

Signals and Systems Homework #1

September 17, 2007

1.1

In each case, the cartesian form is given as $a+jb$. The polar form can be written as $re^{j\theta}$, where $r = \sqrt{a^2 + b^2}$ and $\theta = \arctan \frac{b}{a}$. If θ is in the 2nd or 3rd quadrants, then you need to add or subtract 180 degrees (or π radians).

a

$$\begin{aligned}12 + j4 &= \sqrt{160}e^{j0.3218} \\r &= \sqrt{7^2 + 4^2} = \sqrt{160} \\ \theta &= \arctan \frac{4}{12} = 0.3218 \text{ rad}\end{aligned}$$

b

$$\begin{aligned}3 - j8 &= \sqrt{73}e^{-j1.2120} \\r &= \sqrt{3^2 + 8^2} = \sqrt{73} \\ \theta &= \arctan \frac{-8}{3} = -1.2120 \text{ rad}\end{aligned}$$

c

$$\begin{aligned}-46 + j100 &= \sqrt{12116}e^{j2.0019} \\r &= \sqrt{(-46)^2 + 100^2} = \sqrt{12116} \\ \theta &= \arctan \frac{100}{-46} + \pi = 2.0019 \text{ rad}\end{aligned}$$

d

$$\begin{aligned}-2 - j21 &= \sqrt{445}e^{j4.6174} \\r &= \sqrt{(-2)^2 + (-21)^2} = \sqrt{445} \\ \theta &= \arctan \frac{-2}{-21} + \pi = 4.6174 \text{ rad}\end{aligned}$$

1.2

In each case, the polar form is given as $re^{j\theta}$. The cartesian form is simply $a + jb$, where $a = r \cos \theta$ and $b = r \sin \theta$.

a

$$\begin{aligned} e^{j\frac{\pi}{2}} &= j \\ a &= 1 \cos \frac{\pi}{2} = 0 \\ b &= 1 \sin \frac{\pi}{2} = 1 \end{aligned}$$

b

$$\begin{aligned} 5e^{j\frac{2\pi}{3}} &= -2.5 + j4.3301 \\ a &= 5 \cos \frac{2\pi}{3} = -2.5 \\ b &= 5 \sin \frac{2\pi}{3} = 4.3301 \end{aligned}$$

c

$$\begin{aligned} 11e^{j\frac{-2\pi}{5}} &= 3.399 - j10.4616 \\ a &= 11 \cos \frac{-2\pi}{5} = 3.399 \\ b &= 11 \sin \frac{-2\pi}{5} = 10.4616 \end{aligned}$$

d

$$\begin{aligned} 6.2e^{j3\pi} &= -6.2 \\ a &= 6.2 \cos 3\pi = -6.2 \\ b &= 6.2 \sin 3\pi = 0 \end{aligned}$$

e

$$\begin{aligned} 5e^{j\frac{4\pi}{9}} &= .8682 + j4.9240 \\ a &= 5 \cos \frac{4\pi}{9} = .8682 \\ b &= 5 \sin \frac{4\pi}{9} = 4.9240 \end{aligned}$$

f

$$\begin{aligned} e^{j\frac{\pi}{6}} &= .866 + j0.5 \\ a &= 1 \cos \frac{\pi}{6} = .866 \\ b &= 1 \sin \frac{\pi}{6} = 0.5 \end{aligned}$$

1.3

$$\begin{aligned} z_1 &= -3 - j5 \\ z_2 &= 5 - j7 \end{aligned}$$

First, calculate polar form for z_1 and z_2 :

$$\begin{aligned} z_1 &= -3 - j5 = \sqrt{34}e^{j4.172} \\ r &= \sqrt{-3^2 + -5^2} = \sqrt{34} \\ \theta &= \arctan \frac{-5}{-3} + \pi = 4.172 \text{ rad} \end{aligned}$$

$$\begin{aligned} z_2 &= 5 - j7 = \sqrt{74}e^{-j0.9505} \\ r &= \sqrt{5^2 + -7^2} = \sqrt{74} \\ \theta &= \arctan \frac{-7}{5} + \pi = -.9505 \text{ rad} \end{aligned}$$

Now, calculating $z_1 z_2$ and $\frac{z_1}{z_2}$ in polar form is straight-forward:

$$\begin{aligned} z_1 z_2 &= \sqrt{34}e^{j4.172} \sqrt{74}e^{-j0.9505} \\ &= \sqrt{34 * 74}e^{j(4.172 - 0.9505)} \\ &= \sqrt{2516}e^{j(3.2215)} \end{aligned}$$

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{\sqrt{34}e^{j4.172}}{\sqrt{74}e^{-j0.9505}} \\ &= \sqrt{\frac{34}{74}}e^{j(4.172 - (-0.9505))} \\ &= \sqrt{\frac{34}{74}}e^{j(5.1225)} \end{aligned}$$

Convert polar answers back into cartesian form:

$$\begin{aligned} z_1 z_2 &= \sqrt{2516}e^{j(3.2215)} = -50 - j4 \\ a &= \sqrt{2516} \cos 3.2215 = -50 \\ b &= \sqrt{2516} \sin 3.2215 = -4 \end{aligned}$$

$$\begin{aligned} z_1 z_2 &= \sqrt{\frac{34}{74}}e^{j(5.1225)} = .2703 - j0.6216 \\ a &= \sqrt{\frac{34}{74}} \cos 5.1225 = .2703 \\ b &= \sqrt{\frac{34}{74}} \sin 5.1225 = -.6216 \end{aligned}$$

1.4

$$\begin{aligned}z_1 &= e^{j\frac{\pi}{7}} \\z_2 &= 5e^{-j\frac{3\pi}{5}}\end{aligned}$$

Cartesian form:

$$\begin{aligned}z_1 &= e^{j\frac{\pi}{7}} = .9010 + j.4339 \\a &= 1 \cos \frac{\pi}{7} = .9010 \\b &= 1 \sin \frac{\pi}{7} = .4339\end{aligned}$$

$$\begin{aligned}z_2 &= 5e^{-j\frac{3\pi}{5}} = -1.5451 - j4.7553 \\a &= 5 \cos \frac{-3\pi}{5} = -1.5451 \\b &= 5 \sin \frac{-3\pi}{5} = -4.7553\end{aligned}$$

a

$$\begin{aligned}z_2 - 3z_1 &= -1.5451j4.7553 - 3(.9010 + j.4339) \\&= (-1.5451 - 3(.9010)) + j(-4.7553 - 3(.4339)) \\&= -4.281 - j6.057\end{aligned}$$

b

$$\frac{1}{z_2} = \frac{1}{5e^{-j\frac{3\pi}{5}}} = \frac{1}{5}e^{j\frac{3\pi}{5}}$$

c

$$\begin{aligned}\left(\frac{z_2}{z_1}\right)^2 &= \left(\frac{5e^{-j\frac{3\pi}{5}}}{e^{j\frac{\pi}{7}}}\right)^2 \\&= (5e^{j(-\frac{-3\pi}{5}-\frac{\pi}{7})})^2 \\&= 25e^{-j1.4857}\end{aligned}$$

d

$$z_1^{-\frac{1}{2}} = \left(e^{j\frac{\pi}{7}}\right)^{-\frac{1}{2}} = e^{-j\frac{\pi}{14}}$$

1.5

$$F(\omega) = \frac{2 - 5\omega j}{-3 + \omega j}$$

Cartesian:

$$\begin{aligned} \frac{(2 - 5\omega j)(-3 - \omega j)}{(-3 + \omega j)(-3 - \omega j)} &= \frac{-6 + 5\omega^2 j^2 + 15\omega j - 2\omega j}{9 - \omega^2 j^2} \\ &= \frac{-6 - 5\omega^2 + 13\omega j}{9 + \omega^2} = \frac{-6 - 5\omega^2}{9 + \omega^2} + \frac{13\omega j}{9 + \omega^2} \end{aligned}$$

Real part:

$$\text{Re}(F(\omega)) = \frac{-6 - 5\omega^2}{9 + \omega^2}$$

Imaginary part:

$$\text{Im}(F(\omega)) = \frac{13\omega j}{9 + \omega^2}$$

Polar:

$$\begin{aligned} 2 - 5\omega j &= \sqrt{2^2 + (-5\omega)^2} e^{j(\arctan \frac{-5\omega}{2})} \\ -3 + \omega j &= \sqrt{-3^2 + \omega^2} e^{j(\arctan \frac{\omega}{-3}) + \pi} \\ \frac{2 - 5\omega j}{-3 + \omega j} &= \frac{\sqrt{2^2 + (-5\omega)^2} e^{j(\arctan \frac{-5\omega}{2})}}{\sqrt{-3^2 + \omega^2} e^{j(\arctan \frac{\omega}{-3}) + \pi}} \\ &= \sqrt{\frac{25\omega^2 + 4}{\omega^2 + 9}} e^{j(\arctan \frac{-5\omega}{2} - (\arctan \frac{\omega}{-3}) - \pi)} \end{aligned}$$

Magnitude:

$$|F(\omega)| = \sqrt{\frac{25\omega^2 + 4}{\omega^2 + 9}}$$

Angle:

$$\angle F(\omega) = \arctan \frac{-5\omega}{2} - (\arctan \frac{\omega}{-3}) - \pi$$

1.6

In general: $a \cos \omega t + b \sin \omega t = C \cos(\omega t + \theta)$, where $C = \sqrt{a^2 + b^2}$ and $\theta = \arctan \frac{b}{a}$.

a

$$\begin{aligned} F(t) &= 6 \cos(\omega t) - \sqrt{5} \sin(\omega t) \\ C &= \sqrt{6^2 + (-\sqrt{5})^2} = \sqrt{41} \\ \theta &= \arctan \frac{-\sqrt{5}}{6} = -.3567 \\ F(t) &= \sqrt{41} \cos(\omega t - .3567) \end{aligned}$$

b

$$\begin{aligned} F(t) &= -18 \cos(\omega t) + 2 \sin(\omega t) \\ C &= \sqrt{18^2 + 2^2} = \sqrt{328} \\ \theta &= \arctan \frac{2}{-18} + \pi = 3.039 \\ F(t) &= \sqrt{328} \cos(\omega t + 3.039) \end{aligned}$$