# Homework #5 Solutions
# Signals and Systems, Columbia University

**4.1-4 (b)**

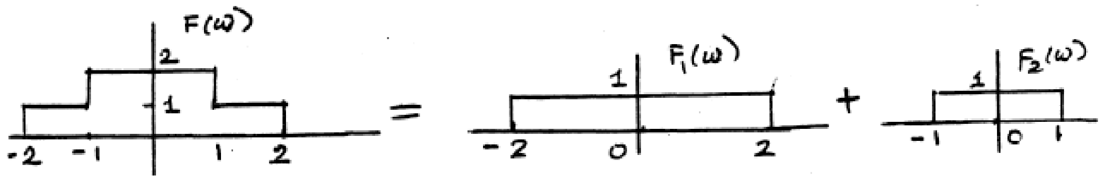
$f(t) = e^{at} \cdot [u(t) - u(t-T)]$

Using Eq. (4.8a),

$$F(w) = \int_0^T e^{at} e^{-jwt} dt = \int_0^T e^{(a-jw)t} dt = \frac{e^{(a-jw)T} - 1}{a - jw}$$

**4.1-6 (b)**

Fig P4.1.6 can be expressed the sum of two functions, $F_1(w)$ and $F_2(w)$




Using Eq.(4.8.b),

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(w) e^{jwt} dw = \frac{1}{2\pi} \int_{-\infty}^{\infty} [F_1(w) + F_2(w)] e^{jwt} dw$$

$$= \frac{1}{2\pi} [\int_{-2}^{2} 1 \cdot e^{jwt} dw + \int_{-1}^{1} 1 \cdot e^{jwt} dw]$$

$$= \frac{\sin 2t + \sin t}{\pi t}$$

**<u>Additional questions</u>**

**Note1:** this question assumes that $x(t) = \sum_{-\infty}^{\infty} X_n e^{j \cdot n \cdot w_0 t}$ ----------------(1)

or, shortly, $x(t) \leftrightarrow X_n$, and the fundamental frequency be $w_0$

**Note2**: Let Xn be Fourier series coefficients of x(t).

1. <u>Linearity</u>

   if $x(t) \leftrightarrow X_n$ and $y(t) \leftrightarrow Y_n$, then
   $ax(t) + by(t) \leftrightarrow aX_n + bY_n$

2. <u>Time-shifting</u>

   $$x(t - t_0) \leftrightarrow X_n e^{-jnw_0 t_0}$$

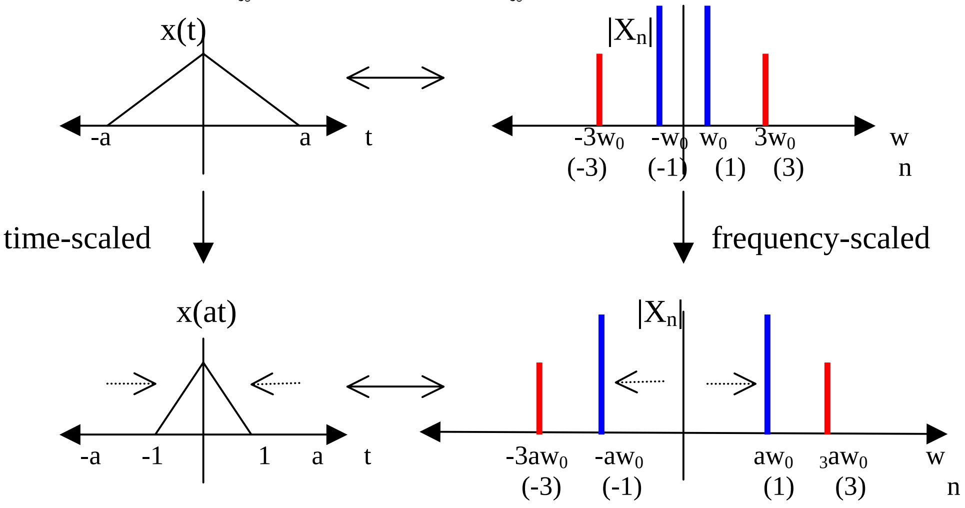
3. <u>Time-reversal</u>

   $$x(-t) \leftrightarrow X_{-n}$$

4. <u>Time-scaling</u>

If x(t) is scaled by a factor a, fourier coefficients of x(at) is calculated like following. For convenience, assume that a > 1 and integer

$$x(at) = \sum_{-\infty}^{\infty} X_n e^{j\cdot n\cdot w_0 at} = \sum_{-\infty}^{\infty} X_n e^{janw_0 t}$$




so, x(at) <-> Xn, but its spectrum is scaled by a factor a.

**A1)** Let x(t) be a periodic signal with Fourier series coefficients Xn. Find the Fourier series expansion of the following signals:

(a) $x(t-t_0) + x(t+t_0)$

Sol)

$$x(t-t_0) = \sum_{-\infty}^{\infty} X_n e^{j\cdot n\cdot w_0(t-t_0)} = \sum_{-\infty}^{\infty} e^{-jnw_0t_0} X_n e^{j\cdot n\cdot w_0 t}$$

or, $x(t-t_0) \leftrightarrow e^{-jnw_0t_0} X_n$ : time shifting property

Similarly, $x(t+t_0) \leftrightarrow e^{jnw_0t_0} X_n$ .

So, $x(t-t_0) + x(t+t_0) \leftrightarrow [e^{-jnw_0t_0} + e^{jnw_0t_0}] \cdot X_n$

$$\leftrightarrow 2\cos(nw_0t_0) \cdot X_n$$

(b) Real { x(t) }

Sol) Real{x{t}} = $[x(t)+x^*(t)]/2$, where $x^*(t)$ is the complex conjugate of x(t)

Using Eq. (1) followed by substituting n with –n,

$$x^*(t) = [\sum_{-\infty}^{\infty} X_n e^{j\cdot n\cdot w_0 t}]^* = \sum_{-\infty}^{\infty} X^*{}_n e^{-j\cdot n\cdot w_0 t} = \sum_{-\infty}^{\infty} X^*{}_{-n} e^{j\cdot n\cdot w_0 t}$$

-----------(2)

so, $$\frac{x(t) + x^*(t)}{2} = \sum_{-\infty}^{\infty} \frac{1}{2}[X_n + X^*{}_{-n}] e^{j\cdot n\cdot w_0 t}$$

or $$\frac{x(t) + x^*(t)}{2} \leftrightarrow= \frac{1}{2}[X_n + X^*{}_{-n}]$$

(c) Imag { x(t) }

Sol) Imag{x{t}} = $[x(t)-x^*(t)]/2j$

Using (b),

$$\frac{x(t) - x^*(t)}{2j} = \sum_{-\infty}^{\infty} \frac{1}{2j}[X_n - X^*{}_{-n}] e^{j\cdot n\cdot w_0 t}$$

or $$\frac{x(t) - x^*(t)}{2j} \leftrightarrow= \frac{1}{2j}[X_n - X^*{}_{-n}]$$

(d) x( -2 t +1 )

Sol1)

Using Eq. (1),

$$x(-2t+1) = \sum_{-\infty}^{\infty} X_n e^{j\cdot n\cdot w_0(-2t+1)} = \sum_{-\infty}^{\infty} X_n e^{jnw_0} e^{-j\cdot 2n\cdot w_0 t}$$

$$= \sum_{-\infty}^{\infty} X_{-n} e^{-jnw_0} e^{j\cdot 2n\cdot w_0 t}$$

so, $x(-2t+1) \leftrightarrow X_{-n} e^{-jnw_0}$ and its spectrum is expanded by a factor 2.

Sol2)

$x(t) \leftrightarrow X_n$

$\rightarrow x(t+1) \leftrightarrow X_n e^{jnw_0}$ by time-shifting property

$\rightarrow x(-t+1) \leftrightarrow X_{-n} e^{-jnw_0}$ by time-reversal property

$\rightarrow x(-2t+1) \leftrightarrow X_{-n} e^{-jnw_0}$ by time-scaling and its spectrum is scaled by a factor 2

**A2)**

If x(t) is real and periodic

**Note:** x(t) is real => $x(t) = x^*(t)$ , by using (1) and (2)

$$\Leftrightarrow \sum_{-\infty}^{\infty} X_n e^{j\cdot n\cdot w_0 t} = \sum_{-\infty}^{\infty} X^*_{-n} e^{j\cdot n\cdot w_0 t}$$ , by comparing coeff's.

$$\Leftrightarrow X_n = X^*_{-n} \text{ ------------(3)}$$

(a) If x(t) is odd, show Xn is imaginary and odd.

Sol) x (t) is odd => x(t) = -x(-t)

$$\Leftrightarrow \sum_{-\infty}^{\infty} X_n e^{jnw_0 t} = -\sum_{-\infty}^{\infty} X_n e^{-jnw_0 t} = \sum_{-\infty}^{\infty} -X_{-n} e^{jnw_0 t}$$

By comparing coefficients

=> $X_n = -X_{-n}$ -----------------(4)

so, $X_n$ is odd

Combining (3) and (4),

$$X_n = -X_{-n} = -X^*_n$$

$\Rightarrow -X_n = X^*_n$ => $X_n$ is imaginary (Q.E.D)

(b) Find the Fourier series coefficients of the odd components of x(t)

$x_o(t) = [x(t) – x(-t)]/2$

Here, $x(-t) = \sum_{-\infty}^{\infty} X_n e^{-jnw_0 t} = \sum_{-\infty}^{\infty} X_{-n} e^{jnw_0 t}$ or, $x(-t) \leftrightarrow X_{-n}$

Using (3),

$$\frac{x(t) - x(-t)}{2} \leftrightarrow \frac{X_n - X_{-n}}{2} = \frac{X_n - X^*_n}{2} = j\,\mathrm{Im}\{X_n\}$$