Stability of Finite Population ALOHA with Variable Packets

Predrag R. Jelenković and Jian Tan

***Abstract*—ALOHA is one of the most basic Medium Access Control (MAC) protocols and represents a foundation for other more sophisticated distributed and asynchronous MAC protocols, e.g., CSMA. In this paper, unlike in the traditional work that focused on mean value analysis, we study the distributional properties of packet transmission delays over an ALOHA channel. We discover a new phenomenon showing that a basic finite population ALOHA model with variable size (exponential) packets is characterized by power law transmission delays, possibly even resulting in zero throughput. These results are in contrast to the classical work that shows exponential delays and positive throughput for finite population ALOHA with fixed packets. Furthermore, we characterize a new stability condition that is entirely derived from the tail behavior of the packet and backoff distributions that may not be determined by mean values. The power law effects and the possible instability might be diminished, or perhaps eliminated, by reducing the variability of packets. However, we show that even a slotted (synchronized) ALOHA with packets of constant size can exhibit power law delays when the number of active users is random. From an engineering perspective, our results imply that the variability of packet sizes and number of active users need to be taken into consideration when designing robust MAC protocols, especially for ad-hoc/sensor networks where other factors, such as link failures and mobility, might further compound the problem.**

***Index Terms*—ALOHA, medium access control, power laws, heavy-tailed distributions, light-tailed distributions, ad-hoc/sensor networks.**

I. Introduction

ALOHA represents one of the first and most basic distributed Medium Access Control (MAC) protocols [1]. It is easy to implement since it does not require any user coordination or complicated controls and, thus, represents a basis for many modern MAC protocols, e.g., Carrier Sense Multiple Access (CSMA). Basically, ALOHA enables multiple users to share a common communication medium (channel) in a completely uncoordinated manner. Namely, a user attempts to send a packet over the common channel and, if there are no other user (packet) transmissions during the same time, the packet is considered successfully transmitted. Otherwise, if the transmissions of more than one packet (user) overlap, we say that there is a collision and the colliding packets need to be retransmitted. Each user retransmits a packet after waiting for an independent (usually exponential/geometric) period of time, making ALOHA entirely decentralized and asynchronous. The desirable properties of ALOHA, including its low complexity and distributed/asynchronous nature, make it especially beneficial for wireless sensor networks with limited resources as well as for wireless ad hoc networks that have difficulty in carrier sensing due to hidden terminal problems and mobility. Furthermore, because of these properties ALOHA represents a basis for many more sophisticated MAC protocols, e.g., CSMA.

Traditionally, the performance evaluation of ALOHA has focused on mean value (throughput) analysis, the examples of which can be found in every standard textbook on networking, e.g., see [3], [11], [10]; for more recent references see [9] and the references therein (due to space limitations, we do not provide comprehensive literature review on ALOHA in this paper). However, it appears that there are no explicit and general studies (more than two users) of the distributional properties of ALOHA, e.g., delay distributions. In this regard, in Subsection II-A, we consider a standard finite population ALOHA model with variable length packets [4], [2] that have an asymptotically exponential tail. Surprisingly, we discover a new phenomenon that the distribution of the number of retransmissions (collisions) and time between two successful transmissions follow power law distributions, as stated in Proposition II.1 of Subsection II-B, Theorem IV.1 of Subsection IV-A on starting behavior as well as Theorem IV.2 of Subsection IV-B on steady state behavior. Based on this observation, we derive new stability conditions for finite population ALOHA with variable packets in Theorem III.1 of Section III. Informally, our theorem shows that when the exponential decay rate of the packet distribution is smaller than the parameter of the exponential backoff distribution and the arrival rate, even the finite population ALOHA may have zero throughput. This is contrary to the common belief that the finite population ALOHA system always has a positive, albeit possibly small, throughput. Furthermore, even when the long term throughput is positive, the high variability of power laws (infinite variance when the power law exponent is less than 2) may cause unstable buffer content (queue sizes), implying periods of very high congestion, long delays, and low throughput. It also may appear counterintuitive that the system is characterized by power laws even though the distributions of all the variables (arrivals, backoffs and packets) of the system are of exponential type. However, this is in line with the results in [5], [12], [6], which show that job completion times in systems with failures where jobs restart from the very beginning exhibit similar power law behavior. Our study in [6] was done in the communication context where job completion times are represented by document/packet transmission delays, e.g., ARQ protocol. It may also be worth noting that [6] reveals

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the existence of power law delays regardless of how light or heavy the packet/document and link failure distributions may be (e.g., Gaussian), as long as they have proportional hazard functions. Furthermore, from a mathematical perspective, Proposition III.1, Theorems IV.1 and IV.2 analyze a more complex setting than the one in [6], [12] and, thus, require a novel proof. Hence, when compared with [6], [12], this paper both discovers a new related phenomenon in a communication MAC layer application area and provides a novel analysis of it.

As already stated in the abstract, the preceding power law phenomenon is a result of combined effects of packet variability and collisions. Hence, one can see easily that the power law delays can be eliminated by reducing the variability of packets. Indeed, for slotted ALOHA with constant size packets the delays are geometrically distributed. However, we show in Section V that, when the number of users sharing the channel is geometrically distributed, the slotted ALOHA exhibits power law delays as well.

In Section VI, we illustrate our results with simulation experiments, which show that the asymptotic power law regime is valid even for relatively small delays and reasonably large probability values. Furthermore, the distribution of packets/number of users in practice might have a bounded support. To this end, we show by a simulation experiment that this situation results in distributions that have power law main body with an exponentiated (stretched) support in relation to the support of the packet size/number of active users. Hence, although exponentially bounded, the delays may be prohibitively long.

In practical applications, we may have combined effects of both variable packets and a random number of users, implying that the delay and congestion is likely to be even worse than predicted by our results. Thus, from an engineering perspective, one has to pay special attention to the packet variability and the number of users when designing robust MAC protocols, especially for ad-hoc/sensor networks where link failures [6], mobility and many other factors might further worsen the performance.

In summary, the rest of the paper is organized as follows. In Section II, we provide the description and the preliminary power law bounds. Then, we present our new stability conditions that are based on packet distribution decay rates in Section III. Further distributional properties for the number of retransmissions and delays are investigated in Section IV. Section V contains the results on power laws in slotted ALOHA with random number of users. Experimental validation of our results can be found in Section VI. The paper is concluded in Section VII. Finally, some of the more technical proofs are postponed to Section VIII.

II. Power Laws in the Finite Population ALOHA with Variable Size Packets

In this section we show that the variability of packet sizes, when coupled with the contention nature of ALOHA, is a cause of power law delays. This study is motivated by the well-known fact that packets in today's Internet have variable sizes. To further emphasize that packet variability is a sole cause of power laws, we assume a finite population ALOHA model where each user can hold (queue) up to one packet at a time since the increased queueing only further exacerbates the problem. In addition, in Section V we show that the user variability in an infinite population model may be a cause of power law delays as well. In the remainder of this section, we describe the model and introduce the necessary notation in Subsection II-A and present the preliminary results in Subsection II-B.

A. Model Description

Consider $M \geq 2$ users sharing a common communication link (channel) of unit capacity. Each user can hold at most one packet in its queue and, when the queue is empty, a new packet is generated after an independent (from all other variables) exponential time with mean $1/\lambda$. Each packet has an independent length that is equal in distribution to a generic random variable L. A user with a newly generated packet attempts its transmission immediately and, if there are no other users transmitting during the same time, the packet is considered successfully transmitted. Otherwise, if the transmissions of more than one packet overlap, we say that there is a collision and the colliding packets need to be retransmitted; for a visual representation of the system see Figure 1. After a collision, each participating user waits (backoffs) for an independent exponential period of time with mean $1/\nu$ and then attempts to retransmit its packet. Each such user continues this procedure until its packet is successfully transmitted and then it generates a new packet after an independent exponential time of mean $1/\lambda$. Let $\{U(t)\}_{t\geq 0}$ denote the number of users that are in backoff state at time t and $\left\{L_i^{(t)}\right\}_{1\leq i\leq U(t)}$ denote the packet sizes of all the $U(t)$ number of active users at time t.

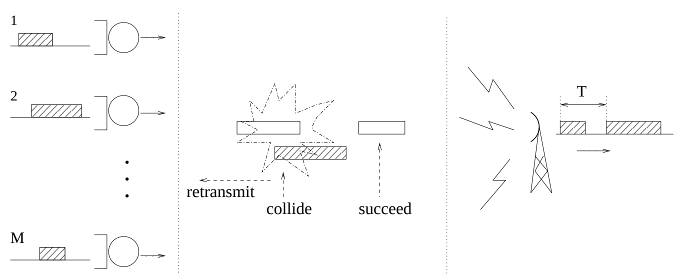

Fig. 1. Finite population ALOHA model with variable packet sizes.

From the perspective of the receiver, let $\{C_i\}_{i\geq 1}$ be an increasing sequence of positive time points when either a collision or successful transmission occurs with $C_0 = 0$. Let $\{D_m\}_{m\geq 1}$ be the sequence of time points when the receiver successfully receives the mth packet and define $T_m = D_m - D_{m-1}$ to be the transmission time for the mth successfully received packet with a convention $D_0 = 0$. Correspondingly, we can define N_m to be the number of (re)transmissions in the interval $(D_{m-1}, D_m]$ for the mth successful transmission.

Now, from the perspective of user $i, , 1 \leq i \leq M$, define $\left\{D_m^{(i)}\right\}_{m\geq 1}$ to be the sequence of time points when user i successfully sends the mth packet and define $T_m^{(i)} = D_m^{(i)} - D_{m-1}^{(i)}$

to be the transmission time for the mth successfully transmitted packet with a convention $D_0^{(i)} = 0$. By the same fashion, we can define $N_m^{(i)}$ to be the number of (re)transmissions in the interval $(D_{m-1}^{(i)}, D_m^{(i)}]$ for the mth successful transmission.

We will study the stability of this model as well as the asymptotic properties of the distributions of N_m, T_m and $N_m^{(i)}, T_m^{(i)}$.

B. Power Law Bounds

In the rest of this subsection, we present preliminary results for the finite ALOHA with variable packets, described in the preceding subsection. Let $x \wedge y = \min(x, y)$, $x \vee y = \max(x, y)$, and $\overset{d}{\le}$, $\overset{d}{\ge}$, $\overset{d}{=}$ denote inequalities and equality in distribution, respectively.

Basically, ALOHA model can be viewed as a state dependent channel with failures where the failure rate depends on the number of backoffed users and the sizes of the packets present in the system. Hence, this model can be viewed as a generalization of the problem stated in [6], [7]. The following proposition shows that the distributions of the number of retransmissions and the delays in our ALOHA model are always sandwiched between two power laws, which is obtained by uniformly bounding the variable collision (failure) rates independently of the state of the channel.

Proposition II.1 *Assume that , for $\mu > 0$,*

$$\lim_{x\to\infty} \frac{\log \mathbb{P}[L > x]}{x} = -\mu, \tag{1}$$

and let $\underline{N}$ and $\overline{N}$ be two random variables with distributions

$$\mathbb{P}\left[\underline{N} > n\right] = \mathbb{E}\left[\left(1 - e^{-L(M-1)(\lambda\wedge\nu)}\right)^n\right] \tag{2}$$

and

$$\mathbb{P}\left[\overline{N} > n\right] = \mathbb{E}\left[\left(1 - \frac{\lambda\wedge\nu}{M(\lambda\vee\nu)} e^{-L(M-1)(\lambda\vee\nu)}\right)^n\right].$$

Then, uniformly for all m and i,

$$\underline{N} \overset{d}{\le} N_m^{(i)} \overset{d}{\le} \overline{N} \tag{3}$$

and

$$\lim_{n\to\infty} \frac{\log \mathbb{P}\left[\underline{N} > n\right]}{\log n} = -\frac{\mu}{(M-1)(\lambda\wedge\nu)}, \tag{4}$$

$$\lim_{n\to\infty} \frac{\log \mathbb{P}\left[\overline{N} > n\right]}{\log n} = -\frac{\mu}{(M-1)(\lambda\vee\nu)}. \tag{5}$$

Similarly, there exist $\underline{T}$ and $\overline{T}$ such that (3), (4) and (5) are satisfied for the corresponding expressions for $T_m^{(i)}$, $\underline{T}$ and $\overline{T}$ (replacing N by T).

Proof: We begin with studying $N_m^{(i)}$. First, we prove the *lower bound*. Note that a collision for user i may occur for two different reasons. Either, when user i attempts to access the channel it collides with the already existing transmission, or after user i successfully starts its transmission it is interrupted later by some other user that tries to access the channel. Now, if $\underline{N}_m^{(i)}$ only counts the collisions due to the second reason, then clearly $N_m^{(i)} \ge \underline{N}_m^{(i)}$. Similarly, if $\underline{T}_m^{(i)}$ is the total time that only measures the delay caused by the collisions of the second type, then $T_m^{(i)} \ge \underline{T}_m^{(i)}$.

Now, consider the system at the moment when user i has successfully initiated its transmission. At that moment a number of users ($\le M-1$) can be in the backoffed state (exponential with rate ν for each user) and the remaining ones are waiting for the new packets to arrive (exponential with rate λ for each user). Hence, the time until another user attempts to access the channel is upper bounded by an exponential time of rate $(M-1)(\lambda\wedge\nu)$. Therefore, given L, the probability that there is a collision of the second type is lower bounded by $1 - e^{-L(M-1)(\lambda\wedge\nu)}$, implying that

$$\begin{aligned}\mathbb{P}\left[N_m^{(i)} > n\right] &\ge \mathbb{P}[\underline{N}_m^{(i)} > n]\\ &\ge \mathbb{E}\left[\left(1 - e^{-L(M-1)(\lambda\wedge\nu)}\right)^n\right]\\ &= \mathbb{P}[\underline{N} > n],\end{aligned} \tag{6}$$

since the repetitions of exponential times of rate $(M-1)(\lambda\wedge\nu)$ are independent due to the memoryless property.

Condition (1) implies that, for any $\epsilon > 0$, there exists x_ϵ such that $\mathbb{P}[L > x] \ge e^{-(\mu+\epsilon)x}$ for all $x \ge x_\epsilon$. Then, if we define random variable L_ϵ with $\mathbb{P}[L_\epsilon > x] = e^{-(\mu+\epsilon)x}, x \ge 0$, we obtain

$$L \overset{d}{\ge} L_\epsilon \mathbf{1}(L_\epsilon > x_\epsilon),$$

resulting in

$$\begin{aligned}\mathbb{P}\left[\underline{N} > n\right] &\ge \mathbb{E}\left[\left(1 - e^{-L_\epsilon \mathbf{1}(L_\epsilon > x_\epsilon)(M-1)(\lambda\wedge\nu)}\right)^n\right]\\ &\ge \mathbb{E}\left[\left(1 - e^{-L_\epsilon(M-1)(\lambda\wedge\nu)}\right)^n \mathbf{1}(L_\epsilon > x_\epsilon)\right].\end{aligned} \tag{7}$$

Noticing that for any $\delta > 0$, there exists $0 < x_\delta < 1$ such that $1 - x \ge e^{-(1+\delta)x}$ for all $0 \le x \le x_\delta$, we can choose x_ϵ large enough, such that

$$\begin{aligned}\mathbb{P}\left[\underline{N} > n\right] &\ge \mathbb{E}\left[e^{-(1+\epsilon)ne^{-L_\epsilon(M-1)(\lambda\wedge\nu)}} \mathbf{1}(L_\epsilon > x_\epsilon)\right]\\ &= \mathbb{E}\left[e^{-(1+\epsilon)ne^{-L_\epsilon(M-1)(\lambda\wedge\nu)}}\right]\\ &\quad - \mathbb{E}\left[e^{-(1+\epsilon)ne^{-L_\epsilon(M-1)(\lambda\wedge\nu)}} \mathbf{1}(L_\epsilon \le x_\epsilon)\right]\\ &\ge \mathbb{E}\left[e^{-(1+\epsilon)ne^{-L_\epsilon(M-1)(\lambda\wedge\nu)}}\right] - \zeta^n,\end{aligned} \tag{8}$$

where $\zeta = e^{-(1+\epsilon)e^{-x_\epsilon(M-1)(\lambda\wedge\nu)}} < 1$.

Now, for any $0 < x < 1$,

$$\mathbb{P}\left[e^{-(\mu+\epsilon)L_\epsilon} < x\right] = \mathbb{P}\left[L_\epsilon > -\frac{\log x}{\mu+\epsilon}\right] = x,$$

implying that $e^{-(\mu+\epsilon)L_\epsilon} \overset{d}{=} U$, where U is a uniform random variable between 0 and 1. Thus, we can derive from (8)

$$\mathbb{P}[\underline{N} > n] \ge \mathbb{E}\left[e^{-n(1+\epsilon)U^{(M-1)(\lambda\wedge\nu)/(\mu+\epsilon)}}\right] - \zeta^n.$$

Now, since

$$\mathbb{E}[e^{-\theta U^{1/\alpha}}] \sim \Gamma(\alpha+1)/\theta^\alpha \tag{9}$$

as $\theta \to \infty$, one can easily obtain

$$\lim_{n\to\infty} \frac{\log \mathbb{P}\left[\underline{N} > n\right]}{\log n} \ge -\frac{\mu+\epsilon}{(M-1)(\lambda\wedge\nu)}, \tag{10}$$

which, by passing $\epsilon \to 0$, proves the lower bound.

Next, we prove the *upper bound*. We observe that a successful transmission has two steps. First, the user has to initiate the transmission successfully (grab the channel). Second, after accessing the channel it has to complete the transmission without interruptions from other users. We will bound these events by independent ones as described below.

After a successful transmission or a collision, user i will attempt to access the channel after an exponential time of rate no smaller than $\lambda \wedge \nu$; each other user will compete to access the channel after an exponential time of rate no larger than $\lambda \vee \nu$. Once user i grabs the channel, its transmission will be successful if the first channel access time of all the other users is larger than L. Note that the first access time of the other users is exponential with rate upper bounded by $(M-1)(\lambda\vee\nu)$. Therefore, given L, the probability of a collision (failure) is upper bounded by

$$1 - \frac{\lambda \wedge \nu}{M(\lambda \vee \nu)} e^{-L(M-1)(\lambda\vee\nu)}.$$

Furthermore, due to the memoryless property of exponential distribution the probability of n successive collisions, given L, can be upper bounded by independent events with probabilities given by the preceding expression. Therefore, after unconditioning, we obtain

$$\begin{aligned}\mathbb{P}\left[N_m^{(i)} > n\right] &\leq \mathbb{E}\left[\left(1 - \frac{\lambda \wedge \nu}{M(\lambda \vee \nu)} e^{-L(M-1)(\lambda\vee\nu)}\right)^n\right] \\ &= \mathbb{P}\left[\overline{N} > n\right]. \end{aligned} \tag{11}$$

Next, using $1-x \leq e^{-x}$ and defining $\zeta \triangleq \lambda\wedge\nu/(M(\lambda\vee\nu))$, we derive, for $x_\epsilon > 0$,

$$\begin{aligned}\mathbb{P}\left[\overline{N} > n\right] &\leq \mathbb{E}\left[e^{-n\zeta e^{-L(M-1)(\lambda\vee\nu)}} \mathbf{1}\left(L > x_\epsilon\right)\right] \\ &\quad + \left(1 - \zeta e^{-x_\epsilon(M-1)(\lambda\vee\nu)}\right)^n \\ &\leq \mathbb{E}\left[e^{-n\zeta e^{-L\mathbf{1}(L>x_\epsilon)(M-1)(\lambda\vee\nu)}}\right] + \eta^n, \end{aligned} \tag{12}$$

where $\eta \triangleq 1 - \zeta e^{-x_\epsilon(M-1)(\lambda\vee\nu)} < 1$. Now, condition (1) implies that, for any $0 < \epsilon < \mu$, we can choose x_ϵ such that $\mathbb{P}[L > x] \leq e^{-(\mu-\epsilon)x}$ for all $x \geq x_\epsilon$. Thus, by defining an exponential random variable L^ϵ with $\mathbb{P}[L^\epsilon > x] = e^{-(\mu-\epsilon)x}, x \geq 0$, we obtain $L\mathbf{1}(L > x_\epsilon) \overset{d}{\leq} L^\epsilon$. Therefore, (12) implies

$$\mathbb{P}\left[\overline{N} > n\right] \leq \mathbb{E}\left[e^{-n\zeta e^{-L^\epsilon(M-1)(\lambda\vee\nu)}}\right] + \eta^n. \tag{13}$$

Similarly as in the proof of the lower bound, we know $e^{-(\mu-\epsilon)L^\epsilon} \overset{d}{=} U$ is a uniform random variable between 0 and 1. Thus, (13) implies

$$\mathbb{P}\left[\overline{N} > n\right] \leq \mathbb{E}\left[e^{-n\zeta U^{(M-1)(\lambda\vee\nu)/(\mu-\epsilon)}}\right] + \eta^n.$$

By (9), we obtain

$$\varlimsup_{n\to\infty} \frac{\log \mathbb{P}\left[\overline{N} > n\right]}{\log n} \leq -\frac{\mu - \epsilon}{(M-1)(\lambda \vee \nu)},$$

which, by passing $\epsilon \to 0$, finishes the proof of the upper bound.

Now, we prove the result for $T_m^{(i)}$. Observe that each attempt for user i to transmit the mth packet consists of two steps. First, user i initiates an attempt to grab the channel; for the jth attempt, denote by $\{X_j\}_{j\geq1}$ the idle period where user i either is waiting for a new packet to arrive or is in its backoff state for the mth packet. Hence, X_1 is exponential with rate λ and $X_j, j > 1$ are exponential with rate ν. Second, after user i makes an attempt to access the channel, it either collides with other users that are transmitting packets or starts transmitting its own packet; for the jth attempt, denote by $\{Y_j\}_{j\geq1}$ the period during which there are no transmissions from other users after user i starts sending the mth packet. Note that if user i fails to grab the channel for the jth attempt, then $Y_j = 0$; if user i successfully grabs the channel for this attempt, then it spends time Y_j transmitting the mth packet without interference from other users. Thus, we have

$$T_m^{(i)} = \sum_{j=1}^{N_m^{(i)}} X_j + \sum_{j=1}^{N_m^{(i)}-1} Y_j + L. \tag{14}$$

Since $\{X_j\}$ is a sequence of exponential random variables with rate equal to either λ or ν, we can always find two i.i.d. exponential sequences, $\{\underline{X}, \underline{X}_j\}_{j\geq1}$ and $\{\overline{X}, \overline{X}_j\}_{j\geq1}$, such that

$$\underline{X}_j \leq X_j \leq \overline{X}_j. \tag{15}$$

Additionally, observe that when user i successfully grabs the channel, Y_j is stochastically upper bounded by an exponential random variable with rate $(M-1)(\lambda \vee \nu)$, and thus, we can construct a sequence of i.i.d. exponential random variables $\{\overline{Y}, \overline{Y}_j\}$ such that

$$Y_j \leq \overline{Y}_j, \tag{16}$$

where $\{\overline{Y}_j\}$ is independent of $\{\overline{X}_j\}$.

First, we prove the upper bound. Using the union bound,

$$\begin{aligned}\mathbb{P}\left[T_m^{(i)} > 2t\right] &\leq \mathbb{P}\left[\sum_{j=1}^{N_m^{(i)}}(X_j + Y_j) + L > 2t\right] \\ &\leq \mathbb{P}\left[\sum_{j=1}^{N_m^{(i)}}(X_j + Y_j) > t, N_m^{(i)} \leq \frac{t}{2\mathbb{E}[\overline{X}+\overline{Y}]}\right] \\ &\quad + \mathbb{P}\left[N_m^{(i)} > \frac{t}{2\mathbb{E}[\overline{X}+\overline{Y}]}\right] + \mathbb{P}\left[L > t\right] \\ &\leq \mathbb{P}\left[\sum_{j=1}^{t/\left(2\mathbb{E}[\overline{X}+\overline{Y}]\right)}(X_j + Y_j) > t\right] \\ &\quad + \mathbb{P}\left[N_m^{(i)} > \frac{t}{2\mathbb{E}[\overline{X}+\overline{Y}]}\right] + \mathbb{P}\left[L > t\right]. \end{aligned}$$

By (3), (15) and (16), we obtain

$$\begin{aligned}\mathbb{P}\left[T_m^{(i)} > 2t\right] &\leq \mathbb{P}\left[\sum_{j=1}^{t/\left(2\mathbb{E}[\overline{X}+\overline{Y}]\right)}(\overline{X}_j + \overline{Y}_j) > t\right] \\ &\quad + \mathbb{P}\left[\overline{N} > \frac{t}{2\mathbb{E}[\overline{X}+\overline{Y}]}\right] + \mathbb{P}\left[L > t\right] \\ &\triangleq I_1 + I_2 + I_3, \end{aligned} \tag{17}$$

which, by defining random variable $\overline{T}$ with the following distribution

$$\mathbb{P}\left[\overline{T} > 2t\right] \triangleq \min\{I_1 + I_2 + I_3, 1\},$$

implies $\mathbb{P}\left[T_m^{(i)} > 2t\right] \leq \mathbb{P}\left[\overline{T} > 2t\right]$, i.e.,

$$T_m^{(i)} \overset{d}{\leq} \overline{T}. \tag{18}$$

For (17), applying Chernoff bound, we derive $I_1 = O(e^{-\eta n})$ for some $\eta > 0$. Condition (1) implies $I_3 = O(e^{-\eta n})$ for some other $\eta > 0$. To compute I_2, using (5), we obtain,

$$\lim_{t\to\infty} \frac{\log \mathbb{P}\left[\overline{N} > \frac{t}{2\mathbb{E}[\overline{X}+\overline{Y}]}\right]}{\log t} = -\frac{\mu}{(M-1)(\lambda \vee \nu)},$$

which, combined with the estimates for I_1 and I_3, implies that

$$\lim_{t\to\infty} \frac{\log \mathbb{P}\left[\overline{T} > t\right]}{\log t} = -\frac{\mu}{(M-1)(\lambda \vee \nu)}. \tag{19}$$

Next, we prove the lower bound. It is easy to obtain

$$\begin{aligned}
\mathbb{P}\left[T_m^{(i)} > t\right] &\geq \mathbb{P}\left[\sum_{j=1}^{N_m^{(i)}-1} (X_j + Y_j) + L > t\right] \\
&\geq \mathbb{P}\left[\sum_{j=1}^{N_m^{(i)}-1} X_j > t\right] \\
&\geq \mathbb{P}\left[\sum_{j=1}^{N_m^{(i)}-1} X_j > t, N_m^{(i)} > \frac{2t}{\mathbb{E}[\underline{X}]} + 1\right] \\
&\geq \mathbb{P}\left[N_m^{(i)} > \frac{2t}{\mathbb{E}[\underline{X}]} + 1\right] \\
&\quad - \mathbb{P}\left[\sum_{j=1}^{N_m^{(i)}-1} X_j \leq t, N_m^{(i)} \geq \frac{2t}{\mathbb{E}[\underline{X}]} + 1\right],
\end{aligned}$$

which, by recalling $X_j \geq \underline{X}_j$ and using (3), yields

$$\begin{aligned}
\mathbb{P}\left[T_m^{(i)} > t\right] &\geq \mathbb{P}\left[N_m^{(i)} > \frac{2t}{\mathbb{E}[\underline{X}]} + 1\right] - \mathbb{P}\left[\sum_{j=1}^{2t/\mathbb{E}[\underline{X}]} X_j \leq t\right] \\
&\geq \mathbb{P}\left[\underline{N} > \frac{2t}{\mathbb{E}[\underline{X}]} + 1\right] - \mathbb{P}\left[\sum_{j=1}^{2t/\mathbb{E}[\underline{X}]} \underline{X}_j \leq t\right] \\
&\triangleq I_1(t) - I_2(t).
\end{aligned}$$

Now, define a random variable $\underline{T}$ with

$$\mathbb{P}\left[\underline{T} > t\right] \triangleq \max\{I_1(t) - I_2(t), 0\},$$

implying

$$T_m^{(i)} \overset{d}{\geq} \underline{T}.$$

Next, by Churnoff bound, we obtain $I_2(t) \leq O\left(e^{-\eta t}\right)$ for some $\eta > 0$. Using (5), we derive

$$\lim_{t\to\infty} \frac{\log \mathbb{P}\left[\underline{N} > \frac{2t}{\mathbb{E}[\underline{X}]} + 1\right]}{\log t} = -\frac{\mu}{(M-1)(\lambda \wedge \nu)},$$

which, combined with the estimates for $I_2(t)$, implies that

$$\lim_{t\to\infty} \frac{\log \mathbb{P}\left[\underline{T} > t\right]}{\log t} = -\frac{\mu}{(M-1)(\lambda \wedge \nu)}. \tag{20}$$

Combining (19) and (20) completes the proof. ■

The following lemma studies the distribution of the number of retransmissions that occur from a point when there is a departure until the system becomes full. For the two sequences $\{C_i\}$ and $\{D_m\}$ defined in Subsection II-A, noting that $\{D_m\}$ is a subsequence of $\{C_i\}$, we can define the position of D_m in $\{C_i\}$ by $h_m \triangleq \min\{i \geq 0 : C_i = D_m\}$. Let N_m^f, $m \geq 0$ be the total number of both collisions and departures until the system becomes full and all the users are backlogged (a collision occurs) for the first time after D_m, i.e., $N_m^f \triangleq \min\{l - h_m : U(C_l+) = M, l \geq h_m\}$, where $U(C_l+)$ represents the right hand limit of $U(t)$ at time C_l. Recall that $\left\{L_i^{(t)}\right\}_{1\leq i\leq U(t)}$ represents the packet sizes of all the $U(t)$ number of active users at time t.

Lemma II.1 *For any finite values $\{L_i^{(D_m)}\}_{1\leq i\leq U(D_m)}$ at time D_m, uniformly for all $m > 0$, we have*

$$\mathbb{P}\left[N_m^f > n\right] = O\left(e^{-\eta\sqrt{n}}\right) \tag{21}$$

where the constant $\eta > 0$ does not depend on $\{L_i^{(D_m)}\}_{1\leq i\leq U(D_m)}$.

Remark 1 We believe that it is possible to prove a tighter exponential bound $\mathbb{P}\left[N_m^f > n\right] = O\left(e^{-\eta n}\right)$, but the preceding Weibull bound suffices for our proofs.

The **proof** of Lemma II.1 is presented in Section VIII.

III. Stability

In this Section, we derive the stability condition of finite population ALOHA with variable packets. Corollaries III.1 and III.2 are based on Proposition II.1; Proposition III.1 studies the distributional properties of the upper bound for the number of (re)transmissions and transmission delay for each successfully received packet observed at the receiver. Using these results, we derive the stability condition in Theorem III.1.

We use $\overline{\underline{\lim}}$ to denote both $\overline{\lim}$ and $\underline{\lim}$, i.e., $\overline{\underline{\lim}}$ means that the corresponding two statements with respect to $\overline{\lim}$ and $\underline{\lim}$ are true. From Proposition II.1, we can easily obtain the following two corollaries. Note that in Corollary III.1 we use $\overline{\underline{\lim}}$ with respect to m since the existence of the stationary region for $N_m^{(i)}$ and $T_m^{(i)}$ is not established. At this point of our analysis, we could not find an easy argument for resolving this, maybe minor, technical issue.

Corollary III.1 *If $\lambda = \nu > 0$, then, as $n \to \infty$,*

$$\overline{\underline{\lim}}_{m\to\infty} \frac{\log \mathbb{P}\left[N_m^{(i)} > n\right]}{\log n} \to -\frac{\mu}{(M-1)\nu}.$$

Corollary III.2 *If $0 < \lambda \leq \nu$ and $\mu > (M-1)\nu$, then the system has a positive throughput. If $\lambda \geq \nu > 0$ and $\mu < (M-1)\nu$, then the system has a zero throughput.*

Proof: Let $N(t) \triangleq \min\{j : \sum_{m=1}^{j} T_m \le t\}$ be the counting process for the number of successfully transmitted packets observed at the receiver from time 0 until time t. By the same fashion, we can define the counting process $N^{(i)}(t) \triangleq \min\{j : \sum_{m=1}^{j} T_m^{(i)} \le t\}$ for user $i, 1 \le i \le M$, which represents the number of successfully transmitted packets observed at user i from time 0 until time t. Clearly, we have

$$N(t) = \sum_{i=1}^{M} N^{(i)}(t) \tag{22}$$

where $N(t), N^{(i)}(t)$ all go to infinity almost surely as $t \to \infty$.

Recalling the proof corresponding to $\overline{T}$ and $\underline{T}$ in Proposition II.1, we can always construct on the same probability space $\{\overline{T}_j\}_{j\ge1}$ and $\{\underline{T}_j\}_{j\ge1}$, two sequences of i.i.d. copies of $\{\overline{T}\}$ and $\{\underline{T}\}$, such that $\underline{T}_j \le T_m^{(i)} \le \overline{T}_j$. Define $\overline{N}^{(i)}(t) \triangleq \min\{j : \sum_{m=1}^{j} \underline{T}_m^{(i)} \le t\}$ and $\underline{N}^{(i)}(t) \triangleq \min\{j : \sum_{m=1}^{j} \overline{T}_m^{(i)} \le t\}$ for user $i, 1 \le i \le M$. By the preceding definitions, we can easily obtain

$$\underline{N}^{(i)}(t) \le N^{(i)}(t) \le \overline{N}^{(i)}(t). \tag{23}$$

Thus, if $\lambda \le \nu$ and $\mu > (M-1)\nu$, then

$$\varliminf_{t\to\infty} \frac{N^{(i)}(t)}{t} \ge \varliminf_{t\to\infty} \frac{\underline{N}^{(i)}(t)}{t} = \frac{1}{\mathbb{E}\left[\overline{T}\right]} > 0, \tag{24}$$

since $\overline{T}$ has a power law tail with index greater than one ($\mathbb{E}\left[\overline{T}\right] < \infty$) by Proposition II.1.

If $\lambda \ge \nu$ and $\mu < (M-1)\nu$, then

$$\varlimsup_{t\to\infty} \frac{N^{(i)}(t)}{t} \le \varlimsup_{t\to\infty} \frac{\overline{N}^{(i)}(t)}{t} = \frac{1}{\mathbb{E}\left[\underline{T}\right]} = 0, \tag{25}$$

since $\underline{T}$ has a power law tail with index smaller than one ($\mathbb{E}\left[\overline{T}\right] = \infty$) by Proposition II.1.

Combining (22), (24) and (25), we finish the proof. ■

Proposition III.1 *For an ALOHA system with finite size packets at $t = 0$ and under condition (1) on asymptotically exponential packet sizes, there exist $\hat{N}$ and $\hat{T}$ such that the number of transmissions N_m and the transmission time T_m satisfy*

$$N_m \overset{d}{\le} \hat{N}, \quad T_m \overset{d}{\le} \hat{T}$$

with

$$\lim_{n\to\infty} \frac{\mathbb{P}[\hat{N} > n]}{\log n} = \lim_{t\to\infty} \frac{\mathbb{P}[\hat{T} > t]}{\log t} = -\frac{\mu}{(M-1)\nu}. \tag{26}$$

Proof: Recalling the definition of N_m^f before Lemma II.1 and using the union bound, we obtain

$$\begin{aligned}
\mathbb{P}[N_m > n] &= \mathbb{P}\left[N_m > n, N_{m-1}^f < N_m\right] \\
&\quad + \mathbb{P}\left[N_m > n, N_{m-1}^f \ge N_m\right] \\
&\le \mathbb{P}\left[N_m - N_{m-1}^f + N_{m-1}^f > n, N_{m-1}^f < N_m\right] \\
&\quad + \mathbb{P}\left[N_{m-1}^f > n\right] \\
&\le \mathbb{P}\left[N_m - N_{m-1}^f > \frac{n}{2}, N_{m-1}^f < N_m\right] \\
&\quad + \mathbb{P}\left[N_{m-1}^f > \frac{n}{2}\right] + \mathbb{P}\left[N_{m-1}^f > \frac{n}{2}\right].
\end{aligned} \tag{27}$$

By Lemma (II.1), we know that for some $0 < \zeta < 1$,

$$\begin{aligned}
\mathbb{P}\left[N_{m-1}^f > \frac{n}{2}\right] + \mathbb{P}\left[N_{m-1}^f > \frac{n}{2}\right] &\le 2\mathbb{P}\left[N_{m-1}^f > \frac{n}{2}\right] \\
&\le 2\zeta^{\sqrt{\frac{n}{2}}}.
\end{aligned} \tag{28}$$

Observe that $N_{m-1}^f < N_m$ implies that there exists time $\sigma < D_m$ at which each user has a packet and is in backoffed status. Thus, by recalling the notation defined for Lemma II.1, we can denote the packet sizes held by M active users at time σ by $L_i^{(\sigma)}, 1 \le i \le M$. In addition, we know that right after time D_{m-1}, one user has just successfully transmitted a packet. Thus, at time σ (when the system is full) there is at least one new packet with size equal in distribution to L in the system. Therefore, we obtain

$$\begin{aligned}
&\mathbb{P}\left[N_m - N_{m-1}^f > \frac{n}{2}, N_{m-1}^f < N_m\right] \\
&\quad \le \mathbb{E}\left[\left(1 - \frac{1}{M}\left(\sum_{i=1}^{M} e^{-L_i^{(D_m-)}(M-1)\nu}\right)\right)^{\lfloor \frac{n}{2} \rfloor}\right] \\
&\quad \le \mathbb{E}\left[\left(1 - \frac{1}{M} e^{-L(M-1)\nu}\right)^{\lfloor \frac{n}{2} \rfloor}\right],
\end{aligned}$$

which, in conjunction with (27) and (28), implies, uniformly for all m,

$$\mathbb{P}[N_m > n] \le \mathbb{E}\left[\left(1 - \frac{1}{M} e^{-L(M-1)\nu}\right)^{\lfloor \frac{n}{2} \rfloor}\right] + 2\zeta^{\sqrt{\frac{n}{2}}}.$$

Now, we can define a random variable $\hat{N}$ which satisfies, for integer n, $\mathbb{P}[\hat{N} > n]$ is equal to

$$\min\left\{1, \mathbb{E}\left[\left(1 - \frac{1}{M} e^{-L(M-1)\nu}\right)^{\lfloor \frac{n}{2} \rfloor}\right] + 2\zeta^{\sqrt{\frac{n}{2}}}\right\},$$

implying

$$N_m \overset{d}{\le} \hat{N}.$$

By using the same approach as in calculating (11), we obtain

$$\lim_{n\to\infty} \frac{\log \mathbb{P}[\hat{N} > n]}{\log n} = -\frac{\mu}{(M-1)\nu},$$

which finishes the proof of the result on $\hat{N}$ in equation (26). The proof for $\hat{T}$ follows similar arguments as in proving the result on $T_m^{(i)}$ in Proposition II-B. ■

Combining Theorem III.1 and Corollary III.2, we obtain the following theorem. Observe that this theorem is slightly more general than Corollary III.2 since it shows that $\mu > (M-1)\nu$ is enough for positive throughput, i.e., the additional condition $\lambda \leq \nu$ in Corollary III.2 is not needed.

Theorem III.1 *Under condition (1), if $\mu > (M-1)\nu$, the ALOHA system has a positive throughput. Conversely, if $\lambda \geq \nu > 0$ and $\mu < (M-1)\nu$, then, the system has a zero throughput.*

Remark 2 For the critical case $\mu = (M-1)\nu$, if L has an exact exponential tail, i.e., $\mathbb{P}[L > x] \sim ce^{-\mu x}$ and $\lambda \geq \nu$, then, the limiting distributions of $N_m^{(i)}$ and $T_m^{(i)}$ would have exact power law tails of index 1, and therefore, have infinite means.

Remark 3 The condition $\lambda \geq \mu$ and $\mu < (M-1)\nu$ yields a zero throughput. However, it appears that one could obtain a positive throughput by decreasing λ for fixed ν and μ in this case. Specifically, we conjecture that the throughput of the system is positive when λ is small enough and $M\mu > (M-1)\nu > \mu$.

Proof: The second statement of this theorem is the same as the second statement of Corollary III.2. Given Proposition III.1, the first statement can be easily derived using basically the same arguments as in the proof of Corollary III.2, and thus we omit the details. ∎

IV. Approximation of the Distributions of N_m and T_m

A. Starting Behavior

In this subsection, we study the number of retransmissions N_m and the transmission delay T_m for the mth successfully transmitted packet observed at the receiver when the system starts from an empty state. This result characterizes the starting behavior of our ALOHA model for small (finite) m. Furthermore, since ALOHA tends to accumulate with time longer packets, it would make sense to define a modified ALOHA which, after a finite (possibly large) number of successful transmissions, refreshes itself by discarding all the packets currently present in the system. Hence, for this modified ALOHA, the following theorem describes the steady state behavior as well.

Theorem IV.1 *Under condition (1), assume that at time $t = 0$ the system is empty $U(0) = 0$, then, for any fixed $m \geq M$, the number of transmissions N_m and the transmission time T_m satisfy*

$$\lim_{n\to\infty} \frac{\mathbb{P}[N_m > n]}{\log n} = \lim_{t\to\infty} \frac{\mathbb{P}[T_m > t]}{\log t} = -\frac{M\mu}{(M-1)\nu}. \quad (29)$$

Remark 4 A special case of this theorem when $U(C_m+) = M$ with all the packets in the system being i.i.d. and equal in distribution to L was proved in Theorem 1 of [8].

Remark 5 Note that this result still holds even if we allow m to be a slowly growing function of n for N_m, e.g., $m = o(\log n)$ (or $m = o(\log t)$ for T_m).

Remark 6 This theorem indicates that the distribution tails of N_m and T_m are essentially power laws when the packet distribution is approximately exponential ($\approx e^{-\mu x}$). Thus, the finite population ALOHA may exhibit high variations, e.g., the system has infinite average transmission time when $0 < M\mu/(M-1)\nu < 1$; and when $1 < M\mu/(M-1)\nu < 2$, the transmission time has finite mean but infinite variance. It might be worth noting that this may even occur when the expected packet length is much smaller than the expected backoff time $\mathbb{E}L \ll 1/\nu$.

Proof of Theorem IV.1: We first prove the logarithmic asymptotics for N_m, based on which a similar result can be proved for T_m.

First, we begin with proving the *lower bound* for N_m. We construct a special event with a positive probability that guides the system from time 0 up to time D_{m-1}. Denote by $\mathscr{E}_1$ the event that only one of the users has packets to send and all the other $M-1$ users are empty from time 0 through time D_{m-1}; additionally, we require that the sizes of these arriving packets be less than a constant $k-1$ with $\mathbb{P}[L \leq k-1] > 0$ and that each new arrival be within a unit interval after the previous departure. This construction implies $D_{m-1} \leq (m-1)k$, and therefore, by time D_{m-1} the probability that the system evolves according to $\mathscr{E}_1$ is lower bounded by

$$\mathbb{P}[\mathscr{E}_1] \geq \left((1-e^{-\lambda})\mathbb{P}[L \leq k-1]\right)^{m-1} e^{-(M-1)\lambda(m-1)k} > 0. \quad (30)$$

Next, immediately after time D_{m-1}, observe that the whole system becomes empty according to our construction. Then, we build another special event $\mathscr{E}_2$ that leads M users to have i.i.d. packets with sizes that are larger than 1 in their buffers after time D_{m-1}.

To this point, we require that each of the M users have a packet with size larger than 1 arriving to the system after D_{m-1} and that their arriving points be within $[D_{m-1}, D_{m-1}+1]$. This event happens with probability $(1-e^{-\lambda})^M \mathbb{P}[L > 1]^M$. Notice that, immediately after the Mth packet arrives, there are either $M-1$ or M users in the backoff status, depending on whether the Mth arrived packet collides with others upon arrival or not. If the Mth packet does not collide with others upon arrival, we require that a retransmission occur within one unit of time after it arrives, which happens with a probability greater than $1-e^{-(M-1)\nu}$. These requirements can guarantee that there exists a time $\tau \in [D_{m-1}, D_{m-1}+2)$ with $\tau = \min\{C_n \mid U(C_n+) = M, C_n > D_{m-1}\}$, at which each user in the system has a packet and is in the backoff status. The probability that the event $\mathscr{E}_2$ happens is lower bounded by

$$\mathbb{P}[\mathscr{E}_2] \geq (1-e^{-\lambda})^M \mathbb{P}[L > 1]^M (1-e^{-(M-1)\nu}) > 0.$$

Now, given $\mathscr{E}_1$ and $\mathscr{E}_2$, we can denote by N_m^* the number of retransmissions between $(\tau, D_m]$, implying $N_m \geq N_m^*$. Then, recalling the notation defined before Lemma II.1 and defining $L_o \triangleq \min\left\{L_1^{(\tau)}, L_2^{(\tau)}, \cdots, L_M^{(\tau)}\right\}$, we obtain

$$\begin{aligned}\mathbb{P}[N_m^* > n \mid \mathscr{E}_1, \mathscr{E}_2] &= \mathbb{E}\left[\left(1 - \frac{1}{M}\left(\sum_{i=1}^{M} e^{-L_i^{(\tau)}(M-1)\nu}\right)\right)^n\right] \\ &\geq \mathbb{E}\left[\left(1 - e^{-L_o(M-1)\nu}\right)^n\right]. \quad (31)\end{aligned}$$

It is easy to check that the complementary cumulative distribution function $\mathbb{P}[L_o \geq x]$ satisfies

$$\lim_{x\to\infty} \frac{\log \mathbb{P}[L_o \geq x]}{x} = -M\mu,$$

which, by using the same technique as in estimating (6), yields

$$\underline{\lim}_{n\to\infty} \frac{\log \mathbb{P}[N_m^* > n \mid \mathscr{E}_1, \mathscr{E}_2]}{\log n} \geq -\frac{M\mu}{(M-1)\nu}.$$

Finally, using $\mathbb{P}[N_m > n] \geq \mathbb{P}[\mathscr{E}_1, \mathscr{E}_2]]\mathbb{P}[N_m^* > n \mid \mathscr{E}_1, \mathscr{E}_2]$ completes the proof of the lower bound for N_m.

Next, we proceed with the proof of the *upper bound* for N_m. Using the same approach as in evaluating (27), we obtain

$$\mathbb{P}[N_m > n] \leq \mathbb{P}\left[N_m - N_{m-1}^f > \frac{n}{2}, N_{m-1}^f < N_m\right] + 2\zeta\sqrt{\tfrac{n}{2}}. \tag{32}$$

Now, we observe that $N_m^f < N_m$ implies that there exists time $\sigma < D_m$ at which each user has a packet at hand and is in the backoff status. Thus, we denote the packet sizes held by M users at time σ by $L_i^{(\sigma)}, 1 \leq i \leq M$. In addition, we know that at time σ the total number of packets, including those still present in the system and those already successfully transmitted, is less than $m + M$ since the system has only M users. Denote the sizes of the first $m + M$ packets arriving to the system by $\{L_1, L_2, \cdots, L_{m+M}\}$ and its order statistics by $L^{(1)} \geq L^{(2)} \geq \cdots \geq L^{(m+M)}$, and we obtain

$$\begin{aligned}
&\mathbb{P}[N_m - N_{m-1}^f > n, N_{m-1}^f < N_m] \\
&\qquad \leq \mathbb{E}\left[\left(1 - \frac{1}{M}\left(\sum_{i=1}^{M} e^{-L_i^{(\sigma)}(M-1)\nu}\right)\right)^n\right] \\
&\qquad \leq \mathbb{E}\left[\left(1 - \frac{1}{M} e^{-L^{(M)}(M-1)\nu}\right)^n\right].
\end{aligned} \tag{33}$$

Since $L^{(M)}$ is the Mth largest value among $L_i, 1 \leq i \leq m + M$, we know

$$\lim_{x\to\infty} \frac{\mathbb{P}[L^{(M)} > x]}{x} = -M\mu.$$

Then, by (32), (33) and using the same approach as in estimating (11), one derives

$$\overline{\lim}_{n\to\infty} \frac{\log \mathbb{P}[N_m > n]}{\log n} \leq -\frac{M\mu}{(M-1)\nu}, \tag{34}$$

which completes the proof of the upper bound.

The proof for the logarithmic asymptotics of T_m is based on similar arguments as in proving $T_m^{(i)}$ in Proposition II.1 and, thus, we omit the details.

■

B. Limiting Steady State Behavior

When the system keeps running for a long period of time, we can show that the preceding upper bound, presented in Theorem III.1, is attainable when $\lambda = \nu > \mu/(M-1)$. In order to study this situation, first we establish the following lemma that characterizes the growth of the packet sizes in the system immediately after a departure at time D_m. Noting that $\lambda = \nu$, we can assume that once a user successfully transmits a packet through the channel, it immediately generates a new packet in its buffer and goes into the backoff state, i.e., we can interpret that the arrival and departure happen at the same time. Therefore, the system evolves as if it always had M packets available and all users remained in the backoff state over the entire operation. Let $\underline{L}(D_m)$ be the minimum of the packet sizes of the other $M-1$ users except the one departing at time D_m.

Lemma IV.1 *Assume that $\lambda = \nu > \mu/(M-1)$ and*

$$\overline{\lim}_{y\to\infty} \sup_{\delta y < x < y} \frac{1}{y-x} \log\left(\frac{\mathbb{P}[L > x]}{\mathbb{P}[L > y]}\right) \leq \mu \tag{35}$$

for $0 < \delta < 1$. Then, there exists $p > 0$ such that for any fixed y,

$$\underline{\lim}_{m\to\infty} \mathbb{P}\left[\underline{L}(D_m) > y\right] > p. \tag{36}$$

Remark 7 We believe that a stronger result $\underline{\lim}_{m\to\infty} \mathbb{P}\left[\underline{L}(D_m) > y\right] = 1$ for all y is also true, but the preceding lemma suffices for our proofs. Furthermore, a careful examination of our proof shows that the result is also true for $\min\{\lambda, \nu\} > \mu/(M-1)$, but we avoid this generalization due to considerable notational complications.

Remark 8 It is easy to see that condition (35) holds for a broad range of distributions from exponential family, e.g., Gamma distribution, $e^{-\mu x}e^{\gamma x^\beta}$ with $0 < \beta < 1$, etc.

The **proof** of Lemma IV.1 is presented in Section VIII. By using this lemma, we can derive the following theorem that characterizes the limiting steady state behavior of our ALOHA model.

Theorem IV.2 *Under condition (35), if $\lambda = \nu > \mu/(M-1)$, we obtain*

$$\begin{aligned}
\lim_{n\to\infty} \overline{\lim}_{m\to\infty} \frac{\mathbb{P}[N_m > n]}{\log n} &= \lim_{t\to\infty} \overline{\lim}_{m\to\infty} \frac{\mathbb{P}[T_m > t]}{\log t} \\
&= -\frac{\mu}{(M-1)\nu}.
\end{aligned} \tag{37}$$

Proof: First, we prove the result for N_m. The upper bound is implied by Proposition III.1 and thus, we only need to prove the lower bound. Recalling the definition of $\underline{L}(D_m)$ in the paragraph before Lemma IV.1 and using Lemma IV.1, we obtain that there exist $p > 0$ and $m_0 > 0$ such that for all $m > m_0$,

$$\mathbb{P}\left[\underline{L}(D_{m-1}) > \frac{\log n}{(M-1)\nu}\right] > p. \tag{38}$$

Since there is a new packet with size equal in distribution to L arriving to the system at time D_{m-1} (see the discussion before Lemma IV.1), and the packet sizes of the other $M-1$

users are lower bounded by $\underline{L}(D_{m-1})$, we obtain

$$\begin{aligned}
\mathbb{P}[N_m > n] &\geq \mathbb{P}\left[N_m > n, \underline{L}(D_{m-1}) > \frac{\log n}{(M-1)\nu}\right] \\
&\geq \mathbb{E}\Bigg[\left(1 - \frac{1}{M}\left(\sum_{i=1}^{M} e^{-L_i^{(D_{m-1})}(M-1)\nu}\right)\right)^n \\
&\qquad \times \mathbf{1}\left(\underline{L}(D_{m-1}) > \frac{\log n}{(M-1)\nu}\right)\Bigg] \\
&\geq \mathbb{E}\Bigg[\Bigg(1 - \frac{1}{M}\Bigg(e^{-L(M-1)\nu} \\
&\qquad\qquad + \sum_{i=1}^{M-1} e^{-\underline{L}(D_{m-1})\cdot(M-1)\nu}\Bigg)\Bigg)^n \\
&\qquad \times \mathbf{1}\left(\underline{L}(D_{m-1}) > \frac{\log n}{(M-1)\nu}\right)\Bigg] \\
&\geq \mathbb{P}\left[\underline{L}(D_{m-1}) > \frac{\log n}{(M-1)\nu}\right] \\
&\qquad \times \mathbb{E}\left[\left(1 - \frac{M-1}{M}\cdot\frac{1}{n} - \frac{1}{M}e^{-L(M-1)\nu}\right)^n\right],
\end{aligned} \tag{39}$$

where we use the independence between the new packet size and $\underline{L}(D_{m-1})$ at time D_{m-1} in the last inequality.

Combining (38) and (39) yields, for n large enough, $\mathbb{P}[N_m > n]$ is lower bounded by

$$\begin{aligned}
&p\left(1 - \frac{M-1}{M}\cdot\frac{1}{n}\right)^n \\
&\quad \times \mathbb{E}\left[\left(1 - \frac{1}{M(1-(M-1)/(Mn))}e^{-L(M-1)\nu}\right)^n\right] \\
&\geq p\left(1 - \frac{M-1}{M}\cdot\frac{1}{n}\right)^n \mathbb{E}\left[\left(1 - e^{-L(M-1)\nu}\right)^n\right],
\end{aligned}$$

which, by noting that

$$\lim_{n\to\infty}\left(1 - \frac{M-1}{M}\cdot\frac{1}{n}\right)^n = e^{-(M-1)/M} > 0$$

and using the same approach as in calculating (6), completes the proof the lower bound. The result on T_m can be proved by using the same approach as in proving the result on T_m in Proposition II.1. ■

V. Power Laws in Slotted ALOHA with Random Number of Users

It is clear from the preceding section that the power law delays arise due to the combination of collisions and packet variability. Hence, it is reasonable to expect an improved performance when this variability is reduced. Indeed, it is easy to see that the delays are geometrically bounded in a slotted ALOHA with constant size packets and a finite number of users. However, in this section we will show that, when the number of users sharing the channel has asymptotically an exponential distribution, the slotted ALOHA exhibits power law delays as well. Situations with random number of users are essentially predominant in practice, e.g., in sensor networks, the number of active sensors in a neighborhood is a random variable since sensors may switch between sleep and active modes, as shown in Figure 2; similarly in ad hoc wireless networks the variability of users may arise due to mobility, new users joining the network, etc.

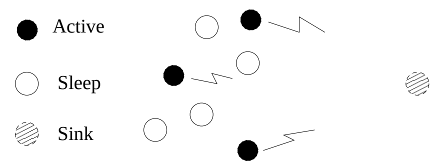

Fig. 2. Random number of active neighbors in a sensor network.

More formally, consider a slotted ALOHA model (e.g., see Section 4.2.2 of [3]) with packets/slots of unit size and a random number of users $M \geq 1$ that are fixed over time. This model can be viewed as a first order approximation of a real system where the number of users change very slowly. Similarly as in Section II, each user holds at most one packet at a time and after a successful transmission a new packet is generated according to an independent Bernoulli process with success probability $1 - e^{-\lambda}, \lambda > 0$. In case of a collision, each colliding user backs off according to an independent geometric random variable with parameter $e^{-\nu}, \nu > 0$. Denote the number of slots where transmissions are attempted but failed and the total time between two successful packet transmissions as N and T, respectively.

Theorem V.1 *If $\lambda = \nu$ and there exists $\alpha > 0$, such that*

$$\lim_{x\to\infty}\frac{\log\mathbb{P}[M > x]}{x} = -\alpha,$$

then, we have

$$\lim_{n\to\infty}\frac{\log\mathbb{P}[N > n]}{\log n} = \lim_{t\to\infty}\frac{\log\mathbb{P}[T > t]}{\log t} = -\frac{\alpha}{\nu}. \tag{40}$$

Remark 9 Similarly as in Theorem IV.1, this result shows that the distributions of N and T are essentially power laws, i.e., $\mathbb{P}[T > t] \approx t^{-\alpha/\nu}$ and, clearly, if $\alpha < \nu$, then $\mathbb{E}N = \mathbb{E}T = \infty$.

Proof: Since $\lambda = \nu$, we can consider a situation where all the users are backlogged, i.e., have a packet to send. In this case the total number of collisions between two successful transmissions is geometrically distributed given M,

$$\mathbb{P}[N > n \mid M] = \left(1 - \frac{Me^{-(M-1)\nu}(1-e^{-\nu})}{1-e^{-M\nu}}\right)^n, \ n \in \mathbb{N},$$

since, given M, $1 - e^{-M\nu}$ is the conditional probability that there is an attempt to transmit a packet, and $1 - e^{-M\nu} - Me^{-(M-1)\nu}(1-e^{-\nu})$ is the conditional probability that there is a collision. Therefore,

$$\mathbb{P}[N > n] = \mathbb{E}\left[\left(1 - \frac{Me^{-(M-1)\nu}(1-e^{-\nu})}{1-e^{-M\nu}}\right)^n\right]. \tag{41}$$

On the other hand, we have

$$\mathbb{P}[T > t] = \mathbb{E}\left[\left(1 - Me^{-(M-1)\nu}(1-e^{-\nu})\right)^t\right], \ t \in \mathbb{N}. \tag{42}$$

Now, following the same arguments as in the proof of Proposition II.1, we can prove (40).

■

Actually, using part i) of Theorem 2.1 in [7], we can compute the exact asymptotics of T under more restrictive conditions.

Theorem V.2 *If $\lambda = \nu$ and $\overline{F}(x) \triangleq \mathbb{P}[M > x]$ satisfies $\overline{F}^{-1}(x) \sim \Phi\left(e^{\nu x}(e^{\nu}x - x)^{-1}\right)$, where $\Phi(\cdot)$ is regularly varying with index $\beta > 0$, then, as $t \to \infty$,*

$$\mathbb{P}[T > t] \sim \frac{\Gamma(\beta+1)}{\Phi(t)}.$$

VI. Simulation Examples

In this section, we illustrate our theoretical results with simulation experiments. In particular, we emphasize the characteristics of the studied ALOHA protocol that may not be immediately apparent from our theorems. For example, in practice, the distributions of packets and number of random users might have bounded supports. We show that this situation may result in truncated power law distributions for the transmission delays. To this end, it is also important to note that the delay distribution has a power law main body with a stretched support in relation to the support of L and M and, thus, may result in very long, although, exponentially bounded delays.

Example 1 (Finite population model) For the finite population model described in Subsection II-A, we compare the starting and steady state behavior in this experiment.

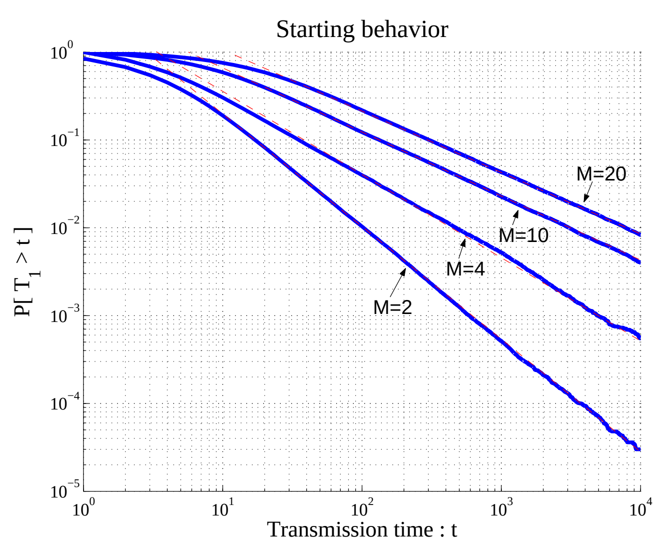

Fig. 3. Starting behavior: transmission time distribution for the first successfully transmitted packet for finite population ALOHA with variable size packets.

First, we verify Theorem IV.1 on the starting behavior by plotting the empirical distribution of time T_1 for the first successful transmission in a system that is initially empty. In this regard, we conduct four experiments for $M = 2, 4, 10, 20$ users, respectively. The packets are assumed i.i.d. exponential with mean 1 and the arrival intervals and backoffs follow an exponential distribution with mean $2/3$. The simulation experiments that each repeatedly measure 10^5 samples are shown in Figure 3, which indicates a power law transmission delay. We can see from the figure that, as M gets large ($M = 10, 20$), the slopes of the distributions that represent the power law exponents on the $\log / \log$ plot are essentially the same, as predicted by our Theorem IV.1.

Next, we compare the starting behavior with the steady state behavior predicted by Theorem IV.2. In this setting, we set $M = 3$ and choose i.i.d. packet sizes that follow an exponential distribution with mean 1. In addition, we assume that arrival intervals and backoffs are exponential with mean 1.5. The starting behavior is represented by repeatedly measuring 10^5 number of the transmission times for the first packet (m=1) in a system that is initially empty and the steady state distribution is obtained by continuously measuring the transmission times of the packets with indexes from $m = 10^5$ to $m = 10^7$. The plot in Figure 4 shows that the transmission time distribution of the first packet for the starting behavior has a slope $-M\mu/((M-1)\nu) = -2.25$, and the steady state transmission time distribution has a slope $-\mu/((M-1)\nu) = -0.75$, as predicted by equations (29) and (37) in the log-log scale, respectively.

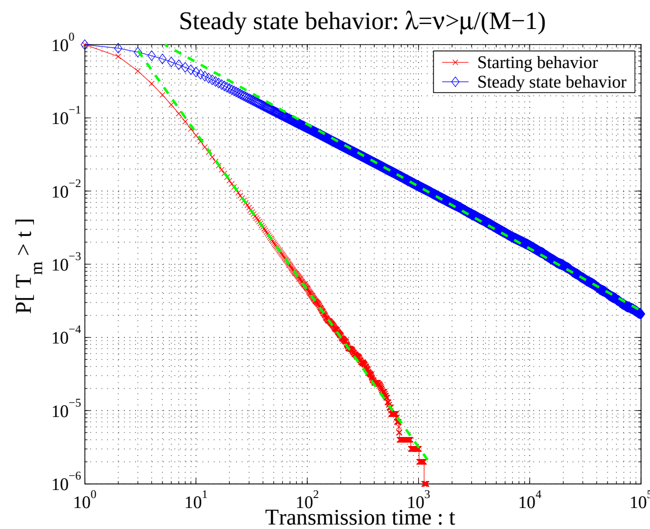

Fig. 4. Comparing starting behavior and steady state behavior for finite population ALOHA with variable size packets.

Example 2 (Random number of users) As stated in Section V, the situation when the number of users M is random may cause heavy-tailed transmission delays even for slotted ALOHA. However, in many practical applications the number of active users M may be bounded, i.e., the distribution $\mathbb{P}[M > x]$ has a bounded support. Thus, from equation (42) it is easy to see that the distribution of T is exponentially bounded. However, this exponential behavior may happen for very small probabilities, while the delays of interest can fall inside the region of the distribution (main body) that behaves as the power law. This example is aimed to illustrate this important phenomenon. Assume that initially $M \geq 1$

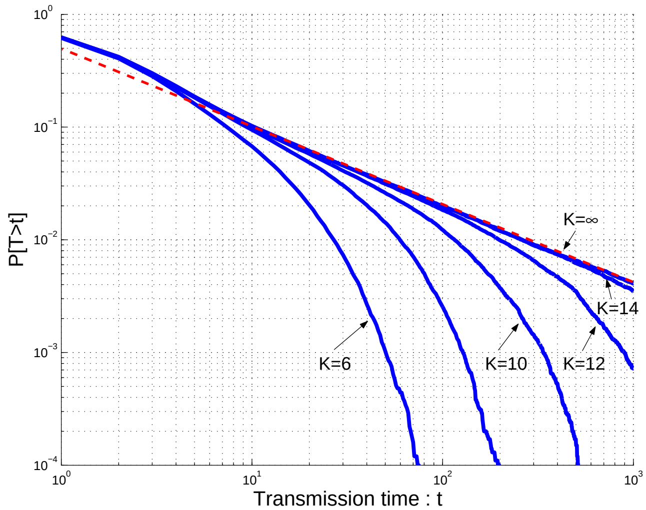

Fig. 5. Illustration of the stretched support of the power law main body when the number of users is $\min(M, K)$, where M is geometrically distributed.

users have unit size packets ready to send and M follows geometric distribution with mean 3. The backoff times of the colliding users and the arrival intervals of the new packets are independent and geometrically distributed with mean 2. We take the number of users to have finite support $[1, K]$ and show how this results in a truncated power law distribution for T in the main body, even though the tails are exponentially bounded. This example is parameterized by K where K ranges from 6 to 14 and for each K we set the number of users to be equal to $M_K = \min(M, K)$. We plot the distribution of $\mathbb{P}[T > t]$, parameterized by K, in Figure 5. From the figure we can see that, when we increase the support of the distributions from $K = 6$ to $K = 14$, the main (power law) body of the distribution of T increases from less than 5 to almost 700. This effect is what we call the stretched support of the main body of $\mathbb{P}[T > t]$ in relation to the support K of M. In fact, it can be rigorously shown that the support of the main body of $\mathbb{P}[T > t]$ grows exponentially fast. Furthermore, it is important to note that, if $K = 14$ and the probabilities of interest for $\mathbb{P}[T > t]$ are bigger than $1/500$, then the result of this experiment is basically the same as for $K = \infty$; see Figure 5.

VII. Concluding Remarks and Further Extensions

In this paper, we show that a basic finite population ALOHA model with exponential packets is characterized by power law transmission delays, possibly even resulting in zero throughput. Based on these results, we establish a new stability condition that is entirely derived from the tail behavior of the packet and backoff distributions.

Note that at any moment of time the finite population ALOHA model from Subsection II-A can be described as a Markov process for the state vector $\left(L_1^{(t)}, L_2^{(t)}, \cdots, L_M^{(t)}\right)$, where $L_i^{(t)}$ is the packet size of user i at time t. However, this Markov process is not easy to analyze in the sense that it has infinitely, possibly uncountably, many states with complicated transitions, where long packets tend to accumulate in the system since the short ones are easier to pass. Hence we conjecture, based on our initial simulation experiments, that in the steady state the system may have multiple functional forms for the power law exponent for different values of λ, ν and μ. The complete characterization of the stability of this Markov process and the full understanding of the spatial interactions and temporal correlations of packet sizes in the system remain a challenging problem. In this paper, we provide a partial picture of the system behavior. Furthermore, from an engineering perspective, it is important to study more sophisticated MAC protocols, including CSMA and RTS/CTS scheme, since ALOHA represents the basis for these more practical MAC protocols.

This power law effect and the possible instability for our ALOHA model might be diminished, or perhaps eliminated, by reducing the variability of packets. However, we show that even a slotted (synchronized) ALOHA with packets of constant size can exhibit power law delays when the number of active users is random. This spatial correlation can have a significant impact on the performance of ALOHA system when users are persistently present over a period of time that is larger than the packet transmission time. A more realistic framework to study this effect could assume that users arrive and depart on a slower time scale.

From the algorithmic perspective, we want to point out that there are other possible ways to reduce the power law delays in ALOHA, for example, adaptive ALOHA decreases by half the retransmission rate after each collision, which might greatly reduce the number of collisions at the expense of possibly low throughput. Hence, finding a right balance between the reduction of power law effects and a good throughput requires further investigation.

VIII. Proofs of Lemmas II.1 and IV.1

Proof of Lemma II.1: Our proof begins with finding a subsequence $\mathcal{C}^{(s)} = \{C_1^{(s)}, C_2^{(s)}, \cdots\}$ from $\mathcal{C} = \{C_{h_m}, C_{h_m+1}, \cdots\}$; recall the definition of h_m preceding the statement of Lemma II.1. The procedure can be described iteratively as follows: initially, set $C_1^{(s)} = C_{h_m}$, and for $j \geq 1$, we denote by $C_{j+1}^{(s)}$ the smallest value in $\mathcal{C}$ that is larger than $C_j^{(s)} + 1$. Based on this subsequence, we define $Y_j, j \geq 1$ to be the number of collisions and departures within each time interval $\left[C_j^{(s)}, C_j^{(s)} + 1\right]$; note that $C_{j+1}^{(s)} > C_j^{(s)} + 1$ by construction. Additionally, let X be the index j of the interval $\left[C_j^{(s)}, C_j^{(s)} + 1\right]$ within which the system reaches the full state for the first time after C_{h_m}, implying $\sum_{j=1}^{X-1} Y_j \leq N_m^f \leq \sum_{j=1}^{X} Y_j$. We will prove that there exists a probability $p > 0$, such that for all $j \geq 1$,

$$\mathbb{P}[X > j] \leq (1-p)^j.$$

To this end, for each interval $\left[C_j^{(s)}, C_j^{(s)} + 1\right], j \geq 1$, we construct a special event $\mathcal{E}_j$ such that on this event the system becomes full at a collision in $\left[C_j^{(s)}, C_j^{(s)} + 1\right]$, i.e., there exists $C_l \in \left[C_j^{(s)}, C_j^{(s)} + 1\right]$ such that $U(C_l+) = M$. Our construction is described as follows. We require that all the backlogged users, including those already in the system immediately after time $C_j^{(s)}$ and the new arrivals in $\left[C_j^{(s)}, C_j^{(s)}+1/2\right]$ that collide with other users, make no retransmissions during the entire interval $\left[C_j^{(s)}, C_j^{(s)}+1/2\right]$, which occurs with a probability lower bounded by $e^{-M\nu/2}$ since there are M users in total and the backoffs are independent and exponential (memoryless). Also, we require that all empty users ($\le M$) observed immediately after time $C_j^{(s)}$ have new arrivals with sizes larger than one ($L>1$) within $\left[C_j^{(s)}, C_j^{(s)}+1/2\right]$, which happens with a probability lower bounded by $(1-e^{-\lambda/2})^M \mathbb{P}[L>1]^M$. Now, if $M-U\left(C_j^{(s)}+\right)$ is even, our construction implies that at time $C_j^{(s)}+1/2$ all the users are backlogged, since two consecutive new arrivals after $C_j^{(s)}$ collide with each other and after that they are not allowed to retransmit before $C_j^{(s)}+1/2$, which implies that there exists $C_l \in \left[C_j^{(s)}, C_j^{(s)}+1/2\right]$ such that $U(C_l+)=M$. On the other hand, if $M-U\left(C_j^{(s)}+\right)$ is odd, at time $C_j^{(s)}+1/2$ there is exactly one user transmitting and the remaining $M-1$ ones are all backlogged. Now, we require that at least one backlogged user retransmit during $\left[C_j^{(s)}+1/2, C_j^{(s)}+1\right]$, which occurs with probability $1-e^{-(M-1)\nu/2}$ due to the memoryless property of the backoff distribution. Clearly, this requirement ensures that the system is full at a collision time within $\left[C_j^{(s)}+1/2, C_j^{(s)}+1\right]$. Thus, irrespective of whether $M-U\left(C_j^{(s)}+\right)$ is even or odd, there exists $C_l \in \left[C_j^{(s)}, C_j^{(s)}+1\right]$ such that $U(C_l+)=M$ on $\mathcal{E}_j$.

Therefore, we can uniformly lower bound the probability of $\mathcal{E}_j$ conditional on $U\left(C_j^{(s)}+\right)$, or equivalently on $M-U\left(C_j^{(s)}+\right)$, almost surely (a.s.) as

$$\begin{aligned}
&\mathbb{P}\left[\mathcal{E}_j \,\middle|\, U\left(C_j^{(s)}+\right)\right] \\
&\quad \ge e^{-M\mu/2}(1-e^{-\lambda/2})^M \mathbb{P}[L>1]^M \left(1-e^{-(M-1)\nu/2}\right) \\
&\quad \triangleq p > 0. \qquad (43)
\end{aligned}$$

Now, observe that $\mathcal{E}_j$ is determined by the value of $U\left(C_j^{(s)}+\right)$ and the future new arrivals and backoff times after time $C_j^{(s)}$. Furthermore, $\{X \ge j\}$ is completely determined by the arrival and backoff processes before time $C_j^{(s)}$. Hence, due to the memoryless property of the backoff and arrival times, the event $\{X \ge j\}$ and $\mathcal{E}_j$ are conditioinally independent given $U\left(C_j^{(s)}+\right)$. Therefore, by using (43), we obtain, a.s.,

$$\begin{aligned}
&\mathbb{P}\left[X \ge j, \mathcal{E}_j \,\middle|\, U\left(C_j^{(s)}+\right)\right] \\
&\quad = \mathbb{P}\left[X \ge j \,\middle|\, U\left(C_j^{(s)}+\right)\right] \mathbb{P}\left[\mathcal{E}_j \,\middle|\, U\left(C_j^{(s)}+\right)\right] \\
&\quad \ge \mathbb{P}\left[X \ge j \,\middle|\, U\left(C_j^{(s)}+\right)\right] p,
\end{aligned}$$

which implies

$$\mathbb{P}\left[X \ge j, \mathcal{E}_j\right] \ge \mathbb{P}\left[X \ge j\right] p. \qquad (44)$$

Thus, by noting that $\{X \ge j\} \cap \mathcal{E}_j \subset \{X = j\}$ and using (44), we obtain

$$\begin{aligned}
\mathbb{P}\left[X \ge j\right] - \mathbb{P}\left[X \ge j+1\right] &= \mathbb{P}\left[X = j\right] \\
&\ge \mathbb{P}\left[X \ge j, \mathcal{E}_j\right] \\
&\ge \mathbb{P}\left[X \ge j\right] p,
\end{aligned}$$

which results in

$$\mathbb{P}\left[X \ge j+1\right] \le \mathbb{P}\left[X \ge j\right](1-p).$$

Iterating on j in the preceding inequality yields

$$\mathbb{P}[X \ge j+1] \le (1-p)^j. \qquad (45)$$

Since the number of collisions and departures Y_j within the interval $\left[C_j^{(s)}, C_j^{(s)}+1\right]$ is bounded by the number of active users $U\left(C_j^{(s)}+\right)$ in the system immediately after time $C_j^{(s)}$ plus the total number of retransmissions and arrivals Z_j within this interval, we obtain

$$Y_j \le U\left(C_j^{(s)}+\right) + Z_j \le M + Z_j. \qquad (46)$$

Then, by noting that Z_j is stochastically smaller than a Poisson random variable with rate $M\max(\nu,\lambda)$ and using (45), (46), we obtain

$$\begin{aligned}
\mathbb{P}[N_m^f > n] &\le \mathbb{P}\left[\sum_{j=1}^{X} Y_j > n\right] \\
&\le \mathbb{P}[X > \sqrt{n}] + \sqrt{n}\,\mathbb{P}[Y_1 > \sqrt{n}] \\
&= O\left(e^{-\eta\sqrt{n}}\right),
\end{aligned}$$

for some $\eta > 0$, which finishes the proof. ■

Proof of Lemma IV.1: Recall the definition of $\{D_m\}_{m\ge 0}$ in Subsection II-A and denote the packet size of the new arrival at time D_m by L_m for all m. First, we prove the case when $M=2$. For $m > w_1 \triangleq \lceil 1/(\mathbb{P}[L>y]) \rceil$, we consider at time D_m a set of departure points $\{D_{m-w_1}, D_{m-w_1+1}, \cdots, D_m\}$. By the explanation before Lemma IV.1, we know that the system has w_1+1 number of arrivals in $[D_{m-w_1}, D_m]$. Define $\tau_{(2,1)} = \min\{j : L_j > y, j \ge m-w_1\}$ and it is easy to see that there exists y_0 such that for all $y > y_0$,

$$\begin{aligned}
\mathbb{P}[\tau_{(2,1)} < m] &= 1 - \mathbb{P}[L \le y]^{w_1} \\
&\ge 1 - (1-\mathbb{P}[L>y])^{1/\mathbb{P}[L>y]} \\
&> 1 - 2e^{-1},
\end{aligned}$$

implying that the event $\mathcal{E}_2$, a packet of size larger than y arriving to the system in $[D_{m-w_1}, D_{m-1}]$, has a positive probability. Now, since $M=2$, denote by $L_j^{(1)}$ the size of the packet that arrives before D_j but is still in the system observed immediately after D_j. Then, define $\mathcal{D}_2 \triangleq \{L_j^{(1)} > y, \tau_{(2,1)} < j \le m\}$, i.e., after packet $\tau_{(2,1)}$ arrives to the system in $[D_{m-w_1}, D_m]$, the size of the remaining packet in the system observed immediately after departure times is always larger than y. Now, we need to show that $\mathbb{P}[\mathcal{D}_2|\mathcal{E}_2]$ is also positive. To this end, we observe that at each point when a departure occurs the new arrival to the system has a packet size equal in distribution to L, and thus, $\mathbb{P}[\mathcal{D}_2|\mathcal{E}_2]$ is lower bounded

$$\geq \mathbb{E}\left[\prod_{i=\tau_{(2,1)}+1}^{m}\left(\mathbf{1}(L_i > y) + \frac{e^{-\nu L_i}}{e^{-\nu L_i}+e^{-\nu y}}\mathbf{1}(L_i \leq y)\right)\right],$$

where $\mathbf{1}(L_i > y) + e^{-\nu L_i}/(e^{-\nu L_i}+e^{-\nu y})\mathbf{1}(L_i \leq y)$ gives the lower bound for the probability that $\underline{L}(D_i)$ is larger than y at D_i. Since $\{L_i\}$ are i.i.d. random variables, we obtain

$$\mathbb{P}[\mathcal{D}_2|\mathcal{E}_2] \geq \left(1 - \mathbb{E}\left[\frac{e^{-\nu y}}{e^{-\nu L_i}+e^{-\nu y}}\mathbf{1}(L_i \leq y)\right]\right)^{w_1}. \quad (47)$$

Now, it is straightforward to see that

$$\begin{aligned}
&\mathbb{E}\left[\frac{e^{-\nu y}}{e^{-\nu L_i}+e^{-\nu y}}\mathbf{1}(L_i \leq y)\right] = \int_0^y \frac{1}{1+e^{\nu(y-x)}}d\mathbb{P}[L \leq x]\\
&\quad = -\frac{1}{1+e^{\nu(y-x)}}\mathbb{P}[L > x]\Big|_0^y + \nu\int_0^y \frac{\mathbb{P}[L > x]e^{\nu(y-x)}dx}{(1+e^{\nu(y-x)})^2}\\
&\quad = \frac{1}{1+e^{\nu y}} - \frac{1}{2}\mathbb{P}[L > y] + \nu\int_0^{\epsilon y}\frac{\mathbb{P}[L > x]e^{\nu(y-x)}dx}{(1+e^{\nu(y-x)})^2}\\
&\qquad + \nu\int_{\epsilon y}^{y}\frac{\mathbb{P}[L > x]e^{\nu(y-x)}dx}{(1+e^{\nu(y-x)})^2}. \qquad (48)
\end{aligned}$$

Since $\nu > \mu$, in the preceding equality, by choosing $0 < \epsilon < 1 - \mu/\nu$, we obtain

$$\nu\int_0^{\epsilon y}\frac{\mathbb{P}[L > x]e^{\nu(y-x)}dx}{(1+e^{\nu(y-x)})^2} \leq \nu e^{-\nu(1-\epsilon)y} = o(\mathbb{P}[L > y]). \quad (49)$$

Next, observe

$$\begin{aligned}
&\nu\int_{\epsilon y}^{y}\frac{\mathbb{P}[L > x]e^{\nu(y-x)}dx}{(1+e^{\nu(y-x)})^2}\\
&\qquad \leq \nu\int_{\epsilon y}^{y}\mathbb{P}[L > x]e^{-\nu(y-x)}dx\\
&\qquad \leq \nu\mathbb{P}[L > y]\int_{\epsilon y}^{y}\frac{\mathbb{P}[L > x]}{\mathbb{P}[L > y]}e^{-\nu(y-x)}dx,
\end{aligned}$$

which, by recalling condition (35), implies that there exist $0 < \delta < \nu - \mu$ and $y_\delta > 0$ such that for all $y > y_\delta$,

$$\begin{aligned}
&\nu\int_{\epsilon y}^{y}\frac{\mathbb{P}[L > x]e^{\nu(y-x)}dx}{(1+e^{\nu(y-x)})^2}\\
&\qquad \leq \nu\mathbb{P}[L > y]\int_{\epsilon y}^{y}e^{(\mu+\delta)(y-x)}e^{-\nu(y-x)}dx\\
&\qquad = O(\mathbb{P}[L > y]). \qquad (50)
\end{aligned}$$

Substituting (48), (49) and (50) into (47), we obtain, for $p_2 > 0$ and y big enough,

$$\mathbb{P}[\mathcal{D}_2|\mathcal{E}_2] > p_2, \quad (51)$$

which finishes the proof of the lemma for $M = 2$.

Now, we prove the case when $M = 3$. For $m > w_2 = \lceil 2/(\mathbb{P}[L > y])\rceil$, consider a set of time points $\mathcal{W} = \{D_{m-w_2}, D_{m-w_2+1}, \cdots, D_m\}$. Define $\tau_{(3,1)} = \min\{j : L_j > y, j \geq m - w_2\}$ and $\tau_{(3,2)} = \min\{j : L_j > y, j > \tau_{(3,1)}\}$. It is easy to see that there exists y_0 such that for all $y > y_0$,

$$\begin{aligned}
\mathbb{P}[\tau_{(3,2)} < m] &\geq \mathbb{P}\left[\tau_{(3,1)} \leq \frac{w}{2}, \tau_{(3,2)} - \tau_{(3,1)} \leq \frac{w}{2}\right]\\
&\geq \left(1 - \mathbb{P}[L \leq y]^{\frac{w}{2}}\right)\left(1 - \mathbb{P}[L \leq y]^{\frac{w}{2}}\right)\\
&\geq \left(1 - 2e^{-1}\right)^2,
\end{aligned}$$

implying that the event $\mathcal{E}_3$, two packets of size larger than y arriving to the system in $[D_{m-w_2}, D_{m-1}]$, has a positive probability.

Since $M = 3$, we can denote by $L_j^{(1)} \geq L_j^{(2)}$ the order statistics of the sizes of the packets excluding the one just arriving to the system at time D_j. Then, we define an event $\mathcal{D}_3$ by $\mathcal{D}_3 \triangleq \left\{L_j^{(1)} > y, \tau_{(3,1)} < j \leq \tau_{(3,2)}\right\} \bigcup \left\{L_j^{(2)} > y, \tau_{(3,2)} < j \leq m\right\}$, i.e., after packet $\tau_{(3,1)}$ arrives and before packet $\tau_{(3,2)}$ comes to the system in $[D_{m-w_2}, D_{m-2}]$, one of the remaining packet sizes in the system observed immediately after each departure time is always larger than y; after packet $\tau_{(3,2)}$ arrives to the system in $[D_{m-w_2}, D_{m-1}]$, all the remaining packet sizes in the system observed immediately after departure times are always larger than y. Now, we need to show that $\mathbb{P}[\mathcal{D}_3|\mathcal{E}_3]$ is positive. To this end, we observe that $\mathbb{P}[\mathcal{D}_3|\mathcal{E}_3]$ is lower bounded by

$$\begin{aligned}
\mathbb{E}\Bigg[&\prod_{i=\tau_{(3,1)}+1}^{\tau_{(3,2)}}\Bigg(\mathbf{1}\left(\{L_i > y\} \cup \{L_i^{(2)} > y\}\right)\\
&\quad + \frac{e^{-2\nu L_i} + e^{-2\nu L_i^{(2)}}}{e^{-2\nu L_i} + e^{-2\nu y} + e^{-2\nu L_j^{(2)}}}\mathbf{1}\left(L_i \leq y, L_i^{(2)} \leq y\right)\Bigg)\\
&\prod_{i=\tau_{(3,2)+1}}^{w_2}\left(\mathbf{1}(L_i > y) + \frac{e^{-2\nu L_i}}{e^{-2\nu L_i} + 2e^{-2\nu y}}\mathbf{1}(L_i \leq y)\right)\Bigg],
\end{aligned}$$

which, by recalling that $\{L_i\}$ are i.i.d. and noting that $\tau_{(3,2)} - \tau_{(3,1)} \leq w_2$, $w_2 - \tau_{(3,2)} \leq w_2$, implies that $\mathbb{P}[\mathcal{D}_3|\mathcal{E}_3]$ is lower bounded by

$$\begin{aligned}
&\left(1 - \mathbb{E}\left[\frac{e^{-2\nu y}}{e^{-2\nu L_i} + e^{-2\nu y}}\mathbf{1}(L_i \leq y, L_i^{(2)} \leq y)\right]\right)^{w_2}\\
&\quad \times\left(1 - \mathbb{E}\left[\frac{2e^{-2\nu y}}{e^{-2\nu L_i} + 2e^{-2\nu y}}\mathbf{1}(L_i \leq y)\right]\right)^{w_2}\\
&\quad \geq \left(1 - \mathbb{E}\left[\frac{2e^{-2\nu y}}{e^{-2\nu L_i} + e^{-2\nu y}}\mathbf{1}(L_i \leq y)\right]\right)^{2w_2}. \qquad (52)
\end{aligned}$$

Then, by using the same approach as in evaluating (47), we can easily obtain, for $p_3 > 0$,

$$\mathbb{P}[\mathcal{D}_3|\mathcal{E}_3] > p_3, \quad (53)$$

which finishes the proof of the case $M = 3$.

The situation $M > 3$, although notationally complicated, follows easily by induction using the same arguments as in proving $M = 2, 3$. For these reasons we omit the details. ■

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