# Signals and Systems
# Homework 4
# Solutions

**3.4-1** Here $T_0 = 2$, so that $\omega_0 = 2\pi/2 = \pi$, and

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi t + b_n \sin n\pi t \qquad -1 \le t \le 1$$

where

$$a_0 = \frac{1}{2}\int_{-1}^{1} t^2\,dt = \frac{1}{3}, \qquad a_n = \frac{2}{2}\int_{-1}^{1} t^2 \cos n\pi t\,dt = \frac{4(-1)^n}{\pi^2 n^2}, \qquad b_n = \frac{2}{2}\int_{-1}^{1} t^2 \sin n\pi t\,dt = 0$$

Therefore

$$f(t) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi t \qquad -1 \le t \le 1$$

Figure S3.4-1 shows $f(t) = t^2$ for all $t$ and the corresponding Fourier series representing $f(t)$ over $(-1, 1)$.

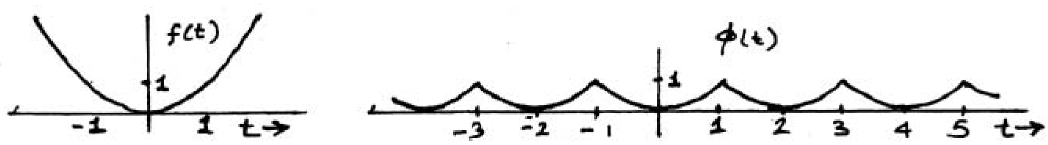



**Fig. S3.4-1**

**3.4-3 (a)** $T_0 = 4$, $\omega_0 = \frac{2\pi}{T_0} = \frac{\pi}{2}$. Because of even symmetry, all sine terms are zero.

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{2}t\right)$$

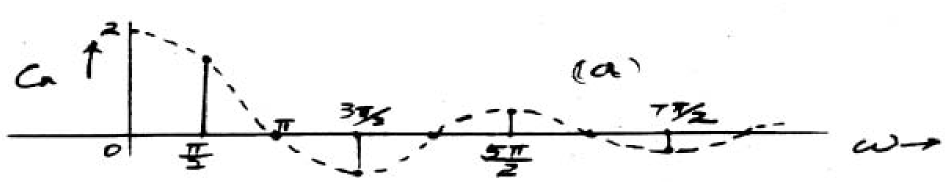
$$a_0 = 0 \text{ (by inspection)}$$

$$a_n = \frac{4}{4}\left[\int_0^1 \cos\left(\frac{n\pi}{2}t\right)\,dt - \int_1^2 \cos\left(\frac{n\pi}{2}t\right)\,dt\right] = \frac{4}{n\pi}\sin\frac{n\pi}{2}$$

Therefore, the Fourier series for $f(t)$ is

$$f(t) = \frac{4}{\pi}\left(\cos\frac{\pi t}{2} - \frac{1}{3}\cos\frac{3\pi t}{2} + \frac{1}{5}\cos\frac{5\pi t}{2} - \frac{1}{7}\cos\frac{7\pi t}{2} + \cdots\right)$$

Here $b_n = 0$, and we allow $C_n$ to take negative values. Figure S3.4-3a shows the plot of $C_n$.

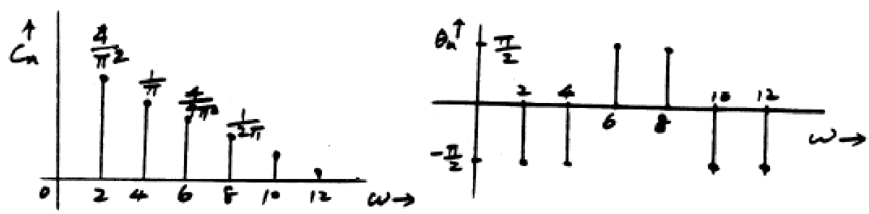


**(d)** $T_0 = \pi$, $\omega_0 = 2$ and $f(t) = \frac{4}{\pi}t$.
$a_0 = 0$ (by inspection).
$a_n = 0$ $(n > 0)$ because of odd symmetry.

$$b_n = \frac{4}{\pi}\int_0^{\pi/4} \frac{4}{\pi} t \sin 2nt \, dt = \frac{2}{\pi n}\left(\frac{2}{\pi n}\sin\frac{\pi n}{2} - \cos\frac{\pi n}{2}\right)$$

$$\begin{aligned} f(t) &= \frac{4}{\pi^2}\sin 2t + \frac{1}{\pi}\sin 4t - \frac{4}{9\pi^2}\sin 6t - \frac{1}{2\pi}\sin 8t + \cdots \\ &= \frac{4}{\pi^2}\cos\left(2t - \frac{\pi}{2}\right) + \frac{1}{\pi}\cos\left(4t - \frac{\pi}{2}\right) + \frac{4}{9\pi^2}\cos\left(6t + \frac{\pi}{2}\right) + \frac{1}{\pi}\cos\left(8t + \frac{\pi}{2}\right) + \cdots \end{aligned}$$

Figure S3.4-3d shows the plot of $C_n$ and $\theta_n$.



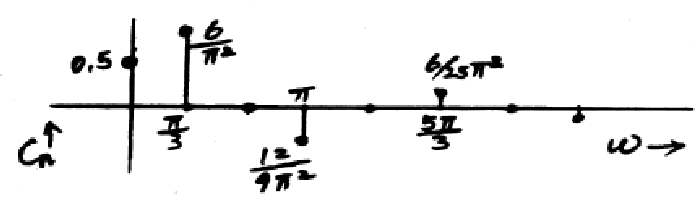

(d)

**(f)** $T_0 = 6$, $\omega_0 = \pi/3$, $a_0 = 0.5$ (by inspection). Even symmetry; $b_n = 0$.

$$\begin{aligned} a_n &= \frac{4}{6}\int_0^3 f(t)\cos\frac{n\pi}{3}\,dt \\ &= \frac{2}{3}\left[\int_0^1 \cos\frac{n\pi}{3}\,dt + \int_1^2 (2-t)\cos\frac{n\pi}{3}t\,dt\right] \\ &= \frac{6}{\pi^2 n^2}\left[\cos\frac{n\pi}{3} - \cos\frac{2n\pi}{3}\right] \end{aligned}$$

$$f(t) = 0.5 + \frac{6}{\pi^2}\left(\cos\frac{\pi}{3}t - \frac{2}{9}\cos\pi t + \frac{1}{25}\cos\frac{5\pi}{3}t + \frac{1}{49}\cos\frac{7\pi}{3}t + \cdots\right)$$

Observe that even harmonics vanish. The reason is that if the dc (0.5) is subtracted from $f(t)$, the resulting function has half-wave symmetry. (See Prob. 3.4-7). Figure S3.4-3f shows the plot of $C_n$.




(f)

**Fig. S3.4-3**

**3.4-8** (a) Here, we need only cosine terms and $\omega_0 = \frac{\pi}{2}$. Hence, we must construct a pulse such that it is an even function of $t$, has a value $t$ over the interval $0 \le t \le 1$, and repeats every 4 seconds as shown in Fig. S3.4-8a. We selected the pulse width $W = 2$ seconds. But it can be anywhere from 2 to 4, and still satisfy these conditions. Each value of $W$ results in different series. Yet all of them converge to $t$ over 0 to 1, and satisfy the other requirements. Clearly, there are infinite number of Fourier series that will satisfy the given requirements. The present choice yields

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{2}\right) t$$

By inspection, we find $a_0 = 1/4$. Because of symmetry $b_n = 0$ and

$$a_n = \frac{4}{4}\int_0^1 t\cos\frac{n\pi}{2}t\,dt = \frac{4}{n^2\pi^2}\left[\cos\left(\frac{n\pi}{2}\right) + \frac{n\pi}{2}\sin\left(\frac{n\pi}{2}\right) - 1\right]$$

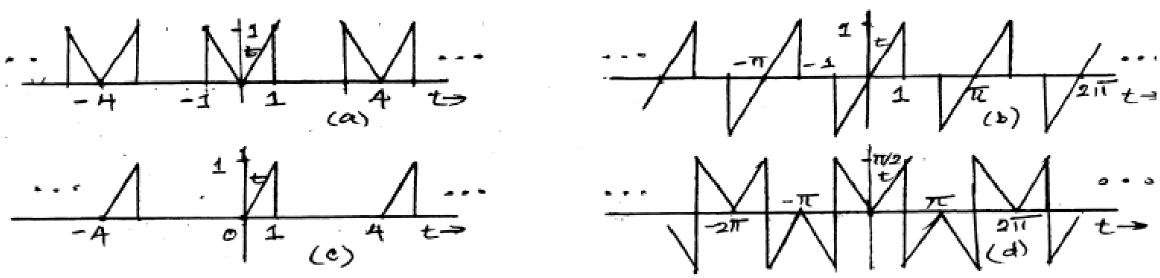
(c) Here, we need both sine and cosine terms and $\omega_0 = \frac{\pi}{2}$. Hence, we must construct a pulse such that it has no symmetry of any kind, has a value $t$ over the interval $0 \le t \le 1$, and repeats every 4 seconds as shown in Fig. S3.4-8c. As usual, the pulse width can be have any value in the range 1 to 4.

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{2}\right) t + b_n \sin\left(\frac{n\pi}{2}\right) t$$

By inspection, $a_0 = 1/8$ and

$$a_n = \frac{2}{4}\int_0^1 t\cos\frac{n\pi}{2}t\,dt = \frac{2}{n^2\pi^2}\left[\cos\left(\frac{n\pi}{2}\right) + \frac{n\pi}{2}\sin\left(\frac{n\pi}{2}\right) - 1\right]$$

$$b_n = \frac{2}{4}\int_0^1 t\sin\frac{n\pi}{2}t\,dt = \frac{2}{n^2\pi^2}\left[\sin\left(\frac{n\pi}{2}\right) - \frac{n\pi}{2}\cos\left(\frac{n\pi}{2}\right)\right]$$




**3.4-9**

| | e | f | h |
|---|---|---|---|
| periodic | no | yes | yes |
| $\omega_0$ | -- | 1/70 | 1 |
| period | -- | 140π | 2π |

**3.5.1**

**(c)**

$$f(t) = D_0 + \sum_{n=-\infty}^{\infty} D_n e^{jnt}, \quad \text{where, by inspection} \qquad D_0 = 0.5$$

$$D_n = \frac{1}{2\pi}\int_0^{2\pi} \frac{t}{2\pi} e^{-jnt}\,dt = \frac{j}{2\pi n}, \quad \text{so that} \quad |D_n| = \frac{1}{2\pi n}, \quad \text{and} \quad \angle D_n = \begin{cases} \frac{\pi}{2} & n > 0 \\ \frac{-\pi}{2} & n < 0 \end{cases}$$

(c)

**3.5-2** In compact trigonometric form, all terms are of cosine form and amplitudes are positive. We can express $f(t)$ as

$$\begin{aligned} f(t) &= 3 + 2\cos\left(2t - \frac{\pi}{6}\right) + \cos\left(3t - \frac{\pi}{2}\right) + \frac{1}{2}\cos\left(5t + \frac{\pi}{3} - \pi\right) \\ &= 3 + 2\cos\left(2t - \frac{\pi}{6}\right) + \cos\left(3t - \frac{\pi}{2}\right) + \frac{1}{2}\cos\left(5t - \frac{2\pi}{3}\right) \end{aligned}$$

From this expression we sketch the trigonometric Fourier spectra as shown in Fig. S3.5-2a. By inspection of these spectra, we sketch the exponential Fourier spectra shown in Fig. S3.5-2b. From these exponential spectra, we can now write the exponential Fourier series as

$$f(t) = 3 + e^{j(2t-\frac{\pi}{6})} + e^{-j(2t-\frac{\pi}{6})} + \frac{1}{2}e^{j(3t-\frac{\pi}{2})} + \frac{1}{2}e^{-j(3t-\frac{\pi}{2})} + \frac{1}{4}e^{j(5t-\frac{2\pi}{3})} + \frac{1}{4}e^{-j(5t-\frac{2\pi}{3})}$$

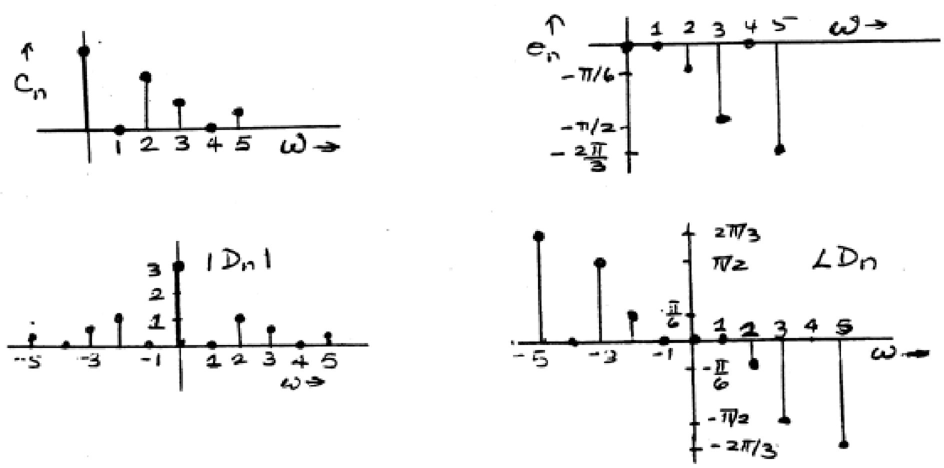


Fig. S3.5-2