Signals And Systems
HW#8
Solutions

4.4-3

$$F_1(\omega) = \text{sinc}(\tfrac{w}{20000}) \quad \text{and} \quad F_2(\omega) = 1$$

Figure S4.4-3 shows $F_1(\omega)$, $F_2(\omega)$, $H_1(\omega)$ and $H_2(\omega)$. Now

$$Y_1(\omega) = F_1(\omega)H_1(\omega)$$
$$Y_2(\omega) = F_2(\omega)H_2(\omega)$$

The spectra $Y_1(\omega)$ and $Y_2(\omega)$ are also shown in Fig. S4.4-3. Because $y(t) = y_1(t)y_2(t)$, the frequency convolution property yields $Y(\omega) = Y_1(\omega) * Y_2(\omega)$. From the width property of convolution, it follows that the bandwidth of $Y(\omega)$ is the sum of bandwidths of $Y_1(\omega)$ and $Y_2(\omega)$. Because the bandwidths of $Y_1(\omega)$ and $Y_2(\omega)$ are 10 kHz, 5 kHz, respectively, the bandwidth of $Y(\omega)$ is 15 kHz.
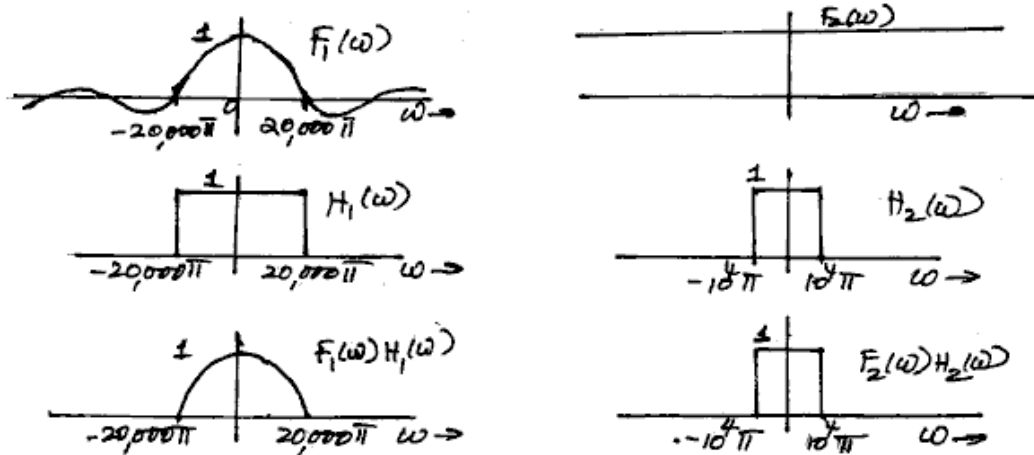

Fig. S4.4-3

4.5-1

$$H(\omega) = e^{-k\omega^2} e^{-j\omega t_0}$$

Using pair 22 (Table 4.1) and time-shifting property, we get

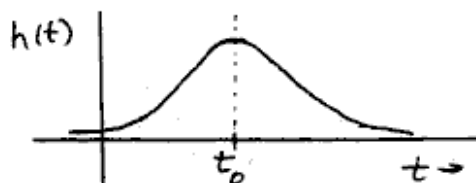$$h(t) = \frac{1}{\sqrt{4\pi k}} e^{-(t-t_0)^2/4k}$$

Figure S4.5-1

This is noncausal. Hence the filter is unrealizable. Also

$$\int_{-\infty}^{\infty} \frac{|\ln|H(\omega)||}{\omega^2 + 1} \, d\omega = \int_{-\infty}^{\infty} \frac{k\omega^2}{\omega^2 + 1} \, d\omega = \infty$$

Hence the filter is noncausal and therefore unrealizable. Since $h(t)$ is a Gaussian function delayed by t_0, it looks as shown in the adjacent figure. Choosing $t_0 = 3\sqrt{2k}$, $h(0) = e^{-4.5} = 0.011$ or 1.1% of its peak value. Hence $t_0 = 3\sqrt{2k}$ is a reasonable choice to make the filter approximately realizable.

4.6-2 Consider a signal

$$f(t) = \text{sinc}(kt) \quad \text{and} \quad F(\omega) = \frac{\pi}{k}\text{rect}(\frac{\omega}{2k})$$

$$E_f = \int_{-\infty}^{\infty} \text{sinc}^2(kt) \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\pi^2}{k^2} \left[\text{rect}\left(\frac{\omega}{2k}\right)\right]^2 \, d\omega$$

$$= \frac{\pi}{2k^2} \int_{-k}^{k} d\omega = \frac{\pi}{k}$$

5.1-2 **(a)**

$$\text{sinc}^2(100\pi t) \Longleftrightarrow 0.01\Delta(\tfrac{\omega}{400\pi})$$

The bandwidth of this signal is $200\,\pi$ rad/s or 100 Hz. The Nyquist rate is 200 Hz (samples/sec)

(b) The Nyquist rate is 200 Hz, the same as in **(a)**, because multiplication of a signal by a constant does not change its bandwidth.

(c)

$$\text{sinc}(100\pi t) + 3\,\text{sinc}^2(60\pi t) \Longleftrightarrow 0.01\,\text{rect}(\tfrac{\omega}{200\pi}) + \tfrac{1}{20}\,\Delta(\tfrac{\omega}{240\pi})$$

The bandwidth of $\text{rect}(\tfrac{\omega}{200\pi})$ is 50 Hz and that of $\Delta(\tfrac{\omega}{240\pi})$ is 60 Hz. The bandwidth of the sum is the higher of the two, that is, 60 Hz. The Nyquist sampling rate is 120 Hz.

(d)

$$\text{sinc}(50\pi t) \Longleftrightarrow 0.02\,\text{rect}(\tfrac{\omega}{100\pi})$$
$$\text{sinc}(100\pi t) \Longleftrightarrow 0.01\,\text{rect}(\tfrac{\omega}{200\pi})$$

The two signals have bandwidths 25 Hz and 50 Hz respectively. The spectrum of the product of two signals is $1/2\pi$ times the convolution of their spectra. From width property of the convolution, the width of the convoluted signal is the sum of the widths of the signals convolved. Therefore, the bandwidth of $\text{sinc}(50\pi t)\text{sinc}(100\pi t)$ is $25 + 50 = 75$ Hz. The Nyquist rate is 150 Hz.

5.1-3

The spectrum of $f(t) = \text{sinc}(200\pi t)$ is $F(\omega) = 0.005\,\text{rect}(\frac{\omega}{400\pi})$. The bandwidth of this signal is 100 Hz (200π rad/s). Consequently, the Nyquist rate is 200 Hz, that is, we must sample the signal at a rate no less than 200 samples/second.

Recall that the sampled signal spectrum consists of $(1/T)F(\omega) = \frac{0.005}{T}\,\text{rect}(\frac{\omega}{400\pi})$ repeating periodically with period equal to the sampling frequency $\mathcal{F}_s$ Hz. We present this information in the following Table for three sampling rates: $\mathcal{F}_s = 150$ Hz (undersampling), 200 Hz (Nyquist rate), and 300 Hz (oversampling).

sampling frequency $\mathcal{F}_s$	sampling interval T	$\frac{1}{T}F(\omega)$	comments
150 Hz	0.006667	$0.75\,\text{rect}\left(\frac{\omega}{400\pi}\right)$	Undersampling
200 Hz	0.005	$\text{rect}\left(\frac{\omega}{400\pi}\right)$	Nyquist Rate
300 Hz	0.003334	$1.5\,\text{rect}\left(\frac{\omega}{400\pi}\right)$	Oversampling

The spectra of $\overline{F}(\omega)$ for the three cases are shown in Fig. S5.1-3. In the first case, we cannot recover $f(t)$ from the sampled signal because of overlapping cycles, which makes it impossible to identify $F(\omega)$ from the corresponding $\overline{F}(\omega)$. In the second and the third case, the repeating spectra do not overlap, and it is possible to recover $F(\omega)$ from $\overline{F}(\omega)$ using a lowpass filter of bandwidth 100 Hz. In the last case the spectrum $\overline{F}(\omega) = 0$ over the band between 100 and 200 Hz. Hence to recover $F(\omega)$, we may use a practical lowpass filter with gradual cutoff between 100 and 200 Hz. The output in the second case is $f(t)$, and in the third case is $1.5f(t)$. The output spectra in the three cases are shown in Fig. S5.1-3.
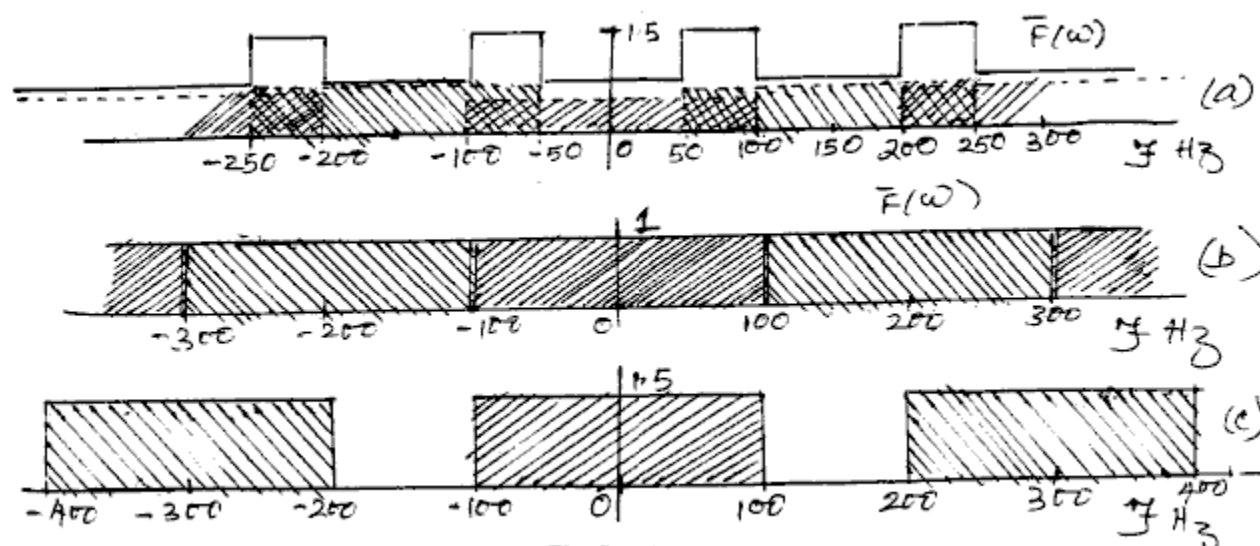

Fig. S5.1-3

5.1-6 The signal $f(t) = \text{sinc}(200\pi t)$ is sampled by a rectangular pulse sequence $p_T(t)$ whose period is 4 ms so that the fundamental frequency (which is also the sampling frequency) is 250 Hz. Hence, $\omega_s = 500\pi$. The Fourier series for $p_T(t)$ is given by

$$p_T(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos n\omega_s t$$

Use of Eqs. (3.66) yields $C_0 = \frac{1}{5}$, $\quad C_n = \frac{2}{n\pi} \sin\left(\frac{n\pi}{5}\right)$, that is,

$$C_0 = 0.2, \quad C_1 = 0.374, \quad C_2 = 0.303, \quad C_3 = 0.202, \quad C_4 = 0.093, \quad C_5 = 0, \cdots$$

Consequently

$$\overline{f}(t) = f(t)p_T(t) = 0.2\, f(t) + 0.374 f(t) \cos 500\pi t + 0.303\, f(t) \cos 1000\pi t + 0.202\, f(t) \cos 1500\pi t + \cdots$$

and

$$\overline{F}(\omega) = 0.2\, F(\omega) + 0.187\, [F(\omega - 500\pi) + F(\omega + 500\pi)]$$
$$+ 0.151\, [F(\omega - 1000\pi) + F(\omega + 1000\pi)]$$
$$+ 0.101\, [F(\omega - 1500\pi) + F(\omega + 1500\pi)] + \cdots$$

In the present case $F(\omega) = 0.005\, \text{rect}(\frac{\omega}{400\pi})$. The spectrum $\overline{F}(\omega)$ is shown in Fig. S5.1-6. Observe that the spectrum consists of $F(\omega)$ repeating periodically at the interval of 500π rad/s (250 Hz). Hence, there is no overlap between cycles, and $F(\omega)$ can be recovered by using an ideal lowpass filter of bandwidth 100 Hz. An ideal lowpass filter of unit gain (and bandwidth 100 Hz) will allow the first term on the he right-side of the above equation to pass fully and suppress all the other terms. Hence the output $y(t)$ is

$$y(t) = 0.2\, f(t)$$

Because the spectrum $\overline{F}(\omega)$ has a zero value in the band from 100 to 150 Hz, we can use an ideal lowpass filter of bandwidth B Hz where $100 < B < 150$. But if $B > 150$ Hz, the filter will pick up the unwanted spectral components from the next cycle, and the output will be distorted.
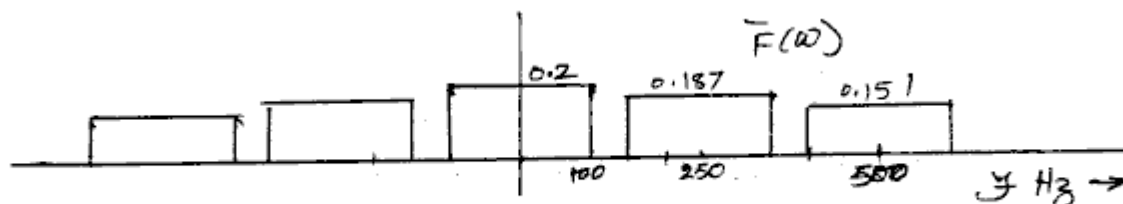

Figure S5.1-6