Chapter 4 System Analysis by Laplace Transform

Fourier transform :

1. Exists only for a restricted class of signals
2. Can not be used to analyze unstable systems.

§4.1. Unilateral Laplace Transform

① Definition:

$$F(s) = \mathcal{L}\{f(t)\}$$

$$= \int_{\underline{0^-}}^{\infty} f(t)\, e^{-st}\, dt$$

where $s = \sigma + j\omega$

↑ complex freq.

Recall

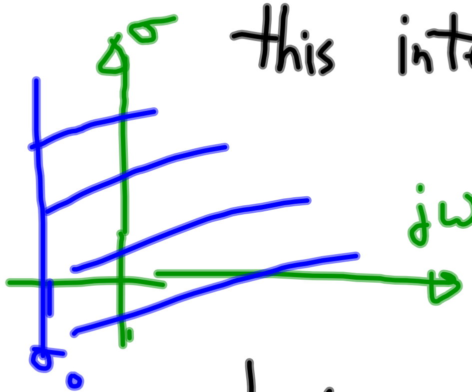
$$F(\omega) = \mathcal{F}\{f(t)\}$$

$$= \int_{\underline{-\infty}}^{+\infty} f(t)\, e^{-j\omega t}\, dt$$

② Region of convergence

$$F(s) = \int_{0^-}^{\infty} \underbrace{\left[f(t)\, e^{-\sigma t} \right]} e^{-j\omega t}\, dt$$

this integral converges if



$$\int_{0^-}^{\infty} \left| f(t)\, e^{-\sigma t} \right| dt < \infty \quad (*)$$

Let σ_0 is the smallest value of σ for which $(*)$ is finite, then the region of convergence for $F(s)$ is $\boxed{Re\{s\} > \sigma_0}$

Ex:

$$\mathcal{L}\{\delta(t)\}$$

$$= \int_{0^-}^{\infty} \delta(t)\, e^{-st}\, dt = 1.$$

$$\mathcal{F}\{\delta(t)\} = 1.$$

Ex:

$$\mathcal{L}\{u(t)\}$$

$$s = \sigma + j\omega$$

$$\mathcal{F}\{u(t)\} = \frac{1}{j\omega} + \pi\delta(\omega)$$

$$= \int_{0^-}^{\infty} u(t)\, e^{-st}\, dt$$

$$= \int_{0^-}^{\infty} e^{-st}\, dt = -\frac{1}{s} e^{-st}\Big|_{0}^{\infty}$$

$$= \frac{1}{s} - \frac{1}{s} e^{-st}\Big|_{t=\infty}$$

for $\mathrm{Re}\{s\} > 0$

$$= \frac{1}{s}$$

Ex:

$$\mathcal{L}\left\{ e^{at} u(t) \right\}$$

$$= \int_{0^-}^{\infty} e^{at} e^{-st} \, dt$$

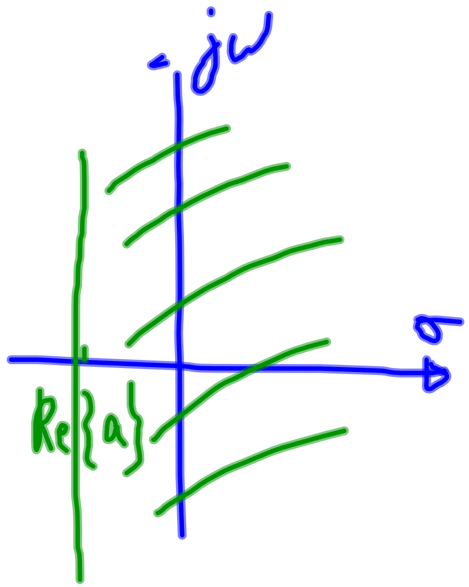
$$= \int_{0^-}^{\infty} e^{(a-s)t} \, dt$$

$$= \frac{1}{a-s} e^{(a-s)t} \Big|_{t=0^-}^{\infty}$$

$$= \frac{1}{s-a} \left[1 - \underbrace{e^{(a-s)t} \Big|_{t=\infty}} \right]$$

$$= \frac{1}{s-a} \qquad \underline{\mathrm{Re}\{s\} > \mathrm{Re}\{a\}}$$

$$\mathcal{F}\left\{ e^{at} u(t) \right\} = \frac{1}{j\omega - a} \qquad \underline{a < 0}$$



§ 4.2. Properties of Laplace Transform

① Linearity:

$$f_1(t) \xleftrightarrow{\mathcal{L}} F_1(s)$$

$$f_2(t) \xleftrightarrow{\mathcal{L}} F_2(s)$$

then $a_1 f_1(t) + a_2 f_2(t) \longleftrightarrow a_1 F_1(s) + a_2 F_2(s)$.

② Time-shifting.

$$f(t) \xleftrightarrow{\mathcal{L}} F(s)$$

the for $t_0 > 0$

$$f(t-t_0)\cdot \underline{\underline{u(t-t_0)}} \longleftrightarrow F(s)\, e^{-st_0}$$

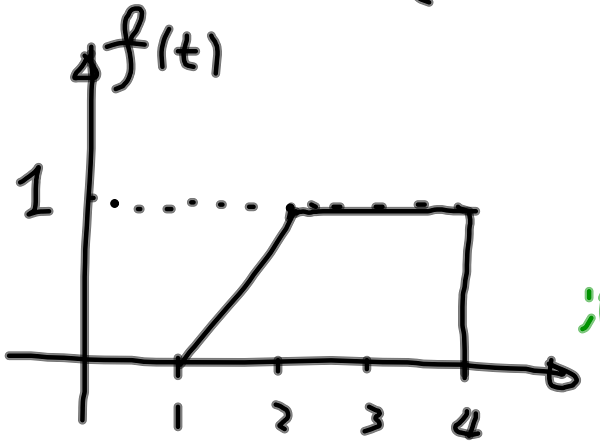
<u>Proof</u>: $\mathcal{L}\{ f(t-t_0)\, u(t-t_0) \}$

$$= \int_{0^-}^{\infty} f(t-t_0)\, u(t-t_0)\, e^{-st}\, dt$$

$$\overset{\tau = t - t_0}{=} \int_{-t_0^-}^{\infty} f(\tau)\, \underline{\underline{u(\tau)}}\, e^{-s(t_0+\tau)}\, d\tau$$

$$= e^{-st_0} \underbrace{\int_{0^-}^{\infty} f(\tau)\, e^{-s\tau}\, d\tau}_{F(s)}$$

Ex: Given $\mathcal{L}\{t\,u(t)\} = \frac{1}{s^2}$



find $\mathcal{L}\{f(t)\}$

$u(t) \leftrightarrow \frac{1}{s}$

$u(t-4) = u(t-4)\cdot u(t-4) \leftrightarrow \frac{1}{s}e^{-4s}$

$$f(t) = (t-1)\left[u(t-1) - u(t-2)\right] + \left[u(t-2) - u(t-4)\right]$$

$$= (t-1)\cdot u(t-1) - (\underline{t-2} + 1)\,u(\underline{t-2}) + u(t-2) - u(t-4)$$

$$= \underbrace{(t-1)u(t-1)}_{\frac{1}{s^2}e^{-s}} - \underbrace{(t-2)u(t-2)}_{\frac{1}{s^2}e^{-2s}} - \underbrace{u(t-4)}_{\frac{1}{s}e^{-4s}}$$

$$= \frac{1}{s^2}e^{-s} - \frac{1}{s^2}e^{-2s} - \frac{1}{s}e^{-4s}$$

③ Frequency - shifting

$$f(t) \xleftrightarrow{\mathcal{L}} F(s).$$

Region of conv. (RoC) for $F(s)$ is $Re\{s\} > \sigma_0$

then $f(t)e^{s_0 t} \longleftrightarrow F(s-s_0)$

the RoC of $F(s-s_0)$ is $Re\{s-s_0\} > \sigma_0$

<u>Proof:</u> $\mathcal{L}\left\{ f(t) e^{s_0 t} \right\}$

$$= \int_{0^-}^{\infty} f(t) e^{s_0 t} e^{-st} dt \Rightarrow Re\{s\} > \sigma_0 + Re\{s_0\}$$

$$= \int_{0^-}^{\infty} f(t) e^{-\underbrace{(s-s_0)}t} dt = F(s-s_0)$$

Ex:

$$\mathcal{L}\left\{\cos\omega_0 t\, u(t)\right\}$$ causal sinusoid.

$$= \mathcal{L}\left\{\frac{1}{2}\underline{e^{j\omega_0 t}u(t)} + \frac{1}{2}\underline{e^{-j\omega_0 t}u(t)}\right\}$$

$$= \frac{1}{2}\cdot\frac{1}{s-j\omega_0} + \frac{1}{2}\cdot\frac{1}{s+j\omega_0}$$

$$= \frac{1}{2}\,\frac{\cancel{2}s}{(s-j\omega_0)(s+j\omega_0)}$$

$$= \frac{s}{s^2+\omega_0^2}$$

④ Scaling

$$f(t) \overset{\mathcal{L}}{\longleftrightarrow} F(s)$$

$$\Rightarrow f(at) \longleftrightarrow \frac{1}{a} F\left(\frac{s}{a}\right), \quad \underline{\underline{a>0}}$$

⑤ Convolution.

$$f_1(t) \longleftrightarrow F_1(s), \quad f_2(t) \longleftrightarrow F_2(s).$$

$$\Longrightarrow f_1(t) * f_2(t) \longleftrightarrow F_1(s) \cdot F_2(s). \ \checkmark$$

Proof: same as in Fourier transform.

⑥ Time differentiation

$$f(t) \leftrightarrow F(s).$$

$$\Rightarrow \frac{d}{dt} f(t) \leftrightarrow s \cdot F(s) - f(0^-)$$

Proof:

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = \int_{0^-}^{\infty} \frac{df(t)}{dt} e^{-st} dt$$

$$= \int_{0^-}^{\infty} e^{-st} df(t) = \underbrace{e^{-st} f(t)\Big|_{0^-}^{\infty}}_{0 - f(0^-)} - \underbrace{\int_{0^-}^{\infty} f(t) de^{-st}}_{s \cdot \underbrace{\int_{0^-}^{\infty} f(t) e^{-st} dt}_{F(s)}}$$

$$= s \cdot F(s) \underbrace{- f(0^-)}$$

$$\mathcal{L}\left\{\frac{d^2 f(t)}{dt}\right\} = \mathcal{L}\left\{\frac{d}{dt}\left(\frac{df(t)}{dt}\right)\right\}$$

$$= s\cdot\underbrace{\mathcal{L}\left\{\frac{df(t)}{dt}\right\}}_{s\cdot F(s) - f(0^-)} - \dot{f}(0^-)$$

$$= s^2\cdot F(s) - s\cdot f(0^-) - \dot{f}(0^-)$$

In general:

$$\mathcal{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n\cdot F(s) - s^{n-1} f(0^-) - s^{n-2}\cdot\dot{f}(0^-) - \cdots - s\cdot f^{(n-2)}(0^-) - f^{(n-1)}(0^-)$$