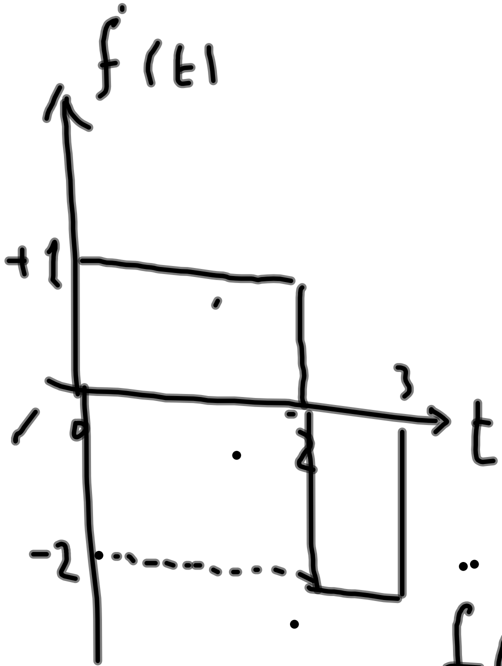
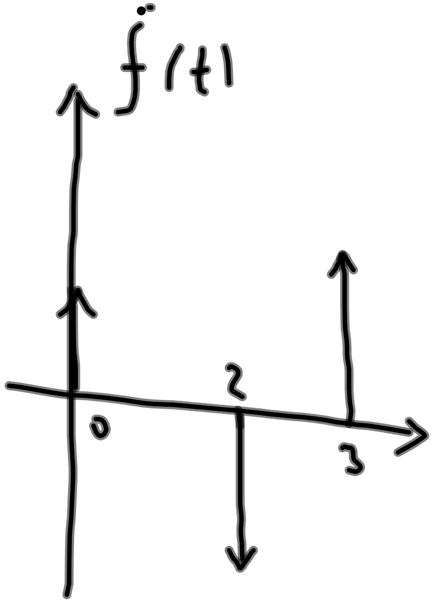
$$\mathcal{L}\left\{\frac{d^n x(t)}{dt^n}\right\} = s^n X(s) - s^{n-1} x(0^-) - \dots - x^{(n-1)}(0^-)$$

Ex: $\mathcal{L}\{f(t)\}$





$$\ddot{f}(t) = \delta(t) - 2\delta(t-2) + 2\delta(t-3)$$

$$L\{\ddot{f}(t)\} = 1 - 3e^{-2s} + 2e^{-3s}$$

$$x(t) \longleftrightarrow X(s)$$

$$x(t-t_0) \longleftrightarrow e^{-t_0 s} X(s)$$

$$L\{\ddot{f}(t)\} = s^2 F(s) - s f(0^-) - \dot{f}(0^-)$$

$$s^2 F(s) = 1 - 3e^{-2s} + 2e^{-3s} \Longrightarrow F(s) = \frac{1}{s^2}\left[1 - 3e^{-2s} - 2e^{-3s}\right]$$

7 Frequency Differntiation:

$$x(t) \longleftrightarrow X(s)$$

$$t\,x(t) \longleftrightarrow -\frac{d}{ds}X(s)$$

Ex:

$$L\{u(t)\} = \frac{1}{s} \Longrightarrow L\{t\,u(t)\} = \frac{-d}{ds}\cdot\frac{1}{s} = \frac{1}{s^2}$$

8 Time Integration:

$$x(t) \longleftrightarrow X(s)$$

$$\begin{cases} \displaystyle\int_{0^-}^{t} x(\tau)\,d\tau \longleftrightarrow X(s)/s \\ \displaystyle\int_{-\infty}^{t} x(\tau)\,d\tau \longleftrightarrow \frac{X(s)}{s} + \frac{\int_{-\infty}^{0^-} x(\tau)\,d\tau}{s} \end{cases}$$

Proof:

$$y(t) = \int_{0^-}^{t} x(\tau)\, d\tau \Rightarrow \frac{d}{dt} y(t) = x(t) \qquad y(0^-) = 0$$

$$X(s) = L\left\{x(t)\right\} = L\left\{\frac{d}{dt} y(t)\right\} = sY(s) - \underbrace{y(0^-)}_{0}$$

$$Y(s) = \frac{X(s)}{s}$$

$$\int_{-\infty}^{t} x(\tau)\,d\tau = \int_{-\infty}^{0^-} x(\tau)\,d\tau + \underbrace{\int_{0^-}^{t} x(\tau)\,d\tau}_{\frac{X(s)}{s}}$$

$$\frac{1}{s}\cdot\int_{-\infty}^{0} x(\tau)\,d\tau$$

9 Frequency Integration

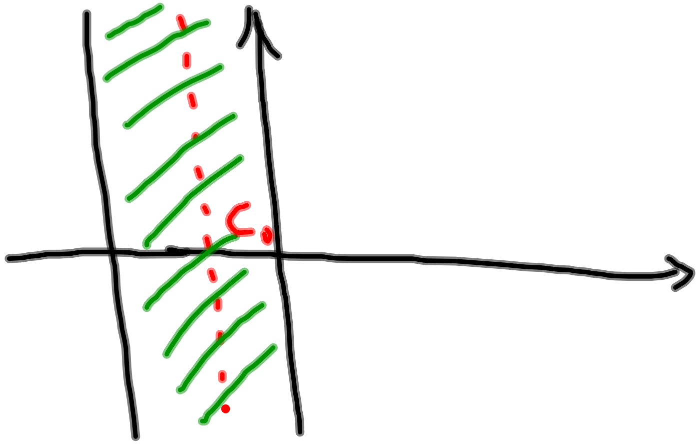
$$x(t) \longleftrightarrow X(s)$$

$$\frac{x(t)}{t} \longleftrightarrow \int_{s}^{+\infty} X(z)\,dz$$

4.3 Inverse Laplace Transform:

$$f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s)\, e^{st}\, ds$$

where $c > \sigma_0$



② Rational function:

$$F(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

$F(s)$: proper if $m < n$

improper if $m \geq n$

improper function: polynomial + proper function

Ex: $F(s) = \dfrac{2s^3 + 9s^2 + 11s + 2}{s^2 + 4s + 3}$

$$2s^3 + 9s^2 + 11s + 2 = (2s+1)(s^2+4s+3) + s - 1$$

$$F(s) = 2s + 1 + \frac{s-1}{s^2+4s+3}$$

3 Partial Expansion of rational Functions.

$$F(s) = \frac{P(s)}{Q(s)} = \frac{b_m s^m + \cdots + b_1 s + b_0}{s^n + \cdots + a_0}$$

Case 1: unrepeated roots for $Q(s)$

$$F(s) = \frac{P(s)}{(s-\lambda_1)\cdots(s-\lambda_n)} = \frac{K_1}{s-\lambda_1} + \cdots + \frac{K_n}{s-\lambda_n}$$

$$k_i \triangleq (s-\lambda_i)\, F(s)\Big|_{s=\lambda_i}$$

$$f(t) = \mathcal{L}^{-1}\left\{ F(s) \right\} = \mathcal{L}^{-1}\left\{ \sum_i \frac{k_i}{s-\lambda_i} \right\}$$

$$= \sum_i k_i e^{\lambda_i t} u(t)$$

$$\mathcal{L}\left\{ e^{\lambda t} u(t) \right\} = \frac{1}{s-\lambda}$$

Ex: $F(s) = \dfrac{2s^2+5}{s^2+3s+2}$

$$= 2 + \frac{-6s+1}{s^2+3s+2} = 2 + \frac{K_1}{s+2} + \frac{K_2}{s+1}$$

$$K_1 = (s+2)\left(\frac{-6s+1}{s^2+3s+2}\right)\Bigg|_{s=-2} = \frac{-6s+1}{s+1}\Bigg|_{s=-2} = -13$$

$$K_2 = \frac{-6s+1}{s+2}\Bigg|_{s=-1} = 7$$

$$F(s) = 2 + \frac{7}{s+2} - \frac{13}{s+1}$$

$$f(t) = 2\delta(t) + 7e^{-2t}u(t) - 13e^{-t}u(t)$$

Ex: $F(s) = \dfrac{s+3+5e^{-2s}}{(s+1)(s+2)}$

$$= \underbrace{\frac{s+3}{(s+1)(s+2)}}_{F_1(s)} + \underbrace{\frac{5}{(s+1)(s+2)}}_{F_2(s)} e^{-2s}$$

$$F_1(s) = \frac{2}{s+1} - \frac{1}{s+2} \longleftrightarrow \left(2e^{-t} - e^{-2t}\right)u(t)$$

$$F_2(s) = \frac{5}{s+1} - \frac{5}{s+2} \longleftrightarrow 5\left(e^{-t} - e^{-2t}\right)u(t)$$

$$f(t) = f_1(t) + f_2(t-2)$$

$$\Rightarrow f(t) = \left(2e^{-t} - e^{-2t}\right) u(t) + 5\left[e^{-(t-2)} - e^{-2(t-2)}\right] u(t-2)$$

Case 2: Complex unrepeated roots:

$$F(s) = \frac{As+B}{s^2+2as+b} = \frac{As+B}{(s-r)(s-r^*)}$$

$$= \frac{C}{s-r} + \frac{C^*}{s-r^*}$$

$$C = F(s)(s-r)\Big|_{s=r}$$

$r = x + jy$

$c = \rho e^{j\theta}$

$\Rightarrow f(t) = 2\rho e^{xt} \cos(yt + \theta)\, u(t)$

Ex: $F(s) = \dfrac{6(s+34)}{s(s^2+10s+34)}$

$$= \frac{K_1}{s} + \frac{c}{s+5+3j} + \frac{c^*}{s+5-3j}$$

$$K_1 = F(s)\cdot s \Big|_{s=0} = 6$$

$$C = F(s)(s+5+3j)\Big|_{s=-5-3j} = \frac{29-3j}{-3-5j} = -3-4j = 5e^{j\left[\tan^{-1}\left(\frac{4}{3}\right)-\pi\right]}$$

$$r = -5 - 3j \rightarrow \begin{cases} x = -5 \\ y = -3 \end{cases}$$

$$c = 5e^{j\left[\tan^{-1}\left(\frac{4}{3}\right) - \pi\right]}$$

$$f(t) = 6u(t) + 10e^{-5t} \cos\left[-3t + \tan^{-1}\left(\frac{4}{3}\right) - \pi\right] u(t)$$

Case 3: Repeated roots:

$$F(s) = \frac{P(s)}{(s-\lambda)^r (s-\alpha_1)\cdots(s-\alpha_j)}$$

$$= \frac{a_0}{(s-\lambda)^r} + \frac{a_1}{(s-\lambda)^{r-1}} + \cdots + \frac{a_{r-1}}{s-\lambda}$$

$$+ \frac{k_1}{s-\alpha_1} + \frac{k_2}{s-\alpha_2} + \cdots + \frac{k_j}{s-\alpha_j}$$

$$a_0 = (s-\lambda)^r \cdot F(s) \Big|_{s=\lambda}$$

$$a_1 = \frac{d}{ds}\left[(s-\lambda)^r F(s) \right]\Big|_{s=\lambda}$$

$$\vdots$$

$$a_m = \frac{1}{m!} \frac{d^m}{ds^m}\left[(s-\lambda)^r F(s) \right]\Big|_{s=\lambda}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-\lambda)^n}\right\} = \frac{1}{(n-1)!}\, t^{n-1} e^{\lambda t} u(t)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-\lambda}\right\} = e^{\lambda t} u(t)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-\lambda)^2}\right\} = t e^{\lambda t} u(t)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-\lambda)^3}\right\} = \frac{1}{2} t^2 e^{\lambda t} u(t)$$