Asymptotic Behavior of Generalized Processor Sharing with Long-Tailed Traffic Sources

Sem Borst[*,**,†], Onno Boxma[*,**], Predrag Jelenković[‡]

[*]CWI
P.O. Box 94079, 1090 GB Amsterdam, The Netherlands

[**]Department of Mathematics & Computing Science
Eindhoven University of Technology
P.O. Box 513, 5600 MB Eindhoven, The Netherlands

[†]Bell Laboratories, Lucent Technologies
P.O. Box 636, Murray Hill, NJ 07974, USA

[‡]Department of Electrical Engineering
Columbia University
New York, NY 10027, USA

Abstract— **We analyze the asymptotic behavior of long-tailed traffic sources under the Generalized Processor Sharing (GPS) discipline. GPS-based scheduling algorithms, such as Weighted Fair Queueing, have emerged as an important mechanism for achieving differentiated quality-of-service in integrated-services networks.**
Under certain conditions, we prove that in an asymptotic sense an individual source with long-tailed traffic characteristics is effectively served at a *constant* rate, which may be interpreted as the maximum feasible average rate for that source to be stable. Thus, asymptotically, the source is only affected by the traffic characteristics of the other sources through their average rate. In particular, the source is essentially immune from excessive activity of sources with 'heavier'-tailed traffic characteristics. This suggests that GPS-based scheduling algorithms provide an effective mechanism for extracting high multiplexing gains, while protecting individual connections.

Keywords— **Generalized Processor Sharing (GPS), long-tailed, queue length asymptotics, regular variation, subexponential, Weighted Fair Queueing (WFQ)**

I. Introduction

Statistical data analysis has provided convincing evidence of long-tailed (subexponential [17]) traffic characteristics in high-speed communication networks. Early indications of the long-range dependence of Ethernet traffic, attributed to long-tailed file size distributions, were reported in [25]. Long-tailed characteristics of the scene length distribution of MPEG video streams were explored in [18], [22].

These empirical findings have encouraged theoretical developments in the modeling and queueing analysis of long-tailed traffic phenomena. Despite significant progress, however, the practical implications are not yet thoroughly understood, in particular issues relating to control and priority mechanisms in the network. To gain a better understanding of those issues, the present paper analyzes the queueing behavior of long-tailed traffic sources under the Generalized Processor Sharing (GPS) discipline. As a design paradigm, GPS is at the heart of commonly-used scheduling algorithms for high-speed switches, such as Weighted Fair Queueing, see for instance Parekh & Gallager [28], [29].

A basic approach in the analysis of long-tailed traffic phenomena is the use of fluid models with long-tailed arrival processes (e.g. On/Off sources with long-tailed On-periods). Fluid models are closely related to the ordinary single-server queue, thus bringing within reach the classical results on regularly-varying (Cohen [15]) or subexponential (Pakes [27], Veraverbeke [30]) behavior of the service and waiting-time distribution in the GI/G/1 queue. We refer to Boxma & Dumas [12] for a comprehensive survey on fluid queues with long-tailed arrival processes. See also Jelenković [19] for an extensive list of references on subexponential queueing models.

As mentioned above, the impact of priority mechanisms on long-tailed traffic phenomena has received relatively little attention. Some recent studies have investigated the effect of the scheduling discipline on the waiting-time distribution in the classical M/G/1 queue, see for instance Anantharam [3]. For FCFS, it is well-known [15] that the waiting-time tail is regularly varying of index $1 - \nu$ iff the service time tail is regularly varying of index $-\nu$. For LCFS preemptive resume as well as for Processor Sharing, the waiting-time tail turns out to be regularly varying of the *same* index as the service time tail [10], [33], although with different pre-factors. In the case of Processor Sharing with several customer classes, Zwart [32] recently showed that the sojourn time distribution of a class-i customer is regularly varying of index $-\nu_i$ iff the service time distribution of that class is regularly varying of index $-\nu_i$, *regardless* of the service time distributions of the other classes. In contrast, for two customer classes with ordinary non-preemptive priority, the tail behavior of the waiting- and sojourn time distributions is determined by the *heaviest* of the (regularly-varying) service time distributions [1], [11].

In the present paper, we consider the Generalized Processor Sharing (GPS) discipline. GPS-based scheduling algorithms, such as Weighted Fair Queueing, have emerged as an important mechanism for achieving differentiated quality-of-service in integrated-services networks. The queueing analysis of GPS is extremely difficult. Interesting partial results for exponential traffic models were obtained in Bertsimas *et al.* [5], Dupuis & Ramanan [16], Massoulié [26], and Zhang *et al.* [31].

Here, we focus on non-exponential traffic models. Extending the results from [7], we show that, under certain conditions, an individual source with long-tailed traffic characteristics is effectively served at a constant rate, which may be interpreted as the critical mean rate for stability. This is strongly reminiscent of the reduced-load equivalence established by Agrawal *et al.* [2]. The result suggests that GPS is effective in extracting high multiplexing gains, while protecting individual connections.

The remainder of the paper is organized as follows. In Section II, we present a detailed model description. In Section III, we briefly discuss some stability issues, and introduce a stability-related notion which will play a crucial role in the analysis. General lower and upper bounds on the buffer content of an individual source are derived in Section IV. We then show, in Section V, that for long-tailed traffic characteristics, the lower and upper bounds have the same asymptotic behavior, yielding exact asymptotic results. In Section VI, we make some concluding remarks.

II. MODEL DESCRIPTION

Consider N sources sharing a link of unit rate. Traffic from the sources is served in accordance with the Generalized Processor Sharing (GPS) discipline, which operates as follows. There are weights $\phi_1, \ldots, \phi_N$ associated with each of the sources, with $\sum_{i=1}^{N} \phi_i = 1$. Denote by $V_i(t)$ the buffer content of source i at time t. Let $\mathbf{V}_i$ be a stochastic variable with as distribution the limiting distribution of $V_i(t)$ for $t \to \infty$ (assuming it exists). If all the sources are backlogged at time t, i.e., $V_i(t) > 0$ for all $i = 1, \ldots, N$, then source i is served at rate ϕ_i. If some of the sources are not backlogged, however, then the excess capacity is redistributed among the backlogged sources in proportion to their respective weights. We refer to Dupuis & Ramanan [16] for a formal description of the evolution of the buffer content process.

Denote by $A_i(r, t)$ the amount of work generated by source i during the time interval $(r, t]$. Define $B_i(r, t)$ as the amount of service received by source i during $(r, t]$. Then the following identity relation holds

$$V_i(t) = V_i(r) + A_i(r, t) - B_i(r, t) \qquad (1)$$

for all $0 \leq r \leq t$.

Remark II.1: Although we use the term 'buffer content' to indicate the workload, we do not make any particular assumptions where traffic physically resides while waiting to be served. Using end-to-end flow control algorithms such as TCP, backlogged sources may for example be instructed to feed traffic into the network only at a rate comparable to the actual service rate so as to avoid excessive buffer overflow due to congestion.

Before describing the traffic model, we first introduce some further notation. For any two real functions $g(\cdot)$ and $h(\cdot)$, we use the notational convention $g(x) \sim h(x)$ as $x \to \infty$ to denote $\lim_{x\to\infty} g(x)/h(x) = 1$, or equivalently, $g(x) = h(x)(1 + o(1))$ as $x \to \infty$. For any stochastic variable $\mathbf{X}$ with distribution function $F(\cdot)$, $\mathbb{E}\mathbf{X} < \infty$, denote by $F^r(\cdot)$ the distribution function of the residual lifetime of $\mathbf{X}$, i.e., $F^r(x) = \frac{1}{\mathbb{E}\mathbf{X}} \int_0^x (1 - F(y)) \mathrm{d}y$.

The classes of *long-tailed*, *subexponential*, *intermediately regularly varying*, and *dominatedly varying* distributions are denoted with the symbols $\mathcal{L}$, $\mathcal{S}$, $\mathcal{IR}$, and $\mathcal{DR}$, respectively. The definitions of these classes are given in Appendix A.

For any $c \geq 0$, denote by $V_i^c(t) := \sup_{0 \leq r \leq t} \{A_i(r, t) - c(t - r)\}$ the buffer content of source i at time t if it were served in isolation at rate c (assuming $V_i^c(0) = 0$). Denote by ρ_i the traffic intensity of source i. For $c > \rho_i$, let $\mathbf{V}_i^c$ be a stochastic variable with as distribution the limiting distribution of $V_i^c(t)$ for $t \to \infty$. Define $W_i^c(x) := \mathbb{P}\{\mathbf{V}_i^c \leq x\}$ and $\bar{W}_i^c(x) := 1 - W_i^c(x)$. In the next sections, we will analyze the asymptotic behavior of sources i where the function $\bar{W}_i^c(\cdot)$ satisfies the following three properties for some value of $\sigma_i > \rho_i$.

Property II.1: For any $c \in (\rho_i, \sigma_i)$, $W_i^c(\cdot) \in \mathcal{L}$, i.e.,

$$\lim_{x\to\infty} \frac{\bar{W}_i^c(x-y)}{\bar{W}_i^c(x)} = 1, \quad \text{for all real } y.$$

Property II.2: For any $c \in (\rho_i, \sigma_i)$, $W_i^c(\cdot) \in \mathcal{DR}$, i.e.,

$$\limsup_{x\to\infty} \frac{\bar{W}_i^c(\eta x)}{\bar{W}_i^c(x)} = G_i^c(\eta) < \infty \text{ for some } \eta \in (0,1)$$

(which implies the property holds for all $\eta > 0$).

Property II.3: For any $c \in (\rho_i, \sigma_i)$,

$$\limsup_{x\to\infty} \frac{\bar{W}_i^{c-\theta}(x)}{\bar{W}_i^c(x)} = H_i^c(\theta) < \infty \text{ for some } \theta \in (0, c-\rho_i)$$

(which implies the property holds for all $\theta > 0$ sufficiently small where we will assume $\lim_{\theta\to 0} H_i^c(\theta) = 1$).

We now describe two traffic scenarios where the function $\bar{W}_i^c(\cdot)$ satisfies the above three properties.

A. Instantaneous arrivals

Here, a source generates instantaneous traffic bursts according to independent renewal processes. The interarrival times between bursts of source i are generally distributed with mean $1/\lambda_i$. The burst sizes of source i have distribution $B_i(\cdot)$ with mean β_i. Thus, the traffic intensity of source i is $\rho_i = \lambda_i \beta_i$. Let $\mathbf{B}_i^r$ be a stochastic variable with distribution $B_i^r(\cdot)$.

The next result is immediate from Pakes [27].

Theorem II.1: If $B_i^r(\cdot) \in \mathcal{S}$, and $\rho_i < c$, then

$$\bar{W}_i^c(x) \sim \frac{\rho_i}{c-\rho_i} \mathbb{P}\{\mathbf{B}_i^r > x\} \qquad \text{as} \quad x \to \infty.$$

Thus, for any finite $\sigma_i > \rho_i$, $\bar{W}_i^c(\cdot)$ satisfies Properties II.1 and II.3 if $B_i^r(\cdot) \in \mathcal{S}$, and Property II.2 if $B_i^r(\cdot) \in \mathcal{DR} \cap \mathcal{L}$.

B. On-Off processes

Here, a source generates traffic according to independent On-Off processes, alternating between On- and Off-periods. The Off-periods of source i are generally distributed with mean $1/\lambda_i$. The On-periods of source i have distribution $A_i(\cdot)$ with mean α_i. While On, source i produces traffic at a constant rate r_i, so the mean burst size is $\alpha_i r_i$. The fraction of time that source i is Off is $p_i = (1/\lambda_i)/(1/\lambda_i + \alpha_i) = 1/(1+\lambda_i\alpha_i)$. The traffic intensity of source i is $\rho_i = (1-p_i)r_i = \lambda_i\alpha_i r_i/(1+\lambda_i\alpha_i)$. Let $\mathbf{A}_i^r$ be a stochastic variable with distribution $A_i^r(\cdot)$.

The next result is established in Jelenković & Lazar [20].

Theorem II.2: If $A_i^r(\cdot) \in \mathcal{S}$, and $\rho_i < c < r_i$, then

$$\bar{W}_i^c(x) \sim p_i \frac{\rho_i}{c-\rho_i} \mathbb{P}\{\mathbf{A}_i^r > x/(r_i - c)\} \qquad \text{as} \quad x \to \infty.$$

Thus, for any $r_i > \sigma_i > \rho_i$, $\bar{W}_i^c(\cdot)$ satisfies Property II.1 if $A_i^r(\cdot) \in \mathcal{S}$, Property II.2 if $A_i^r(\cdot) \in \mathcal{DR} \cap \mathcal{L}$, and Property II.3 if $A_i^r(\cdot) \in \mathcal{IR}$.

III. Stability issues

We first briefly discuss some stability issues. If $\sum_{i=1}^N \rho_i < 1$, then all the sources will be stable, since the GPS discipline is work-conserving. If $\sum_{i=1}^N \rho_i > 1$, then at least one of the sources will be unstable, but others may still be stable. We now identify which sources are stable and which ones are unstable. To avoid technical subtleties, source i is considered 'stable' if the mean service rate is ρ_i, see also Remark III.1 below. For the ease of presentation, we assume the sources are indexed such that

$$\frac{\rho_1}{\phi_1} \leq \ldots \leq \frac{\rho_N}{\phi_N}.$$

Proposition III.1: With the above ordering, the set of stable sources is $S = \{1, \ldots, K^*\}$, with

$$K^* = \max_{k=1,\ldots,N} \left\{ k : \frac{\rho_k}{\phi_k} \leq \frac{1 - \sum_{j=1}^{k-1} \rho_j}{\sum_{j=k}^{N} \phi_j} \right\}.$$

Proof

See Appendix B.

□

It may be verified that $K^* = N$ (i.e. all the sources receive a stable service rate) iff $\sum_{i=1}^N \rho_i \leq 1$. By definition, each of the stable sources $i \in S$ receives a mean service rate ρ_i. Each of the unstable sources $i \notin S$ receives a mean service rate $\phi_i R < \rho_i$, with

$$R = \frac{1}{\sum_{j \notin S} \phi_j} \left(1 - \sum_{j \in S} \rho_j \right).$$

To understand the above formula, notice that the stable sources consume an average aggregate rate $\sum_{j \in S} \rho_j$, leaving an average rate $1 - \sum_{j \in S} \rho_j$ for the unstable sources, which is shared in proportion to the weights ϕ_i.

Remark III.1: For later purposes, we find it convenient to label source i as 'stable' if the mean service rate is ρ_i. In fact, the latter condition is necessary for stability in the usual sense, but not entirely sufficient. A sufficient condition is $\rho_i < \gamma_{ii}$. Indeed, if the buffer of source i never emptied, then it would receive a mean service rate γ_{ii}, so that γ_{ii} is the critical mean rate for stability.

We now introduce a stability-related notion which will play a crucial role in the analysis. Define γ_{iE} as the mean rate at which source i would receive service if the sources $j \in E$ were to continuously claim their full share of the link rate (while the remaining sources $j \notin E$ still acted 'normally'). Now observe that the sources $j \in E$ would in fact show such greedy behavior if they were unstable (which they need not be in reality). So we may determine γ_{iE} by forcing the sources $j \in E$ into the set of unstable sources, and then apply Proposition III.1. The set of sources which would receive a stable service rate if the sources $j \in E$ were to continuously claim their full share of the link rate, is then $S_E = \{1, \ldots, K_E^*\} \setminus E$, with

$$K_E^* = \max_{k=1,\ldots,N} \left\{ k : \frac{\rho_k}{\phi_k} \leq \frac{1 - \sum_{j=1}^{k-1} \rho_j \mathrm{I}_{\{j \notin E\}}}{\sum_{j=k}^{N} \phi_j \mathrm{I}_{\{j \notin E\}} + \sum_{j \in E} \phi_j} \right\}.$$

Thus, $\gamma_{iE} = \rho_i$ for all $i \in S_E$, and $\gamma_{iE} = \phi_i R_E < \rho_i$ for all $i \notin S_E$, with

$$R_E = \frac{1}{\sum_{j \notin S_E} \phi_j} \left(1 - \sum_{j \in S_E} \rho_j \right).$$

To explain the above formula, observe that the sources $j \in S_E$ by definition receive an average aggregate rate $\sum_{j \in S_E} \rho_j$, leaving an average rate $1 - \sum_{j \in S_E} \rho_j$ for the sources $j \notin S_E$, which is shared in proportion to the weights ϕ_i.

IV. Bounds

We now derive some general bounds which we will use in the next section to analyze the asymptotic behavior of the buffer content distribution. We focus on a particular yet arbitrary source i for which we assume $\rho_i < \gamma_{ii}$ to ensure stability.

For any set $E \subseteq \{1, \ldots, N\}$, define

$$\gamma_{iE}(\delta) = (1-\delta)\gamma_{iE} = (1-\delta)\rho_i \qquad \text{for all } i \in S_E,$$

and

$$\gamma_{iE}(\delta) = \phi_i R_E(\delta) \qquad \text{for all } i \notin S_E,$$

with

$$\begin{aligned} R_E(\delta) &= \frac{1}{\sum_{j \notin S_E} \phi_j} \left(1 - \sum_{j \in S_E} \gamma_{jE}(\delta) \right) \\ &= \frac{1}{\sum_{j \notin S_E} \phi_j} \left(1 - (1-\delta) \sum_{j \in S_E} \rho_j \right). \end{aligned}$$

Notice that for $E \neq \emptyset$, $\sum_{i=1}^{N} \gamma_{iE}(\delta) = 1$ for all values of δ.

We first state a basic lemma which will play a central role in deriving the bounds.

Lemma IV.1: For any set $E \subseteq \{1, \ldots, N\}$, $S \supseteq S_E$,

$$\sum_{j \in S} B_j(r,t) \geq \sum_{j \in S} \inf_{r \leq s \leq t} \{A_j(r,s) + \gamma_{jE}(\delta)(t-s)\},$$

for all $\delta \geq \delta_0$ for some $\delta_0 < 0$.

Proof

The proof follows immediately from combining Lemma's C.1 and C.2.

□

We now present a lower bound for the buffer content distribution of source i. For any $j \neq i$, define $U_{ij}^{\delta}(r) := \sup_{s \geq r} \{\gamma_{ji}(\delta)(s-r) - A_j(r,s)\}$. For $\delta > 0$ such that $\gamma_{ij}(\delta) < \rho_j$, let $\mathbf{U}_{ij}^{\delta}$ be a stochastic variable with as distribution the distribution of $U_{ij}^{\delta}(r)$ (which does not depend on r).

Lemma IV.2: (Lower bound) For $\delta > 0$ sufficiently small,

$$\mathbb{P}\{\mathbf{V}_i > x\} \geq \mathbb{P}\{\mathbf{V}_i^{\gamma_{ii}(\delta)} - \sum_{j \neq i} \mathbf{U}_{ij}^{\delta} > x\}. \tag{2}$$

Proof

Notice that $\sum_{j=1}^{N} B_j(r,t) \leq t - r$, so that $B_i(r,t) \leq t - r - \sum_{j \neq i} B_j(r,t)$ for all $0 \leq r \leq t$.

Thus, from (1), for any $0 \leq r \leq t$,

$$V_i(t) \geq A_i(r,t) - (t-r) + \sum_{j \neq i} B_j(r,t). \tag{3}$$

By definition, $S_i \subseteq \{1,\ldots,N\} \setminus \{i\}$. Hence, from Lemma IV.1, for any $\delta \geq 0$,

$$\sum_{j\neq i} B_j(r,t) \geq \sum_{j\neq i} \inf_{r\leq s\leq t}\{A_j(r,s)+\gamma_{ji}(\delta)(t-s)\}. \quad (4)$$

Plugging (4) into (3), for any $\delta \geq 0$ and $0 \leq r \leq t$,

$$\begin{aligned} V_i(t) &\geq A_i(r,t)-(t-r) \\ &+ \sum_{j\neq i} \inf_{r\leq s\leq t}\{A_j(r,s)+\gamma_{ji}(\delta)(t-s)\} \\ &= A_i(r,t)-\gamma_{ii}(\delta)(t-r)-\sum_{j\neq i}\gamma_{ji}(\delta)(t-r) \\ &+ \sum_{j\neq i} \inf_{r\leq s\leq t}\{A_j(r,s)+\gamma_{ji}(\delta)(t-s)\} \\ &\geq A_i(r,t)-\gamma_{ii}(\delta)(t-r) \\ &+ \sum_{j\neq i} \inf_{s\geq r}\{A_j(r,s)-\gamma_{ji}(\delta)(s-r)\} \\ &= A_i(r,t)-\gamma_{ii}(\delta)(t-r)-\sum_{j\neq i} U_{ij}^{\delta}(r). \quad (5) \end{aligned}$$

Denote $r^* := \arg\sup_{0\leq r\leq t}\{A_i(r,t)-\gamma_{ii}(\delta)(t-r)\}$, so that $V_i^{\gamma_{ii}(\delta)}(t) = A_i(r^*,t)-\gamma_{ii}(\delta)(t-r^*)$. Substituting $r=r^*$ in (5) then yields

$$V_i(t) \geq V_i^{\gamma_{ii}(\delta)}(t) - \sum_{j\neq i} U_{ij}^{\delta}(r^*).$$

From the definition it is easily verified that for δ sufficiently small, $\gamma_{ji}(\delta) < \rho_j$ for all $j \neq i$, so that $\mathbf{U}_{ij}^{\delta}$ is well-defined. Thus, in the stationary regime, (2) holds for $\delta > 0$ sufficiently small.

$\square$

We now present an upper bound for the buffer content distribution of source i.

Lemma IV.3: (Upper bound) For $\delta > 0$ sufficiently small,

$$\mathbb{P}\{\mathbf{V}_i > x\} \leq \mathbb{P}\{\mathbf{V}_i^{\gamma_{iE}(-\delta)} + \sum_{j\in S_E} \mathbf{V}_j^{(1+\delta)\rho_j} > x, \quad (6)$$
$$\text{for all sets } E \ni i \text{ with } \gamma_{iE} > \rho_i\}.$$

Proof

Since source i is stable, the distribution of $\mathbf{V}_i$ does not depend on the values of $V_j(0)$. Hence, we may assume $V_j(0) = 0$ for all $j = 1,\ldots,N$.

Denote $S_E^i := S_E \cup \{i\}$. Then from (1),

$$V_i(t) \leq \sum_{j\in S_E^i} V_j(t) = \sum_{j\in S_E^i}\{A_j(0,t)-B_j(0,t)\}. \quad (7)$$

From Lemma IV.1, for any $\delta \leq \delta_0$,

$$\sum_{j\in S_E^i} B_j(0,t) \geq \sum_{j\in S_E^i} \inf_{0\leq s\leq t}\{A_j(0,s)+\gamma_{jE}(-\delta)(t-s)\}, \quad (8)$$

for some $\delta_0 > 0$.

Substituting (8) into (7), for any $\delta \leq \delta_0$,

$$\begin{aligned} V_i(t) &\leq \sum_{j\in S_E^i}\{A_j(0,t)-\inf_{0\leq s\leq t}\{A_j(0,s)+\gamma_{jE}(-\delta)(t-s)\}\} \\ &= \sum_{j\in S_E^i} \sup_{0\leq s\leq t}\{A_j(s,t)-\gamma_{jE}(-\delta)(t-s)\} \\ &= V_i^{\gamma_{iE}(-\delta)}(t) + \sum_{j\in S_E} V_j^{(1+\delta)\rho_j}(t) \end{aligned}$$

for all sets $E \ni i$ with $\gamma_{iE} > \rho_i$.

From the definition it is easily verified that $\gamma_{iE}(-\delta) > \rho_i$ for δ sufficiently small, and hence $\mathbf{V}_i^{\gamma_{iE}(-\delta)}$ is well-defined. Thus, in the stationary regime, (6) holds for $\delta > 0$ sufficiently small.

$\square$

V. Asymptotic behavior

We now use the bounds from the previous section to determine the asymptotic behavior of the buffer content distribution. We consider a source i which satisfies Properties II.1, II.2, and II.3 with $\rho_i < \gamma_{ii} < \sigma_i$, and assume the following condition holds.

Condition 1: For all sets $E \not\ni i$ with $\gamma_{iE\cup\{i\}} \leq \rho_i$,

$$\lim_{x\to\infty} \frac{\prod_{j\in E} \bar{W}_j^{\rho_j(1+\delta)}(x)}{\bar{W}_i^{c}(x)} = 0, \quad \text{for any } \delta > 0, c \in (\rho_i, \sigma_i).$$

Condition 1 postulates that the tail of $\min_{j\in E} \mathbf{V}_j^{\rho_j(1+\delta)}$ is lighter than the tail of $\mathbf{V}_i^{c}$ for all sets $E \not\ni i$ with $\gamma_{iE\cup\{i\}} \leq \rho_i$. In case of instantaneous arrivals, Theorem II.1 then implies that the tail of $\min_{j\in E} \mathbf{B}_j^{r}$ must then be lighter than the tail of $\mathbf{B}_i^{r}$. Similarly, in case of On/Off processes, Theorem II.2 indicates that the tail of $\min_{j\in E} \mathbf{A}_j^{r}$ must then be dominated by the tail of $\mathbf{A}_i^{r}$. The inequality $\gamma_{iE\cup\{i\}} \leq \rho_i$ means that source i could be pushed into instability if the sources $j \in E$ continuously claimed their full share of the capacity. Thus, Condition 1 guarantees that only sources with combined lighter tails could potentially drive source i into instability. In other words, sources with combined heavier tails *cannot* drive source i into instability.

As a special but important case, consider a scenario where some of the sources have regularly varying tails, while the others have exponential tails. Specifically, suppose that $\bar{W}_i^c(\cdot)$ is regularly varying with index $1-\nu_i$ for the sources $i \in R$, $R \subseteq \{1,\ldots,N\}$. For the other sources $j \notin R$, $\bar{W}_j^c(x) = o(x^{-\zeta})$ for any $c > \rho_j$ and some $\zeta > 0$. In this case, for the sources $i \in R$, Condition 1 may be rewritten as follows.

Condition 1'
For all sets $E \subseteq R$, $E \not\ni i$, with $\gamma_{iE} \leq \rho_i$, $\sum_{j \in E} (\nu_j - 1) > \nu_i - 1$.

We now state the main theorem of the paper.

Theorem V.1: Consider a source i which satisfies Properties II.1, II.2, and II.3 with $\rho_i < \gamma_{ii} < \sigma_i$. If Condition 1 holds, then

$$\mathbb{P}\{\mathbf{V}_i > x\} \sim \mathbb{P}\{\mathbf{V}_i^{\gamma_{ii}} > x\} \qquad \text{as} \quad x \to \infty.$$

Before giving the formal proof of Theorem V.1, we first provide an intuitive interpretation and discuss the significance of Condition 1.

The result shows that an individual source with long-tailed traffic characteristics is effectively served at constant rate γ_{ii}. This is strongly reminiscent of the reduced-load equivalence established by Agrawal *et al.* [2]. Remember γ_{ii} is the mean service rate that source i would receive if it continuously claimed capacity. This suggests that the most likely scenario for source i to build a large queue is to generate a large burst, or to experience a long On-period, while the other sources show average behavior. During that congestion period, source i then receives service approximately at rate γ_{ii}.

If Condition 1 does *not* hold, then there is some set $E \not\ni i$ with heavier combined tails than source i and $\gamma_{iE} \leq \rho_i$. We conjecture that the tail behavior of $\mathbf{V}_i$ in that case is determined by the set E^* with the heaviest tails, i.e., $\lim_{x\to\infty} \prod_{j \in E} \bar{W}_j^{\rho_j(1+\delta)}(x) / \prod_{j \in E^*} \bar{W}_j^{\rho_j(1+\delta)}(x) = 0$ for all $E \neq E^*$ with $\gamma_{iE} \leq \rho_i$. Observe that the tail of $\mathbf{V}_i$ is now *heavier* than when source i were served in isolation at a stable rate. The most likely scenario for source i to build a large queue is that the sources $j \in E^*$ generate large bursts, or experience long On-periods, while the other sources, including source i, show average behavior. Source i then receives service approximately at rate $\gamma_{iE^*} \leq \rho_i$, so that the queue will roughly grow at rate $\rho_i - \gamma_{iE^*}$ for a significant period of time.

The conjecture is best illustrated for the case of $N = 2$ sources and regularly varying burst sizes, i.e., $1 - B_i(x) \sim l_i(x) x^{-\nu_i}$ as $x \to \infty$, with $\nu_i > 1$ and $l_i(\cdot)$ slowly varying functions, $i = 1, 2$, see the definition in Appendix A. Suppose that $\rho_1 > \phi_1$, $\rho_1 + \rho_2 < 1$, so that $\rho_2 < \phi_2$. Then $\gamma_{11} = 1 - \rho_2 > \rho_1$, $\gamma_{22} = \phi_2 > \rho_2$, $\gamma_{1\{1,2\}} = \phi_1 < \rho_1$, and $\gamma_{2\{1,2\}} = \phi_2 > \rho_2$. Thus Condition 1 holds for source 2, so that if source 2 satisfies Properties II.1, II.2, and II.3, then $\mathbb{P}\{\mathbf{V}_2 > x\} \sim \mathbb{P}\{\mathbf{V}_2^{\phi_2} > x\}$ as $x \to \infty$. Now suppose that $\nu_1 > \nu_2$. Then Condition 1 does not hold for source 1. We expect that source 1 in fact inherits the tail behavior of source 2, regardless of whether or not it satisfies Properties II.1, II.2, and II.3, and thus experiences heavier tail behavior than when served in isolation.

Note that when source 2 is backlogged, source 1 receives service at rate ϕ_1, so queue 1 will roughly grow at rate $\rho_1 - \phi_1$. When source 2 is not backlogged, queue 1 will drain at rate $1 - \rho_1$. Thus we conjecture that the tail behavior of source 1 is equivalent to the tail behavior of a single On/Off source served at constant rate $1 - \rho_1$, with peak rate $1 - \phi_1$, as On-period distribution the busy-period distribution of source 2, and fraction Off-time $1 - \rho_2/\phi_2$. This conjecture is supported by a result in [8] for a closely related coupled-processors model.

A special but important case is a scenario where $\rho_i < \phi_i$ for all $i = 1, \ldots, N$. It is easily verified that in that case $\gamma_{ii} = 1 - \sum_{j \neq i} \rho_j > 1 - \sum_{j \neq i} \phi_j = \phi_i > \rho_i$. Also, for all sets $E \not\ni i$, $\gamma_{iE \cup \{i\}} > \phi_i > \rho_i$. Thus, Condition 1 holds, so that Theorem V.1 gives $\mathbb{P}\{\mathbf{V}_i > x\} \sim \mathbb{P}\{\mathbf{V}_i^{1 - \sum_{j \neq i} \rho_j} > x\}$ as $x \to \infty$. This result can in fact be obtained using a simpler proof technique and under slightly weaker conditions, see [7] and also Remark V.1 below.

Now suppose each of the sources were served in isolation. Then the required capacity to achieve similar tail behavior is $\sum_{i=1}^N \gamma_{ii} = \sum_{i=1}^N (1 - \sum_{j \neq i} \rho_j) = \sum_{i=1}^N (1 - \rho + \rho_i) = 1 + (N-1)(1-\rho)$. The latter quantity may typically be expected to be significantly larger than 1. This suggests that GPS is effective in extracting high multiplexing gains, while protecting individual connections.

Recall that besides stability, i.e., $\rho_i < \gamma_{ii}$, we also assume $\sigma_i > \gamma_{ii}$ in Theorem V.1. In case of instantaneous arrivals, this assumption is not restrictive. However, in case of On/Off processes, $\sigma_i < r_i$, so that Theorem V.1 does not apply when $r_i \leq \gamma_{ii}$.

If Condition 1 does *not* hold, then we expect the tail behavior of $\mathbf{V}_i$ in that case is still determined by the set E^* as described above. If Condition 1 *does hold*, however, then we conjecture that, possibly under certain additional conditions, the tail behavior is determined by the set E^* with the heaviest tails for which either (i) $\gamma_{iE} < \rho_i$, if $i \notin E$ or

(ii) $\gamma_{iE} < r_i$, if $i \in E$. Observe that the tail of $\mathbf{V}_i$ is now *lighter* than when source i were served in isolation. The most likely scenario for source i to build a large queue is still that the sources $j \in E^*$ generate large bursts or experience long On-periods, while the other sources show average behavior.

We now give the proof of Theorem V.1.

Proof

(Lower bound) From Lemma IV.2, for $\delta > 0$ sufficiently small and any value of y, using independence,

$$\begin{aligned}\mathbb{P}\{\mathbf{V}_i > x\} &\geq \mathbb{P}\{\mathbf{V}_i^{\gamma_{ii}(\delta)} - \sum_{j\neq i} \mathbf{U}_{ij}^{\delta} > x\}\\ &\geq \mathbb{P}\{\mathbf{V}_i^{\gamma_{ii}(\delta)} > x + y, \sum_{j\neq i} \mathbf{U}_{ij}^{\delta} \leq y\}\\ &= \mathbb{P}\{\mathbf{V}_i^{\gamma_{ii}(\delta)} > x + y\}\mathbb{P}\{\sum_{j\neq i} \mathbf{U}_{ij}^{\delta} \leq y\}.\end{aligned}$$

Using the fact that $\bar{W}_i^c(\cdot)$ satisfies Properties II.1 and II.3 for $c \in (\rho_i, \sigma_i)$ and for δ sufficiently small,

$$\liminf_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_{ii}} > x\}} \geq$$

$$\liminf_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_i^{\gamma_{ii}(\delta)} > x\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_{ii}} > x\}} \mathbb{P}\{\sum_{j\neq i} \mathbf{U}_{ij}^{\delta} \leq y\} =$$

$$\mathbb{P}\{\sum_{j\neq i} \mathbf{U}_{ij}^{\delta} \leq y\} \Big/ \limsup_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_i^{\gamma_{ii}} > x\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_{ii}(\delta)} > x\}} =$$

$$\mathbb{P}\{\sum_{j\neq i} \mathbf{U}_{ij}^{\delta} \leq y\} / H_i^{\gamma_{ii}(\delta)}(\gamma_{ii}(\delta) - \gamma_{ii}).$$

Letting $y \to \infty$ and $\delta \downarrow 0$, observing that $\lim_{\delta\downarrow 0} \gamma_{ii}(\delta) = \gamma_{ii}$,

$$\liminf_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_{ii}} > x\}} \geq 1.$$

(Upper bound) Let us index the sets $E \ni i$ for which $\gamma_{iE} > \rho_i$ as $E_0, E_1, \ldots, E_M$. Because $\rho_i < \gamma_{ii}$, we may assume $E_0 = \{i\}$.

Denote $\mathbf{W}_{im}^{\delta} := \mathbf{V}_i^{\gamma_{iE_m}(-\delta)}$ and $\mathbf{Z}_i^{\delta} := \sum_{j\in S_i} \mathbf{V}_j^{(1+\delta)\rho_j}$. Then, from Lemma IV.3, for $\delta > 0$ sufficiently small and any value of y, using independence,

$$\mathbb{P}\{\mathbf{V}_i > x\} \leq$$

$$\mathbb{P}\{\mathbf{V}_i^{\gamma_{ii}(-\delta)} + \mathbf{Z}_i^{\delta} > x,$$

$$\mathbf{W}_{im}^{\delta} + \sum_{j\in S_{E_m}} \mathbf{V}_j^{(1+\delta)\rho_j} > x \; \forall m \geq 1\} \leq$$

$$\mathbb{P}\{\mathbf{V}_i^{\gamma_{ii}(-\delta)} > x - y \text{ or } \mathbf{Z}_i^{\delta} > y,$$

$$\mathbf{W}_{im}^{\delta} > \frac{x}{N} \text{ or } \exists j_m \in S_{E_m} : \mathbf{V}_{j_m}^{(1+\delta)\rho_{j_m}} > \frac{x}{N} \; \forall m \geq 1\} \leq$$

$$\mathbb{P}\{\mathbf{V}_i^{\gamma_{ii}(-\delta)} > x - y\} + \mathbb{P}\{\mathbf{Z}_i^{\delta} > y, \exists m \geq 1 : \mathbf{W}_{im}^{\delta} > \frac{x}{N}\}$$

$$+ \sum_{j_1\in S_{E_1},\ldots,j_M\in S_{E_M}} \mathbb{P}\{\mathbf{V}_{j_m}^{(1+\delta)\rho_{j_m}} > \frac{x}{N} \; \forall m \geq 1\} \leq$$

$$\mathbb{P}\{\mathbf{V}_i^{\gamma_{ii}(-\delta)} > x - y\} + \mathbb{P}\{\mathbf{Z}_i^{\delta} > y\} \sum_{m=1}^{M} \mathbb{P}\{\mathbf{W}_{im}^{\delta} > \frac{x}{N}\}$$

$$+ \sum_{j_1\in S_{E_1},\ldots,j_M\in S_{E_M}} \prod_{j\in\{j_1,\ldots,j_M\}} \mathbb{P}\{\mathbf{V}_j^{(1+\delta)\rho_j} > \frac{x}{N}\}.$$

Thus,

$$\frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_{ii}} > x\}} \leq \frac{\mathbb{P}\{\mathbf{V}_i^{\gamma_{ii}(-\delta)} > x - y\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_{ii}(-\delta)} > x\}} \frac{\mathbb{P}\{\mathbf{V}_i^{\gamma_{ii}(-\delta)} > x\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_{ii}} > x\}}$$

$$+\mathbb{P}\{\mathbf{Z}_i^{\delta} > y\} \sum_{m=1}^{M} \frac{\mathbb{P}\{\mathbf{V}_i^{\gamma_{iE_m}(-\delta)} > \frac{x}{N}\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_{ii}} > \frac{x}{N}\}} \frac{\mathbb{P}\{\mathbf{V}_i^{\gamma_{ii}} > \frac{x}{N}\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_{ii}} > x\}}$$

$$+ \sum_{j_1\in S_{E_1},\ldots,j_M\in S_{E_M}} Q_{i\{j_1,\ldots,j_M\}}(x) \frac{\mathbb{P}\{\mathbf{V}_i^{\gamma_{ii}} > \frac{x}{N}\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_{ii}} > x\}},$$

with

$$Q_{i\{j_1,\ldots,j_M\}}(x) := \frac{\prod_{j\in\{j_1,\ldots,j_M\}} \mathbb{P}\{\mathbf{V}_j^{(1+\delta)\rho_j} > \frac{x}{N}\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_{ii}} > \frac{x}{N}\}}.$$

Using the fact that $\bar{W}_i^c(\cdot)$ satisfies Properties II.1, II.2, and II.3 for $c \in (\rho_i, \sigma_i)$,

$$\limsup_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_{ii}} > x\}} \leq H_i^{\gamma_{ii}}(\gamma_{ii} - \gamma_{ii}(-\delta))$$

$$+\mathbb{P}\{\mathbf{Z}_i^{\delta} > y\} G_i^{\gamma_{ii}}(\frac{1}{N}) \sum_{m=1}^{M} H_i^{\gamma_{ii}}(\gamma_{ii} - \gamma_{iE_m}(-\delta))$$

$$+G_i^{\gamma_{ii}}(\frac{1}{N}) \sum_{j_1\in S_{E_1},\ldots,j_M\in S_{E_M}} \limsup_{x\to\infty} Q_{i\{j_1,\ldots,j_M\}}(x).$$

Now consider a set $\{j_1, \ldots, j_M\}$ with $j_1 \in S_{E_1}, \ldots, j_M \in S_{E_M}$. By definition, $j_1 \notin E_1, \ldots, j_M \notin E_M$, so that $\{i, j_1, \ldots, j_M\} \neq E_1, \ldots, E_M$. Consequently, $\gamma_{i\{i,j_1,\ldots,j_M\}} \leq \rho_i$. Condition 1 then implies that

$$\limsup_{x\to\infty} Q_{i\{j_1,\ldots,j_M\}}(x) = 0.$$

Hence,

$$\limsup_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_{ii}} > x\}} \leq H_i^{\gamma_{ii}}(\gamma_{ii} - \gamma_{ii}(-\delta))$$

$$+\mathbb{P}\{\mathbf{Z}_i^\delta > y\} G_i^{\gamma_{ii}}(\frac{1}{N}) \sum_{m=1}^{M} H_i^{\gamma_{ii}}(\gamma_{ii} - \gamma_{iE_m}(-\delta)).$$

Letting $y \to \infty$ and $\delta \downarrow 0$, observing that $\lim_{\delta \downarrow 0} \gamma_{ii}(-\delta) = \gamma_{ii}$,

$$\limsup_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_{ii}} > x\}} \leq 1.$$

□

Remark V.1: Notice that neither Property II.2 nor Condition 1 were actually used in establishing the lower bound in the proof of Theorem V.1. Now suppose that there is a set $E \ni i$ such that $\gamma_{iE} > \rho_i$ and for any $c \in (\rho_i, \sigma_i)$, $\lim_{x\to\infty} \bar{W}_j^{(1+\delta)\rho_j}(x)/\bar{W}_i^c(x) = 0$ for all $j \in S_E$. In that case, Property II.2 and Condition 1 are not needed in obtaining the upper bound either, so that Properties II.1 and II.3 are sufficient for Theorem V.1 to hold. Two extreme cases where there is such a set $E \ni i$ are: (i) $\rho_i < \phi_i$ with $E = \{1, \ldots, N\}$; (ii) $\lim_{x\to\infty} W_j^{(1+\delta)\rho_j}(x)/W_i^c(x) = 0$ for all $j \in S_i$ with $E = \{i\}$.

VI. Conclusion

We analyzed the asymptotic behavior of long-tailed traffic sources under the Generalized Processor Sharing (GPS) discipline. GPS-based scheduling algorithms, such as Weighted Fair Queueing, have emerged as an important mechanism for achieving differentiated quality-of-service in integrated-services networks.

Under certain conditions, we proved that in an asymptotic sense an individual source with long-tailed traffic characteristics is effectively served at a *constant* rate, which may be interpreted as the maximum feasible average rate for that source to be stable. Thus, asymptotically, the source is only affected by the traffic characteristics of the other sources through their average rate. In particular, the source is essentially immune from excessive activity of sources with 'heavier'-tailed traffic characteristics. This suggests that GPS-based scheduling algorithms provide an effective mechanism for extracting high multiplexing gains, while protecting individual connections.

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Appendix

I. Definitions

Definition 1: A distribution function $F(\cdot)$ on $[0,\infty)$ is called *long-tailed* ($F(\cdot) \in \mathcal{L}$) if

$$\lim_{x\to\infty} \frac{1-F(x-y)}{1-F(x)} = 1, \quad \text{for all real } y.$$

Definition 2: A distribution function $F(\cdot)$ on $[0,\infty)$ is called *subexponential* ($F(\cdot) \in \mathcal{S}$) if

$$\lim_{x\to\infty} \frac{1-F^{2*}(x)}{1-F(x)} = 2,$$

where $F^{2*}(\cdot)$ is the 2-fold convolution of $F(\cdot)$ with itself, i.e., $F^{2*}(x) = \int_0^x F(x-y)F(\mathrm{d}y)$.

The class of subexponential distributions was introduced by Chistyakov [13]. A well-known subclass of $\mathcal{S}$ is the class $\mathcal{R}$ of *regularly-varying* distributions (which contains the Pareto distribution):

Definition 3: A distribution function $F(\cdot)$ on $[0,\infty)$ is called *regularly varying of index* $-\nu$ ($F(\cdot) \in \mathcal{R}_{-\nu}$) if

$$F(x) = 1 - \frac{l(x)}{x^\nu}, \quad \nu \geq 0,$$

where $l : \mathbb{R}_+ \to \mathbb{R}_+$ is a function of slow variation, i.e., $\lim_{x\to\infty} l(\eta x)/l(x) = 1, \eta > 1$.

The class of regularly-varying functions was introduced by Karamata [23]; a key reference is Bingham *et al.* [6]. It is easily seen that $\mathcal{R} \subset \mathcal{S} \subset \mathcal{L}$. Examples of subexponential distributions which do not belong to $\mathcal{R}$ include the Weibull, lognormal, and Benktander distributions (see Klüppelberg [24]). A useful extension of $\mathcal{R}$ is the class $\mathcal{IR}$ of *intermediately regularly-varying* distributions:

Definition 4: A distribution function $F(\cdot)$ on $[0,\infty)$ is called *intermediately regularly varying* ($F(\cdot) \in \mathcal{IR}$) if

$$\lim_{\eta\uparrow 1} \limsup_{x\to\infty} \frac{1-F(\eta x)}{1-F(x)} = 1.$$

A further extension is the class $\mathcal{DR}$ of *dominatedly varying* distributions (see Cline [14]; $\mathcal{R} \subset \mathcal{IR} \subset (\mathcal{DR} \cap \mathcal{L}) \subset \mathcal{S}$):

Definition 5: A distribution function $F(\cdot)$ on $[0,\infty)$ is called *dominatedly varying* ($F(\cdot) \in \mathcal{DR}$) if

$$\limsup_{x\to\infty} \frac{1-F(\eta x)}{1-F(x)} < \infty, \quad \text{for some } \eta \in (0,1).$$

II. Stability issues

We now identify which sources are stable and which ones are unstable. Source i is considered 'stable' if the mean service rate is ρ_i. For the ease of presentation, we assume the sources are indexed such that

$$\frac{\rho_1}{\phi_1} \leq \ldots \leq \frac{\rho_N}{\phi_N}.$$

Define S as the set of stable sources. Denote by γ_i the mean service rate for source i (assuming it exists).

We have $\gamma_i \leq \rho_i$ for all $i = 1,\ldots,N$, with equality for all $i \in S$. Also, if $j \notin S$, then $\frac{\gamma_i}{\phi_i} \leq \frac{\gamma_j}{\phi_j}$ for all $i = 1,\ldots,N$.

In particular, we have $\frac{\gamma_i}{\phi_i} = \frac{\gamma_j}{\phi_j}$ for any pair of sources $i,j \notin S$, so $\gamma_i = \phi_i R$ for all $i \notin S$ for some $R \geq 1$. To determine R, observe that $\sum_{i=1}^N \gamma_i = 1$ if $S \neq \{1,\ldots,N\}$, which gives

$$R = \frac{1}{\sum_{j\notin S} \phi_j} \left(1 - \sum_{j\in S} \rho_j\right).$$

We first prove a lemma that characterizes the structure of the set S.

Lemma B.1: With the above ordering of the sources, the set S is of the form $\{1,\ldots,K\}$ for some K.

Proof

Suppose not, i.e., there are sources i and j, with $i < j$, $i \notin S$, and $j \in S$. Then we have $\gamma_i < \rho_i$, $\gamma_j = \rho_j$, and $\frac{\gamma_i}{\phi_i} \geq \frac{\gamma_j}{\phi_j}$. Thus, $\frac{\rho_i}{\phi_i} > \frac{\rho_j}{\phi_j}$, which would contradict the ordering of the sources.

□

We now prove an auxiliary lemma.

Lemma B.2: With the above ordering of the sources, if

$$\rho_k > \frac{\phi_k}{\sum_{j=k}^{N} \phi_j} \left(1 - \sum_{j=1}^{k-1} \rho_j\right), \tag{9}$$

then

$$\rho_{k+1} > \frac{\phi_{k+1}}{\sum_{j=k+1}^{N} \phi_j} \left(1 - \sum_{j=1}^{k} \rho_j\right). \tag{10}$$

Proof

First observe the equivalence relation

$$\rho_k > \frac{\phi_k}{\sum_{j=k}^{N} \phi_j} \left(1 - \sum_{j=1}^{k-1} \rho_j\right) \Longleftrightarrow \tag{11}$$

$$\rho_k > \frac{\phi_k}{\sum_{j=k+1}^{N} \phi_j} \left(1 - \sum_{j=1}^{k} \rho_j\right).$$

The proof then immediately follows from the fact that $\frac{\rho_k}{\phi_k} \leq \frac{\rho_{k+1}}{\phi_{k+1}}$.

□

The next proposition now identifies the set of stable sources.

Proposition III.1:

With the above ordering of the sources, the set of stable sources is $S = \{1, \ldots, K^*\}$, with

$$K^* = \max_{k=1,\ldots,N} \left\{ k : \frac{\rho_k}{\phi_k} \leq \frac{1 - \sum_{j=1}^{k-1} \rho_j}{\sum_{j=k}^{N} \phi_j} \right\}.$$

Proof

By Lemma B.1, the set S is of the form $\{1, \ldots, L\}$ for some L, so it suffices to show that $L = K^*$. First observe that

$$\rho_{L+1} > \gamma_{L+1} = \frac{\phi_{L+1}}{\sum_{j=L+1}^{N} \phi_j} \left(1 - \sum_{j=1}^{L} \rho_j\right).$$

By Lemma B.2 and the definition of K^*, this implies $L \geq K^*$.

We also have $\gamma_L = \rho_L$ and $\frac{\gamma_L}{\phi_L} \leq \frac{\gamma_{L+1}}{\phi_{L+1}}$. Thus,

$$\rho_L \leq \frac{\phi_L}{\phi_{L+1}} \gamma_{L+1} = \frac{\phi_L}{\sum_{j=L+1}^{N} \phi_j} \left(1 - \sum_{j=1}^{L} \rho_j\right),$$

which is equivalent to

$$\rho_L \leq \frac{\phi_L}{\sum_{j=L}^{N} \phi_j} \left(1 - \sum_{j=1}^{L-1} \rho_j\right).$$

By Lemma B.2 and the definition of K^*, this implies $L \leq K^*$.

□

III. Basic GPS inequalities

Lemma C.1: Let $S \subseteq \{1, \ldots, N\}$ be a set and let α_j, $j \in S$, be numbers such that

$$\alpha_i \frac{1 - \sum_{j \in S} \phi_j}{\phi_i} \leq 1 - \sum_{j \in S} \alpha_j \tag{12}$$

for all $i \in S$.

Then

$$\sum_{j \in S} B_j(r, t) \geq \sum_{j \in S} \inf_{r \leq s \leq t} \{A_j(r, s) + \alpha_j (t - s)\}$$

for all $0 \leq r \leq t$.

Proof

For given values of r, t, define

$$v^* := \max_{r \leq v \leq t} \{v : \sum_{j \in S} B_j(r, v) \geq$$

$$\sum_{j \in S} \inf_{r \leq s \leq v} \{A_j(r, s) + \alpha_j (v - s)\}\}.$$

We need to show that $v^* = t$. Suppose not, i.e., $v^* < t$. Then there must be some source i^* for which

$$B_{i^*}(r, v) < \inf_{r \leq s \leq v} \{A_{i^*}(r, s) + \alpha_{i^*} (v - s)\}$$

for all $v \in (v^*, w^*)$ for some $w^* > v^*$.

Define

$$u^* := \max_{r \le u \le w^*} \{u : B_{i^*}(r,u) \ge \inf_{r \le s \le u} \{A_{i^*}(r,s) + \alpha_{i^*}(u-s)\}\}.$$

First observe that

$$B_{i^*}(u^*, w^*) \le \alpha_{i^*}(w^* - u^*), \tag{13}$$

because otherwise

$$B_{i^*}(r, w^*) = B_{i^*}(r, u^*) + B_{i^*}(u^*, w^*) > \inf_{r \le s \le u^*} \{A_{i^*}(r,s) + \alpha_{i^*}(u^* - s)\} + \alpha_{i^*}(w^* - u^*) \ge \inf_{r \le s \le w^*} \{A_{i^*}(r,s) + \alpha_{i^*}(w^* - s)\},$$

contradicting the definition of w^*.

Further observe that

$$B_i^*(r,u) < \inf_{r \le s \le u} \{A_{i^*}(r,s) + \alpha_{i^*}(u-s)\} \le A_{i^*}(r,u)$$

for all $u \in (u^*, w^*)$, so that source i^* must be continuously backlogged during the interval (u^*, w^*).

Hence, by definition of the GPS discipline,

$$B_{i^*}(u^*, w^*) \ge \frac{\phi_{i^*}}{\phi_j} B_j(u^*, w^*) \tag{14}$$

for all $j = 1, \ldots, N$, and

$$\sum_{j=1}^{N} B_j(u^*, w^*) = w^* - u^*. \tag{15}$$

Using (12)-(15),

$$\sum_{j \in S} B_j(u^*, w^*) \ge \sum_{j \in S} \alpha_j (w^* - u^*). \tag{16}$$

By definition,

$$\sum_{j \in S} B_j(r, u^*) \ge \sum_{j \in S} \inf_{r \le s \le u^*} \{A_j(r,s) + \alpha_j(u^* - s)\}. \tag{17}$$

From (16), (17),

$$\sum_{j \in S} B_j(r, w^*) = \sum_{j \in S} B_j(r, u^*) + \sum_{j \in S} B_j(u^*, w^*) \ge \sum_{j \in S} \inf_{r \le s \le u^*} \{A_j(r,s) + \alpha_j(u^* - s)\} + \sum_{j \in S} \alpha_j(w^* - u^*) \ge \sum_{j \in S} \inf_{r \le s \le w^*} \{A_j(r,s) + \alpha_j(w^* - s),$$

contradicting the definition of v^*, so we must have $v^* = t$ as required.

$\square$

We now show that $\alpha_j = \gamma_{jE}(\delta)$, $j \in S \supseteq S_E$, satisfy (12) for all $\delta \ge \delta_0$ for some $\delta_0 < 0$.

Lemma C.2: For any set $E \subseteq \{1, \ldots, N\}$, $S \supseteq S_E$,

$$\gamma_{iE}(\delta) \frac{1 - \sum_{j \in S} \phi_j}{\phi_i} \le 1 - \sum_{j \in S} \gamma_{jE}(\delta)$$

for all $i \in S$ and $\delta \ge \delta_0$ for some $\delta_0 < 0$.

Proof

Using the definition of $\gamma_{jE}(\delta)$,

$$1 - \sum_{j \in S} \gamma_{jE}(\delta) = 1 - \sum_{j \in S_E} \gamma_{jE}(\delta) - \sum_{j \in S \setminus S_E} \gamma_{jE}(\delta) = \left(1 - \frac{\sum_{k \in S \setminus S_E} \phi_k}{\sum_{k \notin S_E} \phi_k}\right) \left(1 - \sum_{k \in S_E} \gamma_{kE}(\delta)\right) = \frac{1 - \sum_{k \in S} \phi_k}{\sum_{k \notin S_E} \phi_k} \left(1 - \sum_{k \in S_E} \gamma_{kE}(\delta)\right).$$

Thus, we need to show that

$$\gamma_{iE}(\delta) \le \frac{\phi_i}{\sum_{j \notin S_E} \phi_j} \left(1 - \sum_{j \in S_E} \gamma_{jE}(\delta)\right) = \frac{\phi_i}{\sum_{j \notin S_E} \phi_j} \left(1 - \sum_{j \in S_E} (1-\delta)\rho_j\right)$$

for all $i \in S$.

By definition, the above inequality holds with equality for all $i \in S \setminus S_E$.

From the definition of S_E and the equivalence relation (11),

$$\frac{\rho_i}{\phi_i} \le \max_{j \in S_E} \frac{\rho_j}{\phi_j} \le \frac{1 - \sum_{j \in S_E} \rho_j}{\sum_{j \notin S_E} \phi_j}$$

for all $i \in S_E$.

Hence, for all $i \in S$ and $\delta \ge \delta_0$,

$$\gamma_{iE}(\delta) = (1-\delta)\rho_i \le \frac{\phi_i}{\sum_{j \notin S_E} \phi_j} \left(1 - \sum_{j \in S_E} (1-\delta)\rho_j\right),$$

for some $\delta_0 < 0$.

$\square$