§ 4.7. State-space Representations

① state-space description of a system

<u>Def:</u> The <u>state</u> of a system at any time t_0 is the smallest set of numbers $\{x_1(t_0), x_2(t_0), \cdots, x_n(t_0)\}$ that is sufficient to determine the behavior of the system for all time $t > t_0$ when the input to the system is known for $t > t_0$

The variables $x_1, x_2, \cdots, x_n$ are called <u>state variables</u>.

<u>Ex:</u>

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = f(t)$$

<u>state variables:</u>

$$\begin{cases} x_1 = y \\ x_2 = \dot{y} \\ x_3 = \ddot{y} \\ \vdots \\ x_n = y^{(n-1)} \end{cases}$$

state eqn's:

$$\left\{\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\vdots \\
\dot{x}_{n-1} &= x_n \\
\dot{x}_n &= -a_{n-1}x_n - a_{n-2}x_{n-1} - \cdots - a_1 x_2 - a_0 x_1 + f.
\end{aligned}\right.$$

or in matrix form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & & 1 \\ -a_0 & -a_1 & \cdots & & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \cdot f$$

<u>output eqn:</u> $y = x_1$

<u>In general:</u>

(state-space description of a multi-input multi-output LTI system).

<u>state eqn:</u> $\underline{\dot{x}} = \underline{A}\,\underline{x} + \underline{B}\,\underline{f}$

<u>output eqn:</u> $\underline{y} = \underline{C}\,\underline{x} + \underline{D}\,\underline{f}$

state equations from transfer function

state variables: integrator outputs.

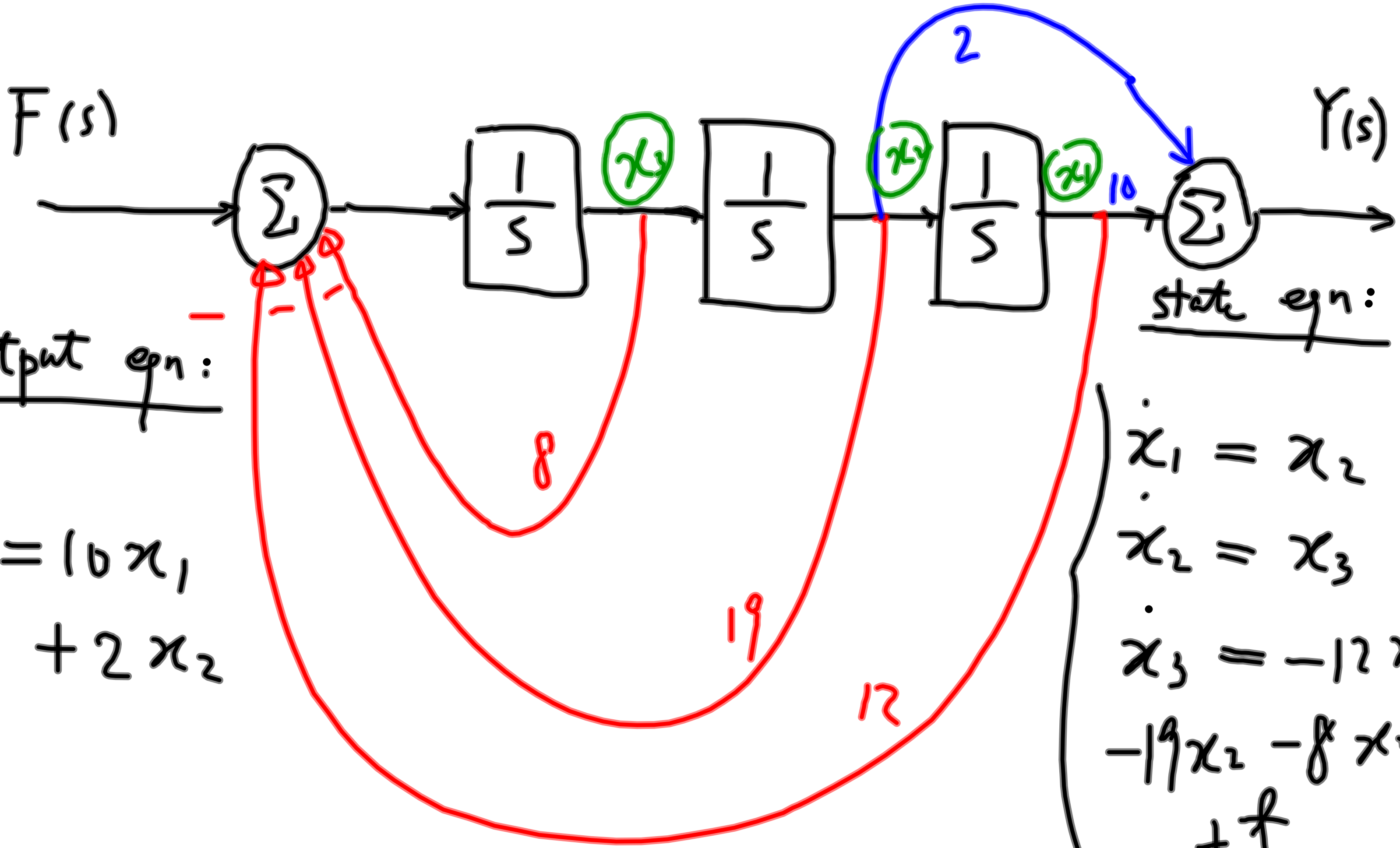
Ex:

$$H(s) = \frac{2s+10}{s^3+8s^2+19s+12}$$

$$= \left(\frac{2}{s+1}\right)\left(\frac{s+5}{s+3}\right)\left(\frac{1}{s+4}\right)$$

$$= \frac{4/3}{s+1} + \frac{-2}{s+3} + \frac{2/3}{s+4}$$

a) direct form



output eqn:

$$y = 10x_1 + 2x_2$$

state eqn:

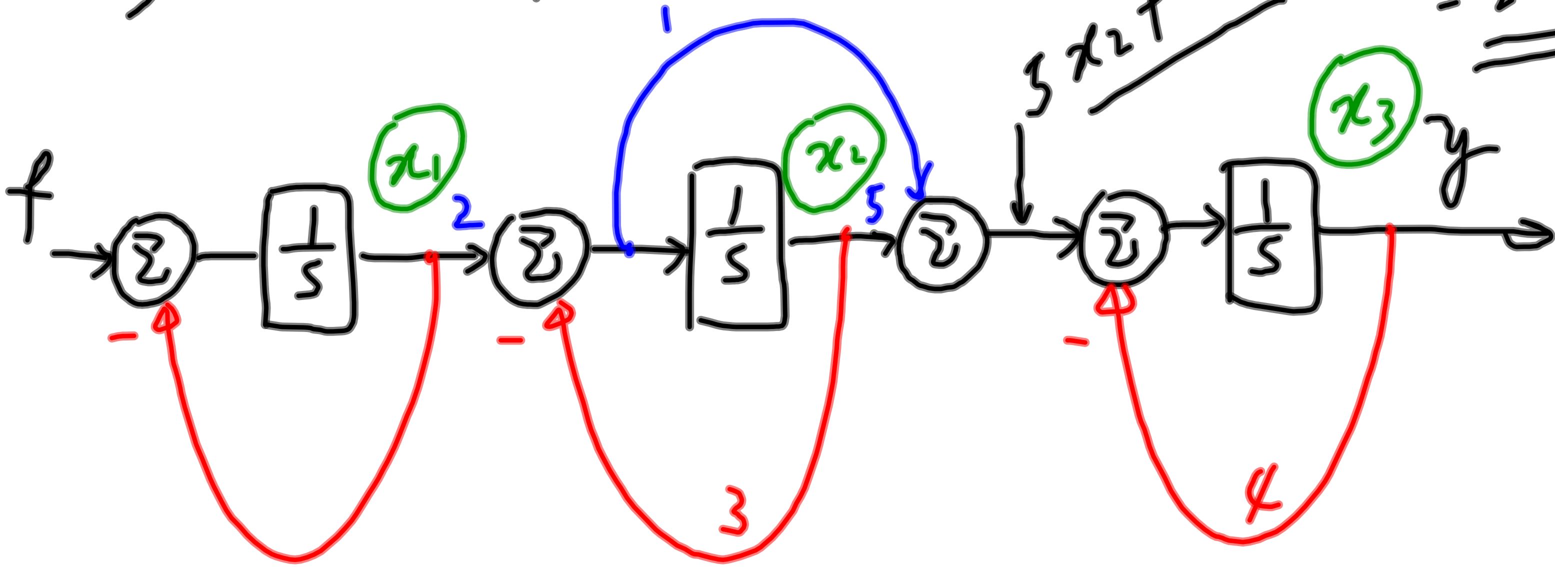
$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -12x_1 - 19x_2 - 8x_3 + f \end{cases}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & -19 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} f$$

$$y = \begin{bmatrix} 10 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

b) serial form

$$3x_2 + \dot{x}_2 = 5x_2 + 2x_1 - 3x_2 = 2x_2 + 2x_1$$



state eqn:

$$\begin{cases} \dot{x}_1 = -x_1 + f \\ \dot{x}_2 = 2x_1 - 3x_2 \\ \dot{x}_3 = 2x_1 + 2x_2 - 4x_3 \end{cases}$$

output eqn:

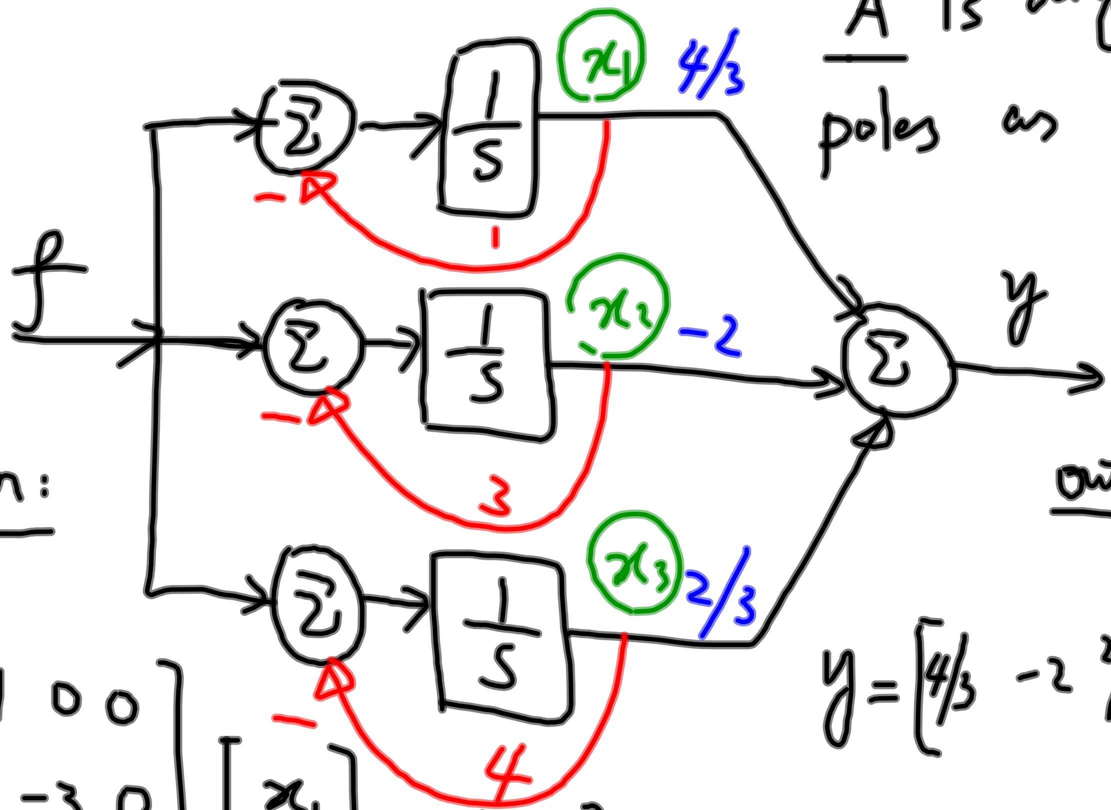
$$y = x_3$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -3 & 0 \\ 2 & 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} f$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

c) parallel form:

Note: In parallel form, $\underline{A}$ is diagonal with poles as diagonal elements.



state eqn:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} f$$

output eqn:

$$y = \begin{bmatrix} 4/3 & -2 & 2/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

§4.8. Frequency Response of an LTI system

Recall: $e^{st} \longrightarrow \boxed{h(t)} \longrightarrow y(t)$ causal

LTI system response to an everlasting exponential

$$y(t) = h(t) * e^{st}$$

$$= \int_{-\infty}^{+\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau$$

$$= e^{st} \boxed{\int_{\substack{-\infty \\ 0^-}}^{+\infty} h(\tau)\, e^{-s\tau}\, d\tau} = \underline{\underline{H(s) \cdot e^{st}}}$$

$\mathcal{L}\{h(t)\}$ — transfer func.

special case: $s = j\omega$

$$\Longrightarrow \quad e^{j\omega t} \longrightarrow \boxed{h(t)} \longrightarrow \overbrace{H(j\omega)}^{\mathcal{F}\{h(t)\}\ \text{– freq. resp.}} \cdot e^{j\omega t}$$

Response to sinusoid – for real-valued $h(t)$:

$$\cos(\omega_0 t + \theta) \longrightarrow \boxed{h(t)} \longrightarrow \underbrace{\left|H(j\omega_0)\right|}_{\text{magnitude resp.}} \cdot \cos\Big(\omega_0 t + \theta + \underbrace{\measuredangle H(j\omega_0)}_{\text{phase resp.}}\Big).$$

Ex: $H(s) = \dfrac{s+0.1}{s+5}$

input $f(t) = \cos 2t$.

output $y(t) = ?$

$$H(j\omega) = \frac{j\omega + 0.1}{j\omega + 5} \Rightarrow \begin{cases} |H(j\omega)| = \dfrac{\sqrt{\omega^2 + 0.01}}{\sqrt{\omega^2 + 25}} \\ \angle H(j\omega) = \tan^{-1}\left(\dfrac{\omega}{0.1}\right) - \tan^{-1}\left(\dfrac{\omega}{5}\right) \end{cases}$$

$\Longrightarrow \omega = 2$: $|H(j2)| = 0.372$

$\angle H(j2) = 65.3°$

$\therefore y(t) = 0.372 \cos(2t + 65.3°)$