$$f(t) = 2e^{-t}u(t) + 3t^2 e^{-2t}u(t)$$

$$-2te^{-2t}u(t) - 2e^{-2t}u(t)$$

$$= \left[2e^{-t} + (3t^2 - 2t - 2)e^{-2t}\right]u(t).$$

§ 4.4. System Analysis via Laplace Transform

Ex: $(D^2+5D+6)\,y(t) = (D+1)\,f(t)$

$y(0^-)=2, \quad \dot{y}(0^-)=1.$

input: $f(t)=e^{-4t}u(t).$

Find the output $y(t)$.

$$\frac{d^2y(t)}{dt^2}+5\frac{dy(t)}{dt}+6y(t)=\frac{df(t)}{dt}+f(t)$$

$$\Rightarrow \left[s^2\cdot Y(s)-s\underbrace{y(0^-)}_{2}-\underbrace{\dot{y}(0^-)}_{1}\right]+5\left[s\cdot Y(s)-\underbrace{y(0^-)}_{2}\right]+6\cdot Y(s)$$

$$=\left[s\cdot F(s)-f(0^-)\right]+F(s)$$

$$\Rightarrow \left(s^2 \cdot Y(s) - 2s - 1\right) + 5\left(sY(s) - 2\right) + 6Y(s)$$

$$= (s+1) \cdot F(s) = \frac{s+1}{s+4}$$

$$\Rightarrow Y(s) = \frac{2s + 11 + \frac{s+1}{s+4}}{s^2 + 5s + 6}$$

total response

$$= \frac{2s^2 + 20s + 45}{(s+2)(s+3)(s+4)}$$

$$\Downarrow$$

$$= \frac{13/2}{s+2} - \frac{3}{s+3} - \frac{3/2}{s+4}$$

$$\therefore y(t) = \left(\frac{13}{2} e^{-2t} - 3e^{-3t} - \frac{3}{2} e^{-4t}\right) u(t)$$

Recall:

$$\text{total response} = \underbrace{\text{zero-input resp.}}_{\text{initial cond.}} + \underbrace{\text{zero-state resp.}}_{\text{input}}$$

$$\Rightarrow s^2 \cdot Y(s) + 5s\,Y(s) + 6\,Y(s)$$

$$= \left[s \cdot y(0^-) + \dot{y}(0^-) + 5\,y(0^-) \right] + (s+1)F(s).$$

$$\Rightarrow Y(s) = \underbrace{\frac{(s+5)\,y(0^-) + \dot{y}(0^-)}{s^2+5s+6}}_{Y_{zi}(s)} + \underbrace{\frac{s+1}{s^2+5s+6}F(s)}_{Y_{zs}(s)}.$$

$$= \frac{2s+11}{(s+2)(s+3)} + \frac{s+1}{(s+2)(s+3)(s+4)}$$

$$= \left(\frac{7}{s+2} - \frac{5}{s+3}\right) + \left(\frac{-1/2}{s+2} + \frac{2}{s+3} - \frac{3/2}{s+4}\right)$$

$$\Rightarrow y(t) = \underbrace{\left(7\cdot e^{-2t} - 5e^{-3t}\right)u(t)}_{y_{zi}(t)}$$

$$+ \underbrace{\left(-\frac{1}{2}e^{-2t} + 2e^{-3t} - \frac{3}{2}e^{-4t}\right)u(t)}_{y_{zs}(t)}$$

Finding the impulse response $h(t)$ of an LTI system

$$\overbrace{\left(D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0\right)}^{Q(D)} y(t)$$

$$= \underbrace{\left(b_m D^m + b_{m-1} D^{m-1} + \cdots + b_1 D + b_0\right)}_{P(D)} f(t)$$

<u>Recall:</u> $$\underbrace{y_{zs}(t)}_{\text{zero-state resp.}} = \underbrace{h(t)}_{\text{impulse resp.}} * \underbrace{f(t)}_{\text{causal input}}$$

$$\Rightarrow Y_{zs}(s) = H(s) \cdot F(s) \Rightarrow H(s) = \frac{Y_{zs}(s)}{F(s)}$$

But for zero state resp.

$$y(0^-) = \dot{y}(0^-) = \cdots = y^{(n-1)}(0^-) = 0$$

and since input is causal

$$f(0^-) = \dot{f}(0^-) = \cdots = f^{(n-1)}(0^-) = 0.$$

$$\text{diff eqn} \Rightarrow Q(s) \cdot Y_{zs}(s) = P(s) \cdot F(s)$$

$$\Rightarrow H(s) = \frac{Y_{zs}(s)}{F(s)} = \frac{P(s)}{Q(s)}.$$

transfer function

$$\therefore h(t) = \mathcal{L}^{-1}\left\{ \frac{P(s)}{Q(s)} \right\}$$

Ex.

$$\frac{d^2 y(t)}{dt^2} + 5\cdot\frac{dy(t)}{dt} + 6y(t) = \frac{df(t)}{dt} + f(t)$$

$$H(s) = \frac{s+1}{s^2+5s+6} = \frac{s+1}{(s+2)(s+3)}$$

$$= \frac{-1}{s+2} + \frac{2}{s+3}$$

$$\Rightarrow h(t) = \left(2e^{-3t} - e^{-2t}\right)u(t).$$

Summary: LTI system Analysis

$$Q(D)\cdot y(t) = P(D)\cdot f(t).$$

- transfer func. $H(s) = \dfrac{P(s)}{Q(s)}.$

  impulse resp: $h(t) = \mathcal{L}^{-1}\{H(s)\}.$

- zero-state resp: $Y_{zs}(s) = H(s)\,F(s).$

  $y_{zs}(t) = \mathcal{L}^{-1}\{Y_{zs}(s)\}$

- zero-input resp: $Y_{zi}(s) = \dfrac{g(y(0^-), \dot y(0^-), \dots, y^{(n-1)}(0^-))}{Q(s)}$

  $y_{zi}(t) = \mathcal{L}^{-1}\{Y_{zi}(s)\}$

- total resp.

  $y(t) = y_{zs}(t) + y_{zi}(t)$

§ 4.5. System Stability.

Intuitively, stability means without external input, the system output due to initial conditions.

$$y_{zi}(t) \longrightarrow 0, \quad \text{as } t \to \infty.$$

Definitions:

- A system is <u>asymptotically stable</u> if

  $$y_{zi}(t) \to 0 \quad \text{as } t \to \infty.$$

- A system is <u>unstable</u> if

  $$|y_{zi}(t)| \to \infty \quad \text{as } t \to \infty$$

- A system is <u>marginally stable</u> if

  $y_{zi}(t)$ remains bounded as $t \to \infty$

  (approaches neither 0 nor ∞)

Conditions on stability:

$$H(s) = \frac{P(s)}{Q(s)}.$$

poles: roots of $Q(s)$.

zeros: roots of $P(s)$.

- $H(s)$ is asymptotically stable iff all its poles are in LHP.
- $H(s)$ is unstable if

  (a) at least one pole is in RHP

  or (b) there are repeated poles on imaginary axis.

· $H(s)$ is marginally stable if no poles is on RHP and there are unrepeated poles on imaginary axis.