Signals And Systems
HW#9
Solutions

6.1-3

(c)

$$F(s) = \frac{(s+1)^2}{s^2 - s - 6} = \frac{(s+1)^2}{(s+2)(s-3)}$$

This is an improper fraction with $b_n = b_2 = 1$. Therefore

$$F(s) = 1 + \frac{a}{s+2} + \frac{b}{s-3} = 1 - \frac{0.2}{s+2} + \frac{3.2}{s-3}$$

$$f(t) = \delta(t) + (3.2e^{3t} - 0.2e^{-2t})u(t)$$

(e)

$$F(s) = \frac{2s+1}{(s+1)(s^2+2s+2)} = \frac{-1}{s+1} + \frac{As+B}{s^2+2s+2}$$

Multiply both sides by s and let $s \to \infty$. This yields

$$0 = -1 + A \implies A = 1$$

Setting $s = 0$ on both sides yields

$$\tfrac{1}{2} = -1 + \tfrac{B}{2} \implies B = 3$$

$$F(s) = -\frac{1}{s+1} + \frac{s+3}{s^2+2s+2}$$

In the second fraction, $A = 1$, $B = 3$, $a = 1$, $c = 2$, $b = \sqrt{2-1} = 1$.

$$r = \sqrt{\tfrac{2+9-6}{2-1}} = \sqrt{5} \qquad \theta = \tan^{-1}(\tfrac{-2}{1}) = -63.4^\circ$$

$$f(t) = [-e^{-t} + \sqrt{5}e^{-t}\cos(t - 63.4^\circ)]u(t)$$

(g)

$$F(s) = \frac{1}{(s+1)(s+2)^4} = \frac{1}{s+1} + \frac{k_1}{s+2} + \frac{k_2}{(s+2)^2} + \frac{k_3}{(s+2)^3} - \frac{1}{(s+2)^4}$$

Multiplying both sides by s and let $s \to \infty$. This yields

$$0 = 1 + k_1 \implies k_1 = -1$$

$$\frac{1}{(s+1)(s+2)^4} = \frac{1}{s+1} - \frac{1}{s+2} + \frac{k_2}{(s+2)^2} + \frac{k_3}{(s+2)^3} - \frac{1}{(s+2)^4}$$

Setting $s = 0$ and -3 on both sides yields

$$\tfrac{1}{16} = 1 - \tfrac{1}{2} + \tfrac{k_2}{4} + \tfrac{k_3}{8} - \tfrac{1}{16} \implies 4k_2 + 2k_3 = -6$$
$$-\tfrac{1}{2} = -\tfrac{1}{2} + 1 + k_2 - k_3 - 1 \implies k_2 - k_3 = 0$$

Solving these two equations simultaneously yields $k_2 = k_3 = -1$. Therefore

$$F(s) = \frac{1}{s+1} - \frac{1}{s+2} - \frac{1}{(s+2)^2} - \frac{1}{(s+2)^3} - \frac{1}{(s+2)^4}$$

$$f(t) = [e^{-t} - (1 + t + \frac{t^2}{2} + \frac{t^3}{6})e^{-2t}]u(t)$$

6.1-3

(i)

$$F(s) = \frac{s^3}{(s+1)^2(s^2+2s+5)} = \frac{k}{s+1} - \frac{1/4}{(s+1)^2} + \frac{As+B}{s^2+2s+5}$$

Multiply both sides by s and let $s \to \infty$ to obtain

$$1 = k + A$$

Setting $s = 0$ and 1 yields

$$0 = k - \tfrac{1}{4} + \tfrac{B}{5} \implies 20k + 4B = 5$$
$$\tfrac{1}{32} = \tfrac{k}{2} - \tfrac{1}{16} + \tfrac{A+B}{8} \implies 16k + 4A + 4B = 3$$

Solving these three equations in k, A and B yields $k = \frac{3}{4}$, $A = \frac{1}{4}$ and $B = -\frac{5}{2}$.

$$F(s) = \frac{3/4}{s+1} - \frac{1/4}{(s+1)^2} + \frac{1}{4}\left(\frac{s-10}{s^2+2s+5}\right)$$

For the last fraction in parenthesis, $A = 1$, $B = -10$, $a = 1$, $c = 5$, $b = \sqrt{5-1} = 2$.

$$r = \sqrt{\frac{5+100+20}{5-1}} = 5.59 \qquad \theta = \tan^{-1}\left(\tfrac{11}{4}\right) = 70^\circ$$

Therefore

$$\begin{aligned} f(t) &= [(\tfrac{3}{4} - \tfrac{1}{4}t)e^{-t} + \tfrac{5.59}{4}e^{-t}\cos(2t+70^\circ)]u(t) \\ &= [\tfrac{1}{4}(3-t) + 1.3975\cos(2t+70^\circ)]e^{-t}u(t) \end{aligned}$$

6.2-1

(b)

$$f(t) = e^{-(t-\tau)}u(t-\tau)$$

$$F(s) = \frac{1}{s+1}e^{-s\tau}$$

(c)

$$f(t) = e^{-(t-\tau)}u(t) = e^{\tau}e^{-t}u(t)$$

Therefore $\qquad F(s) = e^{\tau}\dfrac{1}{s+1}$

(d)

$$f(t) = e^{-t}u(t-\tau) = e^{-\tau}e^{-(t-\tau)}u(t-\tau)$$

Observe that $e^{-(t-\tau)}u(t-\tau)$ is $e^{-t}u(t)$ delayed by τ. Therefore

$$F(s) = e^{-\tau}\left(\frac{1}{s+1}\right)e^{-s\tau} = \left(\frac{1}{s+1}\right)e^{-(s+1)\tau}$$

6.2-3

(d)

$$F(s) = \frac{e^{-s} + e^{-2s} + 1}{s^2 + 3s + 2} = (e^{-s} + e^{-2s} + 1)\left[\frac{1}{s^2 + 3s + 2}\right]$$
$$= (e^{-s} + e^{-2s} + 1)\left[\frac{1}{s+1} - \frac{1}{s+2}\right]$$

$$F(s) = (e^{-s} + e^{-2s} + 1)\hat{F}(s)$$

where

$$\hat{F}(s) = \frac{1}{s+1} - \frac{1}{s+2} \quad \text{and} \quad \hat{f}(t) = (e^{-t} - e^{-2t})u(t)$$

Moreover

$$f(t) = \hat{f}(t-1) + \hat{f}(t-2) + \hat{f}(t)$$
$$= [e^{-(t-1)} - e^{-2(t-1)}]u(t-1) + [e^{-(t-2)} - e^{-2(t-2)}]u(t-2) + (e^{-t} - e^{-2t})u(t)$$

6.2-4 (a)

$$g(t) = f(t) + f(t - T_0) + f(t - 2T_0) + \cdots$$

and

$$G(s) = F(s) + F(s)e^{-sT_0} + F(s)e^{-2sT_0} + \cdots$$
$$= F(s)[1 + e^{-sT_0} + e^{-2sT_0} + e^{-3sT_0} + \cdots$$
$$= \frac{F(s)}{1 - e^{-sT_0}} \qquad |e^{-sT_0}| < 1 \text{ or } \operatorname{Re} s > 0$$

(b)

$$f(t) = u(t) - u(t-2) \quad \text{and} \quad F(s) = \frac{1}{s}(1 - e^{-2s})$$
$$G(s) = \frac{F(s)}{1 - e^{-8s}} = \frac{1}{s}\left(\frac{1 - e^{-2s}}{1 - e^{-8s}}\right)$$

6.3-1 (a)

$$(s^2+3s+2)Y(s) = s\left(\frac{1}{s}\right)$$

$$Y(s) = \frac{1}{s^2+3s+2} = \frac{1}{s+1} - \frac{1}{s+2}$$

$$y(t) = (e^{-t} - e^{-2t})u(t)$$

(b)

$$(s^2Y(s) - 2s - 1) + 4(sY(s) - 2) + 4Y(s) = (s+1)\frac{1}{s+1}$$

or

$$(s^2+4s+4)Y(s) = 2s+10$$

and

$$Y(s) = \frac{2s+10}{s^2+4s+4} = \frac{2s+10}{(s+2)^2} = \frac{2}{s+2} + \frac{6}{(s+2)^2}$$

$$y(t) = (2+6t)e^{-2t}u(t)$$

The above answers were obtained using the Laplace Transform method, taking into account initial conditions. On the following page, I compute zero input and zero state responses separately. Zero input responses are obtained using substitution of t=0 into the characteristic equation's solution, while zero state solutions are obtained using the Laplace transform method, but with all initial conditions set to 0.

6.3-1 Finding zero-state (zs) and zero input (zi) responses:

(a) $y(0^-) = \dot{y}(0^-) = 0$ so $\boxed{y_{zi} = 0}$

$\boxed{y_{zs}(t) = y(t) = (e^{-t} - e^{-2t})\,u(t)}$

(b) $y(0^-) = 2$ $\dot{y}(0^-) = 1$ With 0 input,

$(D^2 + 4D + 4)\,y(t) = 0$

Char. eqn has ~~roots~~ double root at $\lambda = -2$:

$2 = y_{zi}(0) = (Ae^{-2t} + Bte^{-2t})\big|_{t=0}$ $y(0^-) = 2 \Rightarrow$ $A=2$

$1 = \dot{y}_{zi}(0) = (-2Ae^{-2t} - 2Bte^{-2t} + Be^{-2t})\big|_{t=0} = \boxed{B - 2A = 1} \Rightarrow B = 5$

So $\boxed{y_{zi}(t) = (2e^{-2t} + 5te^{-2t})\,u(t)}$

For zero state, $(s^2+4s+4)\,Y_{zs}(s) = (s+1)\left(\frac{1}{s+1}\right)$

$Y_{zs}(s) = \frac{1}{s^2+4s+4} = \frac{1}{(s+2)^2} \Rightarrow \boxed{y_{zs}(t) = te^{-2t}u(t)}$

As a check, $y_{zi}(t) + y_{zs}(t) = (2e^{-2t} + 6te^{-2t})\,u(t) = y(t)$ gotten above using Laplace transform method.

(6.3-5) (b) $\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} - 11\frac{dy}{dt} + 6y(t) = 3\frac{d^2f}{dt^2} + 7\frac{df}{dt} + 5f(t)$

So $\boxed{H(s) = \frac{3s^2+7s+5}{s^3+6s^2-11s+6}} = \frac{1.7714}{s+7.56} + \frac{1.2286s+.4754}{s^2-1.56s+.7937}$

Using pair 5 of Table 6.1 for 1st term and
Using pair 10c of Table 6.1 for the 2nd term,

get $\boxed{h(t) = \mathcal{L}^{-1}(H(s)) = \left[1.7714\,e^{-7.56t} + 3.5504e^{.78t}\cos(.4304t - 1.207)\right]u(t)}$

(c) $\frac{d^4y}{dt^4} + 4\frac{dy}{dt} = 3\frac{df}{dt} + 2f(t) \Rightarrow \boxed{H(s) = \frac{3s+2}{s^4+4s}}$

$H(s) = \frac{.2302}{s+1.5874} + \frac{.5}{s} + \frac{-.73s+1.52}{s^2-1.5874s+2.5198}$ using 2, 5, 10c from Table 6.1

$\boxed{h(t) = \mathcal{L}^{-1}(H(s)) = \left[.2302e^{-1.5874t} + .5 + 1.0028\,e^{.7937t}\cos(1.3747t - 2.4716)\right]u(t)}$