on-periods, while the other flows show roughly average behavior.

We now give the proof of theorem 4.1.

Proof of theorem 4.1 (Lower bound). From lemma 4.5, using independence, for $\delta > 0$ sufficiently small and any y,

$$\mathbb{P}\{\mathbf{V}_i > x\} \geqslant \mathbb{P}\left\{\mathbf{V}_i^{\gamma_i(\delta)} > x + y, \sum_{j\neq i} \mathbf{Z}_j^{\rho_j(1-\delta)} \leqslant y\right\}$$
$$= \mathbb{P}\left\{\mathbf{V}_i^{\gamma_i(\delta)} > x + y\right\} \mathbb{P}\left\{\sum_{j\neq i} \mathbf{Z}_j^{\rho_j(1-\delta)} \leqslant y\right\}.$$

Thus

$$\frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}} \geqslant \frac{\mathbb{P}\{\mathbf{V}_i^{\gamma_i(\delta)} > x + y\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x + y\}} \frac{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x + y\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}} \mathbb{P}\left\{\sum_{j\neq i} \mathbf{Z}_j^{\rho_j(1-\delta)} \leqslant y\right\}.$$

Using the fact that $\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}$ satisfies properties 3.1 and 3.2,

$$\liminf_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}} \geqslant F_i^{\gamma_i}\big(\gamma_i(\delta) - \gamma_i\big) \mathbb{P}\left\{\sum_{j\neq i} \mathbf{Z}_j^{\rho_j(1-\delta)} \leqslant y\right\}.$$

Letting $y \to \infty$, and then $\delta \downarrow 0$, we have

$$\liminf_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}} \geqslant 1.$$

(*Upper bound*). We first consider the case that condition (i) of assumption 4.1 applies.

From property (4) and lemma 4.6, taking $E = \{i\}$, using independence, for $\delta > 0$ sufficiently small and any y,

$$\mathbb{P}\{\mathbf{V}_i > x\} \leqslant \mathbb{P}\left\{\mathbf{V}_i^{\phi_i} > x,\ \mathbf{V}_i^{\gamma_i(-\delta)} + \sum_{j\in S_i} \mathbf{V}_j^{\rho_j(1+\delta)} > x\right\}$$
$$\leqslant \mathbb{P}\left\{\mathbf{V}_i^{\phi_i} > x, \mathbf{V}_i^{\gamma_i(-\delta)} > x - y \text{ or } \sum_{j\in S_i} \mathbf{V}_j^{\rho_j(1+\delta)} > y\right\}$$
$$\leqslant \mathbb{P}\left\{\mathbf{V}_i^{\gamma_i(-\delta)} > x - y\right\} + \mathbb{P}\left\{\mathbf{V}_i^{\phi_i} > x, \sum_{j\in S_i} \mathbf{V}_j^{\rho_j(1+\delta)} > y\right\}$$
$$= \mathbb{P}\left\{\mathbf{V}_i^{\gamma_i(-\delta)} > x - y\right\} + \mathbb{P}\left\{\mathbf{V}_i^{\phi_i} > x\right\} \mathbb{P}\left\{\sum_{j\in S_i} \mathbf{V}_j^{\rho_j(1+\delta)} > y\right\}.$$

Thus

$$\frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}} \leqslant \frac{\mathbb{P}\{\mathbf{V}_i^{\gamma_i(-\delta)} > x - y\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x - y\}} \frac{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x - y\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}} + \frac{\mathbb{P}\{\mathbf{V}_i^{\phi_i} > x\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}} \mathbb{P}\left\{\sum_{j \in S_i} \mathbf{V}_j^{\rho_j(1+\delta)} > y\right\}.$$

Using the fact that $\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}$ satisfies properties 3.1 and 3.3,

$$\limsup_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}} \leqslant G_i^{\gamma_i}\big(\gamma_i - \gamma_i(-\delta)\big) + G_i^{\gamma_i}(\gamma_i - \phi_i)\mathbb{P}\left\{\sum_{j \in S_i} \mathbf{V}_j^{\rho_j(1+\delta)} > y\right\}.$$

Letting $y \to \infty$, and then $\delta \downarrow 0$, we obtain

$$\limsup_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}} \leqslant 1.$$

We now consider the case that condition (ii) of assumption 4.1 applies.

Let us index the sets $E \ni i$ for which $\gamma_{iE} > \rho_i$ as $E_1, \ldots, E_M$. Note that $M \geqslant 1$ as $\gamma_i > \rho_i$. From lemma 4.6, using independence, for $\delta > 0$ sufficiently small and any y,

$$\begin{aligned}
\mathbb{P}\{\mathbf{V}_i > x\} &\leqslant \mathbb{P}\left\{\mathbf{V}_i^{\gamma_{iE}(-\delta)} + \sum_{j \in S_E} \mathbf{V}_j^{\rho_j(1+\delta)} > x \text{ for all sets } E \ni i \text{ with } \gamma_{iE} > \rho_i\right\} \\
&= \mathbb{P}\Bigg\{\mathbf{V}_i^{\gamma_i(-\delta)} + \sum_{j \in S_i} \mathbf{V}_j^{\rho_j(1+\delta)} > x, \\
&\qquad \mathbf{V}_i^{\gamma_{iE_m}(-\delta)} + \sum_{j \in S_{E_m}} \mathbf{V}_j^{\rho_j(1+\delta)} > x \ \forall m = 1, \ldots, M\Bigg\} \\
&\leqslant \mathbb{P}\Bigg\{\mathbf{V}_i^{\gamma_i(-\delta)} > x - y \text{ or } \sum_{j \in S_i} \mathbf{V}_j^{\rho_j(1+\delta)} > y,\ \mathbf{V}_i^{\gamma_{iE_m}(-\delta)} > \frac{x}{N} \\
&\qquad \text{or } \exists j_m \in S_{E_m}\colon \mathbf{V}_{j_m}^{\rho_{j_m}(1+\delta)} > \frac{x}{N} \ \forall m = 1, \ldots, M\Bigg\} \\
&\leqslant \mathbb{P}\big\{\mathbf{V}_i^{\gamma_i(-\delta)} > x - y\big\} + \mathbb{P}\left\{\sum_{j \in S_i} \mathbf{V}_j^{\rho_j(1+\delta)} > y,\ \exists m\colon \mathbf{V}_i^{\gamma_{iE_m}(-\delta)} > \frac{x}{N}\right\} \\
&\quad + \mathbb{P}\left\{\exists j_m \in S_{E_m}\colon \mathbf{V}_{j_m}^{\rho_{j_m}(1+\delta)} > \frac{x}{N} \ \forall m = 1, \ldots, M\right\} \\
&\leqslant \mathbb{P}\big\{\mathbf{V}_i^{\gamma_i(-\delta)} > x - y\big\} + \mathbb{P}\left\{\sum_{j \in S_i} \mathbf{V}_j^{\rho_j(1+\delta)} > y\right\} \sum_{m=1}^{M} \mathbb{P}\big\{\mathbf{V}_i^{\gamma_{iE_m}(-\delta)} > x/N\big\} \\
&\quad + \sum_{j_1 \in S_{E_1}, \ldots, j_M \in S_{E_M}} \prod_{j \in \{j_1, \ldots, j_M\}} \mathbb{P}\left\{\mathbf{V}_j^{\rho_j(1+\delta)} > \frac{x}{N}\right\}.
\end{aligned}$$

Thus

$$\begin{aligned}\frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}} \leqslant{}& \frac{\mathbb{P}\{\mathbf{V}_i^{\gamma_i(-\delta)} > x - y\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x - y\}} \frac{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x - y\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}} \\ &+ \mathbb{P}\left\{\sum_{j \in S_i} \mathbf{V}_j^{\rho_j(1+\delta)} > y\right\} \sum_{m=1}^{M} \frac{\mathbb{P}\{\mathbf{V}_i^{\gamma_{iE_m}(-\delta)} > x/N\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x/N\}} \frac{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x/N\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}} \\ &+ \sum_{j_1 \in S_{E_1}, \dots, j_M \in S_{E_M}} \frac{\prod_{j \in \{j_1, \dots, j_M\}} \mathbb{P}\{\mathbf{V}_j^{\rho_j(1+\delta)} > x/N\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x/N\}} \frac{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x/N\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}}.\end{aligned}$$

Using the fact that $\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}$ satisfies properties 3.1, 3.3, and 4.1,

$$\begin{aligned}\limsup_{x \to \infty} \frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}} \leqslant{}& G_i^{\gamma_i}\big(\gamma_i - \gamma_i(-\delta)\big) \\ &+ H_i^{\gamma_i}\left(\frac{1}{N}\right) \mathbb{P}\left\{\sum_{j \in S_i} \mathbf{V}_j^{\rho_j(1+\delta)} > y\right\} \sum_{m=1}^{M} G_i^{\gamma_i}\big(\gamma_i - \gamma_{iE_m}(-\delta)\big) \\ &+ H_i^{\gamma_i}\left(\frac{1}{N}\right) \sum_{j_1 \in S_{E_1}, \dots, j_M \in S_{E_M}} \limsup_{x \to \infty} \frac{\prod_{j \in \{j_1, \dots, j_M\}} \mathbb{P}\{\mathbf{V}_j^{\rho_j(1+\delta)} > x\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}}.\end{aligned}$$

Now consider a set $\{j_1, \dots, j_M\}$ with $j_1 \in S_{E_1}, \dots, j_M \in S_{E_M}$. By definition, $j_1 \notin E_1, \dots, j_M \notin E_M$, so that $\{i, j_1, \dots, j_M\} \neq E_1, \dots, E_M$, $\{i\}$. Consequently, $\gamma_{i\{i, j_1, \dots, j_M\}} \leqslant \rho_i$. Condition (ii) of assumption 4.1 then implies that

$$\limsup_{x \to \infty} \frac{\prod_{j \in \{j_1, \dots, j_M\}} \mathbb{P}\{\mathbf{V}_j^{\rho_j(1+\delta)} > x\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}} = 0.$$

Hence,

$$\begin{aligned}\limsup_{x \to \infty} \frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}} \leqslant{}& G_i^{\gamma_i}\big(\gamma_i - \gamma_i(-\delta)\big) \\ &+ H_i^{\gamma_i}\left(\frac{1}{N}\right) \mathbb{P}\left\{\sum_{j \in S_i} \mathbf{V}_j^{\rho_j(1+\delta)} > y\right\} \sum_{m=1}^{M} G_i^{\gamma_i}\big(\gamma_i - \gamma_{iE_m}(-\delta)\big).\end{aligned}$$

Letting $y \to \infty$, and then $\delta \downarrow 0$, we obtain

$$\limsup_{x \to \infty} \frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}} \leqslant 1. \qquad \square$$

5. Induced burstiness

As in the previous sections, we focus on a particular flow i for which we assume $\rho_i < \gamma_i$ to ensure stability. However, now we consider the case that assumption 4.1 does *not* hold. Instead, we assume there exists a 'dominant' set E^* consisting of just a single flow k^*. Thus, $\gamma_{ik^*} < \rho_i$, which in fact implies that $\gamma_{jk^*} < \rho_j$ for all $j > i$. In addition, we assume that $\gamma_{jk^*} > \rho_j$ for all $j < i$, so that $S_{k^*} = \{1, \ldots, i-1\} \setminus \{k^*\}$.

5.1. Bounds

We start with deriving bounds for the workload distribution which we will use in the next subsection to analyze the tail behavior. We first introduce some additional notation. Denote

$$\psi_i := \frac{\phi_i}{\sum_{j=i}^N \phi_j}, \qquad \chi_i := \frac{\phi_i}{\phi_{k^*} + \sum_{j=i}^N \phi_j}, \qquad \xi_i := 1 - \sum_{j=1, j\neq k^*}^{i-1} \rho_j - \frac{\rho_i}{\psi_i}.$$

It is easily verified from the stability condition $\rho_i < \gamma_i$ that

$$\rho_i < \psi_i \left(1 - \sum_{j=1}^{i-1} \rho_j\right),$$

so that $\xi_i > \rho_{k^*}$. Define

$$Q_{k^*}^\delta(t) := \sup_{0\leqslant s\leqslant t} \left\{\psi_i \left[B_{k^*}^{\gamma_{k^*}(\delta)}(s,t) - \gamma_{k^*}(\delta)(t-s)\right] + \left(\rho_i(1-\delta) - \gamma_{ik^*}(\delta)\right)(t-s)\right\}, \quad (9)$$

with $B_k^{\gamma_{k^*}(\delta)}(s,t)$ as in (2).

As $S_{k^*} = \{1, \ldots, i-1\} \setminus \{k^*\}$,

$$\begin{aligned}\gamma_{ik^*}(\delta) &= \frac{\phi_i}{\phi_{k^*} + \sum_{j=i}^N \phi_j}\left(1 - (1-\delta) \sum_{j=1, j\neq k^*}^{i-1} \rho_j\right) \\ &= \psi_i\left(1 - \frac{\phi_{k^*}}{\phi_{k^*} + \sum_{j=i}^N \phi_j}\right)\left(1 - (1-\delta) \sum_{j=1, j\neq k^*}^{i-1} \rho_j\right) \\ &= \psi_i\left(1 - \gamma_{k^*}(\delta) - (1-\delta) \sum_{j=1, j\neq k^*}^{i-1} \rho_j\right).\end{aligned}$$

Thus, (9) may be rewritten as

$$Q_{k^*}^\delta(t) = \psi_i \sup_{0\leqslant s\leqslant t} \left\{B_{k^*}^{\gamma_{k^*}(\delta)}(s,t) - c(\delta)(t-s)\right\}, \quad (10)$$

with

$$c(\delta) := \gamma_{k^*}(\delta) + \frac{\gamma_{ik^*}(\delta) - \rho_i(1-\delta)}{\psi_i} = (1-\delta)\xi_i + \delta.$$

For δ not too small, let $\mathbf{Q}_{k^*}^\delta$ be a stochastic variable having the limiting distribution of $Q_{k^*}^\delta(t)$ for $t \to \infty$.

We first present a lower bound for the workload distribution of flow i.

Lemma 5.1 (Lower bound). For $\delta > 0$ sufficiently small,

$$\mathbb{P}\{\mathbf{V}_i > x\} \geqslant \mathbb{P}\left\{\mathbf{Q}_{k^*}^{\delta} - \sum_{j \neq k^*} \mathbf{Z}_j^{\rho_j(1-\delta)} > x\right\}. \tag{11}$$

The proof of the above lemma may be found in [13].

We now provide an upper bound for the workload distribution of flow i. Define $s^* := \sup\{s \leqslant t | V_{k^*}^{\gamma_{k^*}(-\delta)}(s) = 0\}$ (or, equivalently, $s^* := \arg\sup_{0 \leqslant s \leqslant t}\{A_{k^*}(s, t) - \gamma_{k^*}(-\delta)(t - s)\}$). For all $j = 1, \ldots, N$, denote

$$W_j^{\delta}(t) := V_j^{\rho_j(1+\delta)}(t) + \frac{\phi_{k^*}}{\sum_{j=i}^{N} \phi_j} V_j^{\rho_j(1+\delta)}(s^*).$$

For $\delta > 0$, let $\mathbf{W}_j^{\delta}$ be a stochastic variable with the limiting distribution of $W_j^{\delta}(t)$ for $t \to \infty$.

Lemma 5.2 (Upper bound). For $\delta > 0$ sufficiently small,

$$\mathbb{P}\{\mathbf{V}_i > x\} \leqslant \mathbb{P}\left\{\mathbf{Q}_{k^*}^{-\delta} + \mathbf{V}_i^{\rho_i(1+\delta)} + \chi_i \sum_{j=1, j \neq k^*}^{i-1} \mathbf{W}_j^{\delta} > x\right\}. \tag{12}$$

The proof of the above lemma may be found in [13].

5.2. Asymptotic behavior

We now use the bounds from the previous subsection to determine the tail distribution of the workload. We first prove an auxiliary lemma. For conciseness, denote $\mathbf{P}_{k^*}^r := (\mathbf{P}_{k^*}^{\gamma_{k^*}})^r$, $\mathbf{Q}_{k^*} := \mathbf{Q}_{k^*}^0$.

Lemma 5.3. If $S_{k^*}^r(\cdot) \in \mathcal{IR}$, $U_{k^*}(\cdot)$ is an exponential distribution, and $r_{k^*} > \gamma_{k^*}$ in case of fluid input, then

$$\mathbb{P}\{\mathbf{Q}_{k^*} > x\} \sim \frac{\gamma_{k^*} - \rho_{k^*}}{\gamma_{k^*}} \frac{\rho_{k^*}}{\xi_i - \rho_{k^*}} \mathbb{P}\left\{\mathbf{P}_{k^*}^r > \frac{x}{\rho_i - \gamma_{ik^*}}\right\}, \tag{13}$$

with $\mathbb{P}\{\mathbf{P}_{k^*}^r > x/(\rho_i - \gamma_{ik^*})\}$ as in theorems 2.3 and 2.6, respectively.

Proof. Notice from (10) that, up to a multiplicative factor ψ_i, $Q_{k^*}^{\delta}(t)$ represents the workload at time t in a queue with service rate $c(\delta)$ fed by the departure process of a queue of service rate $\gamma_{k^*}(\delta)$ fed by flow k^*. The departure process of the latter queue is an on–off process with as on- and off-periods the busy and idle periods associated with the workload process $V_{k^*}^{\gamma_{k^*}(\delta)}(t)$. During the on-periods, traffic is generated at constant rate $\gamma_{k^*}(\delta)$ (for δ sufficiently small in case of fluid input so that $\gamma_{k^*}(\delta) < r_{k^*}$). The fraction of off-time is $1 - \rho_{k^*}/\gamma_{k^*}(\delta)$. The on- and off-periods are independent because

$U_{k^*}(\cdot)$ is an exponential distribution. As in the proof of theorem 2.6, it may be shown that $S^r_{k^*}(\cdot) \in \mathcal{IR}$ implies $P^r_{k^*}(\cdot) \in \mathcal{IR}$. Hence, from theorem 2.4,

$$\mathbb{P}\left\{\mathbf{Q}^{\delta}_{k^*} > x\right\} \sim \frac{\gamma_{k^*}(\delta) - \rho_{k^*}}{\gamma_{k^*}(\delta)} \frac{\rho_{k^*}}{(1-\delta)\xi_i + \delta - \rho_{k^*}} \mathbb{P}\left\{\mathbf{P}^r_{k^*} > \frac{x}{\rho_i(1-\delta) - \gamma_{ik^*}(\delta)}\right\}, \quad (14)$$

and, in particular, (13) follows. □

In accordance with the discussion in the previous section, we make the following assumption.

Assumption 5.1. In addition to $S_{k^*} = \{1, \ldots, i-1\} \setminus \{k^*\}$, each of the following two conditions holds:

(i) $\mathbb{P}\{\mathbf{S}^r_i > x\} = \mathrm{o}(\mathbb{P}\{\mathbf{S}^r_{k^*} > x\})$ as $x \to \infty$;

(ii) For all sets $E \not\ni i$, $E \neq \{k^*\}$, with $\gamma_{iE\cup\{i\}} \leqslant \rho_i$, for any $\delta > 0$,

$$\prod_{j \in E} \mathbb{P}\left\{\mathbf{V}^{\rho_j(1+\delta)}_j > x\right\} = \mathrm{o}\big(\mathbb{P}\{\mathbf{Q}_{k^*} > x\}\big) \quad \text{as } x \to \infty.$$

As indicated earlier, if $S^r_j(\cdot) \in \mathcal{IR}$, then according to theorems 2.1 and 2.4, for $c > \rho_j$, $\mathbb{P}\{\mathbf{V}^c_j > x\} \sim K^c_j \mathbb{P}\{\mathbf{S}^r_j > x\}$ for some constant $0 < K^c_j < \infty$. If $S_j(\cdot)$ is light-tailed, i.e., $\mathbb{P}\{\mathbf{S}_j > x\} = \mathrm{o}(\mathrm{e}^{-\kappa_1 x})$ for some $\kappa_1 > 0$, then for $c > \rho_j$, $\mathbb{P}\{\mathbf{V}^c_j > x\} = \mathrm{o}(\mathrm{e}^{-\kappa_2 x})$ for some $\kappa_2 > 0$. Also, according to theorems 2.3 and 2.6, if $S^r_{k^*}(\cdot) \in \mathcal{IR}$, then $\mathbb{P}\{\mathbf{Q}_{k^*} > x\} \sim K\mathbb{P}\{\mathbf{S}^r_{k^*} > x\}$ for some constant $K > 0$. Thus, a sufficient requirement for condition (ii) of assumption 5.1 to hold is $S^r_{k^*}(\cdot) \in \mathcal{IR}$, and for all sets $E \subseteq \{1, \ldots, N\}$, $E \neq \{k^*\}$, with $\gamma_{iE\cup\{i\}} \leqslant \rho_i$, either $S_j(\cdot)$ is light-tailed for some $j \in E$, or $S^r_j(\cdot) \in \mathcal{IR}$ for all $j \in E$ and $\prod_{j\in E} \mathbb{P}\{\mathbf{S}^r_j > x\} = \mathrm{o}(\mathbb{P}\{\mathbf{S}^r_{k^*} > x\})$ as $x \to \infty$.

Now consider the special case where for some set $R \subseteq \{1, \ldots, N\}$ the flows $j \in R$, in particular flow k^*, have regularly varying tails with index $-\nu_j$, whereas the flows $j \notin R$ have exponential tails. In that case, the sufficient condition indicated above may be expressed as follows: for all sets $E \subseteq R$ with $\gamma_{iE\cup\{i\}} \leqslant \rho_i$, $\sum_{j\in E}(\nu_j - 1) > \nu_{k^*} - 1$. Condition (i) then reduces to $i \notin R$ or $\nu_i > \nu_{k^*}$.

We now give the main result of this section.

Theorem 5.1. If $S^r_{k^*}(\cdot) \in \mathcal{IR}$, $U_{k^*}(\cdot)$ is an exponential distribution, and assumption 5.1 holds, then

$$\mathbb{P}\{\mathbf{V}_i > x\} \sim \mathbb{P}\{\mathbf{Q}_{k^*} > x\}.$$

Before giving the formal proof of theorem 5.1, we first provide an intuitive interpretation. As alluded to earlier, the result suggests that the most likely way for flow i to build a large queue is that flow k^* generates a large burst or experiences a long on-period, while the other flows, including flow i itself, show roughly average behavior. Specifically, when flow k^* generates a large amount of traffic, so it becomes backlogged for a long period of time, it receives service approximately at rate γ_{k^*}. Thus it experiences a busy period as if it were served at constant rate γ_{k^*}.

During that congestion period, the flows $j \neq k^*$ receive service approximately at rate γ_{jk^*}, while they generate traffic at average rate ρ_j. Thus, the queue of flow $j \notin S_{k^*} = \{i, \ldots, N\}$, in particular flow i, grows roughly at rate $\rho_j - \gamma_{jk^*} > 0$.

By the time the long congestion period ends, the flows $j \geqslant i$ have built large queues, and then start to receive service approximately at rate

$$\frac{\phi_j}{\sum_{j=i}^{N} \phi_j}\left(1 - \sum_{j=1, j\neq k^*}^{i-1} \rho_j\right).$$

The queue of flow i then starts to drain roughly at rate $\psi_i(1-\sum_{j=1,j\neq k^*}^{i-1} \rho_j) - \rho_i = \psi_i \xi_i$, and is the first to empty among the flows $j \geqslant i$.

In conclusion, the queue of flow i grows at rate $\rho_i - \gamma_{ik^*}$ when flow k^* is backlogged. When flow k^* is not backlogged, the queue of flow i drains at rate $\psi_i \xi_i$.

Thus, the queue of flow i behaves as that of a queue with service rate $\psi_i \xi_i$ fed by an on–off process with as on- and off-periods the busy and idle periods of flow k^* when served at constant rate γ_{k^*}. During the on-periods, traffic is produced at rate $\rho_i - \gamma_{ik^*} + \psi_i \xi_i = \psi_i \gamma_{k^*}$. This is reflected in theorem 5.1 if we use lemma 5.3 to interpret the right-hand side.

In preparation for the proof of theorem 5.1, we first state an auxiliary lemma.

Lemma 5.4. If $S_{k^*}^r(\cdot) \in \mathcal{IR}$, and $\mathbb{P}\{\mathbf{S}_i^r > x\} = \mathrm{o}(\mathbb{P}\{\mathbf{S}_{k^*}^r > x\})$ as $x \to \infty$, then for any $c > \rho_i$, $\mathbb{P}\{\mathbf{V}_i^c > x\} = \mathrm{o}(\mathbb{P}\{\mathbf{Q}_{k^*} > x\})$ as $x \to \infty$.

Proof. For any $\varepsilon > 0$, construct the stochastic variable $\mathbf{S}_i^\varepsilon$ with distribution

$$\mathbb{P}\{\mathbf{S}_i^\varepsilon > x\} = \min\{1, \mathbb{P}\{\mathbf{S}_i > x\} + \varepsilon \mathbb{P}\{\mathbf{S}_{k^*} > x\}\}.$$

Denote by $V_i^{c,\varepsilon}(t)$ the workload at time t in a queue with service rate c fed by flow i where the stochastic variable $\mathbf{S}_i$ in the arrival process is replaced by $\mathbf{S}_i^\varepsilon$. For $\varepsilon > 0$ sufficiently small, let $\mathbf{V}_i^{c,\varepsilon}$ be a stochastic variable having the limiting distribution of $V_i^{c,\varepsilon}(t)$ for $t \to \infty$. (Note that $\mathbb{E}\mathbf{S}_i^\varepsilon \leqslant \mathbb{E}\mathbf{S}_i + \varepsilon \mathbb{E}\mathbf{S}_{k^*}$, so that the queue is stable for $\varepsilon > 0$ sufficiently small.)

Clearly, $\mathbf{S}_i^\varepsilon$ is stochastically larger than $\mathbf{S}_i$, so that for $\varepsilon > 0$ sufficiently small,

$$\mathbb{P}\{\mathbf{V}_i^c > x\} \leqslant \mathbb{P}\{\mathbf{V}_i^{c,\varepsilon} > x\}. \tag{15}$$

Also,

$$\mathbb{P}\{(\mathbf{S}_i^\varepsilon)^r > x\} \sim \varepsilon \frac{\mathbb{E}\mathbf{S}_{k^*}}{\mathbb{E}\mathbf{S}_i^\varepsilon} \mathbb{P}\{\mathbf{S}_{k^*}^r > x\},$$

which implies that $\mathbb{P}\{(\mathbf{S}_i^{\varepsilon})^r > x\} \in \mathcal{IR}$. Hence, by theorems 2.1, 2.3, 2.4, 2.6, and lemma 5.3,

$$\limsup_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_i^{c,\varepsilon} > x\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x\}} \leqslant \varepsilon K, \tag{16}$$

for some finite constant K independent of ε.

The lemma follows by combining (15) and (16) and letting $\varepsilon \downarrow 0$. □

We now give the proof of theorem 5.1.

Proof of theorem 5.1. Using (13) and the fact that $P_{k^*}^r(\cdot) \in \mathcal{IR}$ (which implies $P_{k^*}^r(\cdot) \in \mathcal{L}$), for any y,

$$\lim_{x\to\infty} \frac{\mathbb{P}\{\mathbf{Q}_{k^*} > x - y\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x\}} = 1, \tag{17}$$

and

$$\limsup_{x\to\infty} \frac{\mathbb{P}\{\mathbf{Q}_{k^*} > x/N\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x\}} = F < \infty. \tag{18}$$

Also using (14),

$$\lim_{x\to\infty} \frac{\mathbb{P}\{\mathbf{Q}_{k^*}^{\delta} > x\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x\}} = G(\delta), \tag{19}$$

with $\lim_{\delta\to 0} G(\delta) = 1$.

(*Lower bound*). From lemma 5.1, using independence, for $\delta > 0$ sufficiently small and any y,

$$\begin{aligned}\mathbb{P}\{\mathbf{V}_i > x\} &\geqslant \mathbb{P}\Big\{\mathbf{Q}_{k^*}^{\delta} > x + y, \sum_{j\neq k^*} \mathbf{Z}_j^{\rho_j(1-\delta)} \leqslant y\Big\} \\ &= \mathbb{P}\big\{\mathbf{Q}_{k^*}^{\delta} > x + y\big\}\mathbb{P}\Big\{\sum_{j\neq k^*} \mathbf{Z}_j^{\rho_j(1-\delta)} \leqslant y\Big\}.\end{aligned}$$

Thus

$$\frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x\}} \geqslant \frac{\mathbb{P}\{\mathbf{Q}_{k^*}^{\delta} > x + y\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x + y\}} \frac{\mathbb{P}\{\mathbf{Q}_{k^*} > x + y\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x\}} \mathbb{P}\Big\{\sum_{j\neq k^*} \mathbf{Z}_j^{\rho_j(1-\delta)} \leqslant y\Big\}.$$

Using (17), (19),

$$\liminf_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x\}} \geqslant G(\delta)\mathbb{P}\Big\{\sum_{j\neq k^*} \mathbf{Z}_j^{\rho_j(1-\delta)} \leqslant y\Big\}.$$

Letting $y \to \infty$, and then $\delta \downarrow 0$, we have

$$\liminf_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x\}} \geqslant 1.$$

(*Upper bound*). Let us index the sets $E \ni i$ for which $\gamma_{iE} > \rho_i$ as $E_1, \ldots, E_M$. Note that $M \geqslant 1$ as $\gamma_i > \rho_i$. It is easily verified from the fact that $S_{k^*} = \{1, \ldots, i-1\} \setminus \{k^*\}$ that $k^* \notin E_m$, $k^* \in S_{E_m}$ for all $m = 1, \ldots, M$.

From lemmas 4.6, 5.2, using independence, for $\delta > 0$ sufficiently small and any y,

$$\begin{aligned}
\mathbb{P}\{\mathbf{V}_i > x\} &\leqslant \mathbb{P}\Bigg\{\mathbf{Q}_{k^*}^{-\delta} + \mathbf{V}_i^{\rho_i(1+\delta)} + \sum_{j=1, j\neq k^*}^{i-1} \mathbf{W}_j^{\delta} > x, \\
&\qquad \mathbf{V}_i^{\gamma_{iE}(-\delta)} + \sum_{j\in S_E} \mathbf{V}_j^{\rho_j(1+\delta)} > x \quad \text{for all sets } E \ni i \text{ with } \gamma_{iE} > \rho_i\Bigg\} \\
&= \mathbb{P}\Bigg\{\mathbf{Q}_{k^*}^{-\delta} + \mathbf{V}_i^{\rho_i(1+\delta)} + \sum_{j=1, j\neq k^*}^{i-1} \mathbf{W}_j^{\delta} > x, \mathbf{V}_i^{\gamma_{iE_m}(-\delta)} \\
&\qquad + \sum_{j\in S_{E_m}} \mathbf{V}_j^{\rho_j(1+\delta)} > x \ \forall m = 1, \ldots, M\Bigg\} \\
&= \mathbb{P}\Bigg\{\mathbf{Q}_{k^*}^{-\delta} + \mathbf{V}_i^{\rho_i(1+\delta)} + \sum_{j=1, j\neq k^*}^{i-1} \mathbf{W}_j^{\delta} > x, \mathbf{V}_i^{\gamma_{iE_m}(-\delta)} \\
&\qquad + \mathbf{V}_{k^*}^{\rho_{k^*}(1+\delta)} + \sum_{j\in S_{E_m}, j\neq k^*} \mathbf{V}_j^{\rho_j(1+\delta)} > x \ \forall m = 1, \ldots, M\Bigg\} \\
&\leqslant \mathbb{P}\Bigg\{\mathbf{Q}_{k^*}^{-\delta} > x - y \ \text{ or } \ \mathbf{V}_i^{\rho_i(1+\delta)} + \sum_{j=1, j\neq k^*}^{i-1} \mathbf{W}_j^{\delta} > y, \mathbf{V}_i^{\gamma_{iE_m}(-\delta)} > \frac{x}{N} \\
&\qquad \text{or } \ \mathbf{V}_{k^*}^{\rho_{k^*}(1+\delta)} > \frac{x}{N} \ \text{ or} \\
&\qquad \exists j_m \in S_{E_m},\, j_m \neq k^*\colon\ \mathbf{V}_{j_m}^{\rho_{j_m}(1+\delta)} > \frac{x}{N} \ \forall m = 1, \ldots, M\Bigg\} \\
&\leqslant \mathbb{P}\big\{\mathbf{Q}_{k^*}^{-\delta} > x - y\big\} + \mathbb{P}\bigg\{\exists m\colon\ \mathbf{V}_i^{\gamma_{iE_m}(-\delta)} > \frac{x}{N}\bigg\} \\
&\qquad + \mathbb{P}\Bigg\{\mathbf{V}_{k^*}^{\rho_{k^*}(1+\delta)} > \frac{x}{N}, \mathbf{V}_i^{\rho_i(1+\delta)} + \sum_{j=1, j\neq k^*}^{i-1} \mathbf{W}_j^{\delta} > y\Bigg\} \\
&\qquad + \mathbb{P}\bigg\{\exists j_m \in S_{E_m},\, j_m \neq k^*\colon\ \mathbf{V}_{j_m}^{\rho_{j_m}(1+\delta)} > \frac{x}{N} \ \forall m = 1, \ldots, M\bigg\} \\
&\leqslant \mathbb{P}\big\{\mathbf{Q}_{k^*}^{-\delta} > x - y\big\} + \sum_{m=1}^{M} \mathbb{P}\bigg\{\mathbf{V}_i^{\gamma_{iE_m}(-\delta)} > \frac{x}{N}\bigg\} \\
&\qquad + \mathbb{P}\bigg\{\mathbf{V}_{k^*}^{\rho_{k^*}(1+\delta)} > \frac{x}{N}\bigg\}\mathbb{P}\Bigg\{\mathbf{V}_i^{\rho_i(1+\delta)} + \sum_{j=1, j\neq k^*}^{i-1} \mathbf{W}_j^{\delta} > y\Bigg\} \\
&\qquad + \sum_{j_1\in S_{E_1}\setminus\{k^*\}, \ldots, j_M\in S_{E_M}\setminus\{k^*\}} \ \prod_{j\in\{j_1,\ldots,j_M\}} \mathbb{P}\bigg\{\mathbf{V}_j^{\rho_j(1+\delta)} > \frac{x}{N}\bigg\}.
\end{aligned}$$

Thus

$$
\begin{aligned}
&\frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x\}} \\
&\quad \leqslant \frac{\mathbb{P}\{\mathbf{Q}_{k^*}^{-\delta} > x-y\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x-y\}} \frac{\mathbb{P}\{\mathbf{Q}_{k^*} > x-y\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x\}} + \sum_{m=1}^{M} \frac{\mathbb{P}\{\mathbf{V}_i^{\gamma_{iE_m}(-\delta)} > x/N\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x/N\}} \frac{\mathbb{P}\{\mathbf{Q}_{k^*} > x/N\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x\}} \\
&\qquad + \frac{\mathbb{P}\{\mathbf{V}_{k^*}^{\rho_{k^*}(1+\delta)} > x/N\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x/N\}} \frac{\mathbb{P}\{\mathbf{Q}_{k^*} > x/N\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x\}} \mathbb{P}\left\{\mathbf{V}_i^{\rho_i(1+\delta)} + \sum_{j=1, j\neq k^*}^{i-1} \mathbf{W}_j^{\delta} > y\right\} \\
&\qquad + \sum_{j_1 \in S_{E_1}\setminus\{k^*\}, \ldots, j_M \in S_{E_M}\setminus\{k^*\}} \frac{\prod_{j\in\{j_1,\ldots,j_M\}} \mathbb{P}\{\mathbf{V}_j^{\rho_j(1+\delta)} > x/N\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x/N\}} \frac{\mathbb{P}\{\mathbf{Q}_{k^*} > x/N\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x\}}.
\end{aligned}
$$

According to theorems 2.3, 2.6, and lemma 5.3,

$$
\limsup_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_{k^*}^{\rho_{k^*}(1+\delta)} > x/N\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x/N\}} = H(\delta) < \infty.
$$

Using (17)–(19), and lemma 5.4,

$$
\begin{aligned}
&\limsup_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x\}} \\
&\quad \leqslant G(-\delta) + FH(\delta)\mathbb{P}\left\{\mathbf{V}_i^{\rho_i(1+\delta)} + \sum_{j=1, j\neq k^*}^{i-1} \mathbf{W}_j^{\delta} > y\right\} \\
&\qquad + F \sum_{j_1 \in S_{E_1}\setminus\{k^*\}, \ldots, j_M \in S_{E_M}\setminus\{k^*\}} \limsup_{x\to\infty} \frac{\prod_{j\in\{j_1,\ldots,j_M\}} \mathbb{P}\{\mathbf{V}_j^{\rho_j(1+\delta)} > x\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x\}}.
\end{aligned}
$$

Now consider a set $\{j_1, \ldots, j_M\}$ with $j_1 \in S_{E_1}\setminus\{k^*\}, \ldots, j_M \in S_{E_M}\setminus\{k^*\}$. By definition $j_1 \notin E_1, \ldots, j_M \notin E_M$, so that $\{i, j_1, \ldots, j_M\} \neq E_1, \ldots, E_M, \{k^*\}$. Consequently, $\gamma_{i\{i,j_1,\ldots,j_M\}} \leqslant \rho_i$. Condition (ii) of assumption 5.1 then implies that

$$
\limsup_{x\to\infty} \frac{\prod_{j\in\{j_1,\ldots,j_M\}} \mathbb{P}\{\mathbf{V}_j^{\rho_j(1+\delta)} > x\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x\}} = 0.
$$

Hence,

$$
\limsup_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x\}} \leqslant G(-\delta) + FH(\delta) \limsup_{x\to\infty} \mathbb{P}\left\{\mathbf{V}_i^{\rho_i(1+\delta)} + \sum_{j=1, j\neq k^*}^{i-1} \mathbf{W}_j^{\delta} > y\right\}.
$$

Letting $y \to \infty$, and then $\delta \downarrow 0$, we obtain

$$
\limsup_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x\}} \leqslant 1. \qquad \square
$$

6. Conclusion

We analyzed the queueing behavior of long-tailed traffic flows under the Generalized Processor Sharing (GPS) discipline. We showed a sharp dichotomy in qualitative behavior, depending on the relative values of the weight parameters. For certain weight combinations, an individual flow with long-tailed traffic characteristics is effectively served at a *constant* rate. The effective service rate may be interpreted as the maximum average traffic rate for the flow to be stable, which is only influenced by the traffic characteristics of the other flows through their average rates. This indicates that GPS-based scheduling algorithms offer a potential mechanism for obtaining substantial multiplexing gains, while protecting indvidual flows. For other weight combinations however, a flow may be strongly affected by the activity of 'heavier'-tailed flows, and may inherit their traffic characteristics, causing induced burstiness. The stark contrast in qualitative behavior highlights the great significance of the weight parameters.

In the present paper we focused on the workload of an individual flow at a single node. Some of the results may be extended to 'bottle-neck nodes' in feed-forward networks [41]. It would also be interesting to examine delays or loss probabilities in case of finite buffers.

With class aggregation, the flows that we considered may actually be *macro*-flows, each consisting of several *micro*-flows, which at a lower level may be served on a FCFS basis, or also according to GPS. It would be interesting to investigate the behavior of the micro-flows in such hierarchical situations.

In section 5 we found that a light-tailed flow whose weight is 'too small' could be strongly affected by a heavy-tailed flow. The case of a light-tailed flow whose weight is 'large enough' to be protected is analyzed in [14,15].

A final issue concerns the behavior of an on–off flow whose peak rate r_i is smaller than the effective service rate γ_i. In that case, other flows too need to show anomalous activity for the workload of flow i to grow, which means that the tail behavior may become 'less heavy-tailed' or even light-tailed. This phenomenon may be viewed as somewhat dual to the induced burstiness described above.

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Appendix A. Stability issues

We now identify which flows are stable and which ones are unstable. Flow i is considered 'stable' if the mean service rate is ρ_i. For ease of presentation, we assume the flows are indexed such that

$$\frac{\rho_1}{\phi_1} \leqslant \cdots \leqslant \frac{\rho_N}{\phi_N}.$$

Define S^* as the set of stable flows. Denote by γ_i the mean service rate for flow i (assuming it exists).

We have $\gamma_i \leqslant \rho_i$ for all $i = 1, \ldots, N$, with equality for all $i \in S^*$. Also, if $j \notin S^*$, then $\gamma_i/\phi_i \leqslant \gamma_j/\phi_j$ for all $i = 1, \ldots, N$.

In particular, we have $\gamma_i/\phi_i = \gamma_j/\phi_j$ for any pair of flows $i, j \notin S^*$, so $\gamma_i = \phi_i R$ for all $i \notin S^*$ for some $R \geqslant 1$. To determine R, observe that $\sum_{i=1}^{N} \gamma_i = 1$ if $S^* \neq \{1, \ldots, N\}$, which gives

$$R = \frac{1}{\sum_{j \notin S^*} \phi_j} \left(1 - \sum_{j \in S^*} \rho_j \right).$$

We first prove a lemma that characterizes the structure of the set S^*.

Lemma A.1. With the above ordering of the flows, the set S^* is of the form $\{1, \ldots, K\}$ for some K.

Proof. Suppose not, i.e., there are flows i and j, with $i < j$, $i \notin S^*$, and $j \in S^*$. Then we have $\gamma_i < \rho_i$, $\gamma_j = \rho_j$, and $\gamma_i/\phi_i \geqslant \gamma_j/\phi_j$. Thus, $\rho_i/\phi_i > \rho_j/\phi_j$, which would contradict the ordering of the flows. □

We now prove an auxiliary lemma.

Lemma A.2. With the above ordering of the flows, if

$$\rho_k > \frac{\phi_k}{\sum_{j=k}^{N} \phi_j} \left(1 - \sum_{j=1}^{k-1} \rho_j \right), \tag{A.1}$$

then

$$\rho_{k+1} > \frac{\phi_{k+1}}{\sum_{j=k+1}^{N} \phi_j} \left(1 - \sum_{j=1}^{k} \rho_j \right). \tag{A.2}$$

Proof. First observe the equivalence relation

$$\rho_k > \frac{\phi_k}{\sum_{j=k}^{N} \phi_j} \left(1 - \sum_{j=1}^{k-1} \rho_j \right) \iff \rho_k > \frac{\phi_k}{\sum_{j=k+1}^{N} \phi_j} \left(1 - \sum_{j=1}^{k} \rho_j \right). \tag{A.3}$$

The proof then immediately follows from the fact that $\rho_k/\phi_k \leqslant \rho_{k+1}/\phi_{k+1}$. □

The next lemma now identifies the set of stable flows.

Lemma 4.1. With the above ordering of the flows, the set of stable flows is $S^* = \{1, \ldots, K^*\}$, with

$$K^* = \max_{k=1,\ldots,N} \left\{ k\colon \frac{\rho_k}{\phi_k} \leqslant \frac{1 - \sum_{j=1}^{k-1} \rho_j}{\sum_{j=k}^{N} \phi_j} \right\}.$$

Proof. By lemma A.1, the set S^* is of the form $\{1, \ldots, L\}$ for some L, so it suffices to show that $L = K^*$. First observe that

$$\rho_{L+1} > \gamma_{L+1} = \frac{\phi_{L+1}}{\sum_{j=L+1}^{N} \phi_j} \left(1 - \sum_{j=1}^{L} \rho_j \right).$$

By lemma A.2 and the definition of K^*, this implies $L \geqslant K^*$. We also have $\gamma_L = \rho_L$ and $\gamma_L / \phi_L \leqslant \gamma_{L+1} / \phi_{L+1}$. Thus,

$$\rho_L \leqslant \frac{\phi_L}{\phi_{L+1}} \gamma_{L+1} = \frac{\phi_L}{\sum_{j=L+1}^{N} \phi_j} \left(1 - \sum_{j=1}^{L} \rho_j \right),$$

which is equivalent to

$$\rho_L \leqslant \frac{\phi_L}{\sum_{j=L}^{N} \phi_j} \left(1 - \sum_{j=1}^{L-1} \rho_j \right).$$

By lemma A.2 and the definition of K^*, this implies $L \leqslant K^*$. □

Appendix B. Proof of lemma 4.5

Lemma 4.5 (Lower bound). For $\delta > 0$ sufficiently small,

$$\mathbb{P}\{\mathbf{V}_i > x\} \geqslant \mathbb{P}\left\{ \mathbf{V}_i^{\gamma_i(\delta)} - \sum_{j \neq i} \mathbf{Z}_j^{\rho_j(1-\delta)} > x \right\}.$$

Proof. From (1),

$$V_i(t) \geqslant A_i(r, t) - B_i(r, t) \quad \text{for all } 0 \leqslant r \leqslant t. \tag{B.1}$$

Note that $\sum_{j=1}^{N} B_j(r, t) \leqslant t - r$, so that

$$B_i(r, t) \leqslant t - r - \sum_{j \neq i} B_j(r, t). \tag{B.2}$$

By definition, $i \notin S_i$. Hence, from lemma 4.3, for any $\delta \geqslant 0$,

$$\sum_{j \neq i} B_j(r, t) \geqslant \sum_{j \neq i} \inf_{r \leqslant s \leqslant t} \bigl\{ A_j(r, s) + \gamma_{ji}(\delta)(t - s) \bigr\}. \tag{B.3}$$

Combining (B.1)–(B.3), for any $\delta \geqslant 0$ and $0 \leqslant r \leqslant t$,

$$\begin{aligned}
V_i(t) &\geqslant A_i(r,t) - (t-r) + \sum_{j \neq i} \inf_{r \leqslant s \leqslant t} \bigl\{A_j(r,s) + \gamma_{ji}(\delta)(t-s)\bigr\} \\
&= A_i(r,t) - \gamma_i(\delta)(t-r) - \sum_{j \neq i} \gamma_{ji}(\delta)(t-r) \\
&\quad + \sum_{j \neq i} \inf_{r \leqslant s \leqslant t} \bigl\{A_j(r,s) + \gamma_{ji}(\delta)(t-s)\bigr\} \\
&\geqslant A_i(r,t) - \gamma_i(\delta)(t-r) + \sum_{j \neq i} \inf_{r \leqslant s \leqslant t} \bigl\{A_j(r,s) - \gamma_{ji}(\delta)(s-r)\bigr\} \\
&= A_i(r,t) - \gamma_i(\delta)(t-r) - \sum_{j \neq i} \sup_{r \leqslant s \leqslant t} \bigl\{\gamma_{ji}(\delta)(s-r) - A_j(r,s)\bigr\} \\
&\geqslant A_i(r,t) - \gamma_i(\delta)(t-r) - \sum_{j \neq i} Z_j^{\gamma_{ji}(\delta)}(r). \qquad \text{(B.4)}
\end{aligned}$$

Define $r^* := \arg\sup_{0 \leqslant r \leqslant t}\{A_i(r,t) - \gamma_i(\delta)(t-r)\}$, so that $V_i^{\gamma_i(\delta)}(t) = A_i(r^*,t) - \gamma_i(\delta)(t-r^*)$. Taking $r = r^*$ in (B.4) then yields

$$V_i(t) \geqslant V_i^{\gamma_i(\delta)}(t) - \sum_{j \neq i} Z_j^{\gamma_{ji}(\delta)}(r^*).$$

By definition, $\gamma_{ji}(\delta) = \rho_j(1-\delta)$ for all $j \in S_i$. Also, $\gamma_{ji}(\delta) > \gamma_{ji}$ with $\gamma_{ji}(\delta) \downarrow \gamma_{ji}$ for $\delta \downarrow 0$ for all $j \notin S_i$. In particular, $\gamma_i(\delta) > \rho_i$, because $\gamma_i > \rho_i$. Since $\gamma_{ji} < \rho_j$ for $j \notin S_i$, $j \neq i$, we also have that for δ sufficiently small, $\gamma_{ji}(\delta) < \rho_j(1-\delta)$ for $j \notin S_i$, $j \neq i$. Hence, for δ sufficiently small, $\gamma_{ji}(\delta) \leqslant \rho_j(1-\delta)$ for all $j \neq i$, so that

$$V_i(t) \geqslant V_i^{\gamma_i(\delta)}(t) - \sum_{j \neq i} Z_j^{\rho_j(1-\delta)}(r^*),$$

as $Z_j^c(r)$ is increasing in c.

Note that r^*, $V_i^{\gamma_i(\delta)}(t)$ only depend on $A_i(s,t)$, not on $A_j(s,t)$, $j \neq i$, and are thus independent of $Z_j^{\rho_j(1-\delta)}(r^*)$. Hence, for $\delta > 0$ sufficiently small,

$$\begin{aligned}
\mathbb{P}\bigl\{V_i(t) > x \mid r^*\bigr\} &\geqslant \mathbb{P}\Biggl\{V_i^{\gamma_i(\delta)}(t) - \sum_{j \neq i} Z_j^{\rho_j(1-\delta)}(r^*) > x \mid r^*\Biggr\} \\
&= \mathbb{P}\Biggl\{V_i^{\gamma_i(\delta)}(t) - \sum_{j \neq i} \mathbf{Z}_j^{\rho_j(1-\delta)} > x \mid r^*\Biggr\}.
\end{aligned}$$

Thus, in the stationary regime the stated lower bound holds for $\delta > 0$ sufficiently small. $\square$

Appendix C. Proof of lemma 4.6

Lemma 4.6 (Upper bound). For $\delta > 0$ sufficiently small,

$$\mathbb{P}\{\mathbf{V}_i > x\} \leqslant \mathbb{P}\left\{\mathbf{V}_i^{\gamma_{iE}(-\delta)} + \psi_{iE} \sum_{j \in S_E} \mathbf{V}_j^{\rho_j(1+\delta)} > x \text{ for all sets } E \ni i \text{ with } \gamma_{iE} > \rho_i\right\}.$$

Proof. Define $r^* := \sup\{r \leqslant t \mid V_i(r) = 0\}$. Then $V_i(r^*) = 0$, so from (1),

$$V_i(t) \leqslant A_i(r^*, t) - B_i(r^*, t). \tag{C.1}$$

Also, $V_i(r) > 0$ for all $r \in (r^*, t]$, i.e., flow i is continuously backlogged during the interval $(r^*, t]$. Hence, by definition of the GPS discipline,

$$B_i(r^*, t) \geqslant \frac{\phi_i}{\phi_j} B_j(r^*, t)$$

for all $j = 1, \ldots, N$, and

$$\sum_{j=1}^{N} B_j(r^*, t) = t - r^*.$$

Thus, for any subset $S \subseteq \{1, \ldots, N\}$,

$$B_i(r^*, t) \geqslant \frac{\phi_i}{\sum_{j \notin S} \phi_j} \sum_{j \notin S} B_j(r^*, t), \tag{C.2}$$

and

$$\sum_{j \notin S} B_j(r^*, t) = t - r^* - \sum_{j \in S} B_j(r^*, t). \tag{C.3}$$

Substituting (C.3) into (C.2), using (1),

$$\begin{aligned}
B_i(r^*, t) &\geqslant \frac{\phi_i}{\sum_{j \notin S} \phi_j} \left[t - r^* - \sum_{j \in S} B_j(r^*, t)\right] \\
&= \frac{\phi_i}{\sum_{j \notin S} \phi_j} \left[t - r^* - \sum_{j \in S} \left[V_j(r^*) + A_j(r^*, t) - V_j(t)\right]\right] \\
&\geqslant \frac{\phi_i}{\sum_{j \notin S} \phi_j} \left[t - r^* - \sum_{j \in S} \left[V_j(r^*) + A_j(r^*, t)\right]\right].
\end{aligned}$$

In particular, for any subset $E \subseteq \{1, \ldots, N\}$, $\delta > 0$, using lemma 4.4,

$$B_i(r^*, t) \geqslant \psi_{iE}\left[t - r^* - \sum_{j \in S_E} \left[V_j^{\rho_j(1+\delta)}(r^*) + A_j(r^*, t)\right]\right]. \tag{C.4}$$

Substituting (C.4) into (C.1),

$$
\begin{aligned}
V_i(t) &\leqslant A_i(r^*,t) - \psi_{iE}\Big[t - r^* - \sum_{j\in S_E}\big[V_j^{\rho_j(1+\delta)}(r^*) + A_j(r^*,t)\big]\Big] \\
&= A_i(r^*,t) - \psi_{iE}\Big(1 - \sum_{j\in S_E}\rho_j(1+\delta)\Big)(t-r^*) \\
&\quad + \psi_{iE}\Big[\sum_{j\in S_E}\big[V_j^{\rho_j(1+\delta)}(r^*) + A_j(r^*,t)\big] - \sum_{j\in S_E}\rho_j(1+\delta)(t-r^*)\Big] \\
&= A_i(r^*,t) - \gamma_{iE}(-\delta)(t-r^*) \\
&\quad + \psi_{iE}\sum_{j\in S_E}\Big[V_j^{\rho_j(1+\delta)}(r^*) + A_j(r^*,t) - \rho_j(1+\delta)(t-r^*)\Big] \\
&\leqslant V_i^{\gamma_{iE}(-\delta)}(t) + \psi_{iE}\sum_{j\in S_E}V_j^{\rho_j(1+\delta)}(t).
\end{aligned}
$$

From the definition it is easily seen that for $\delta > 0$, $\gamma_{iE}(-\delta) < \gamma_{iE}$ with $\gamma_{iE}(\delta) \uparrow \gamma_{iE}$ for $\delta \downarrow 0$. Since $\gamma_{iE} > \rho_i$, we have that $\gamma_{iE}(-\delta) > \rho_i$ for δ sufficiently small, and hence $\mathbf{V}_i^{\gamma_{iE}(-\delta)}$ is well-defined.

Thus, in the stationary regime the stated upper bound holds for $\delta > 0$ sufficiently small. $\square$

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