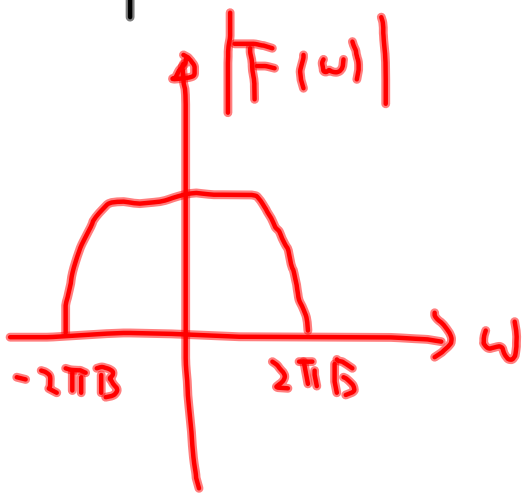
§ 3.5. The Sampling Theorem

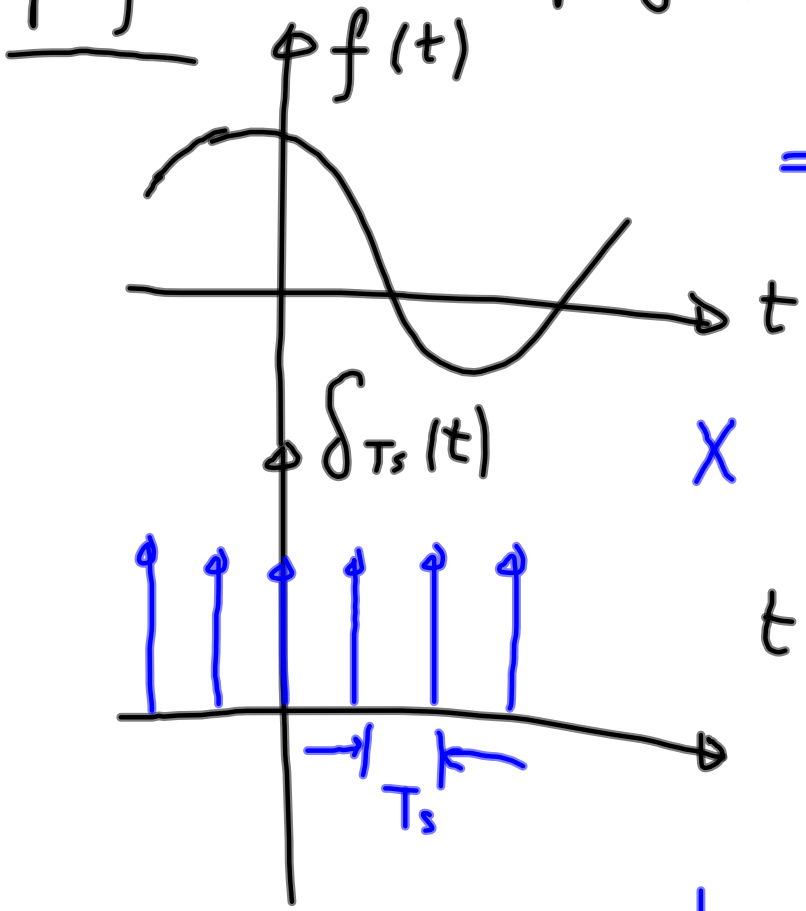
① The Nyquist Theorem

If the spectrum of a signal $f(t)$ is bandlimited to B Hz, i.e., $|F(\omega)| = 0$ for $|\omega| > 2\pi B$, then $f(t)$ can be reconstructed exactly from its samples taken uniformly at a rate $\underline{F_s > 2B}$ samples per second.

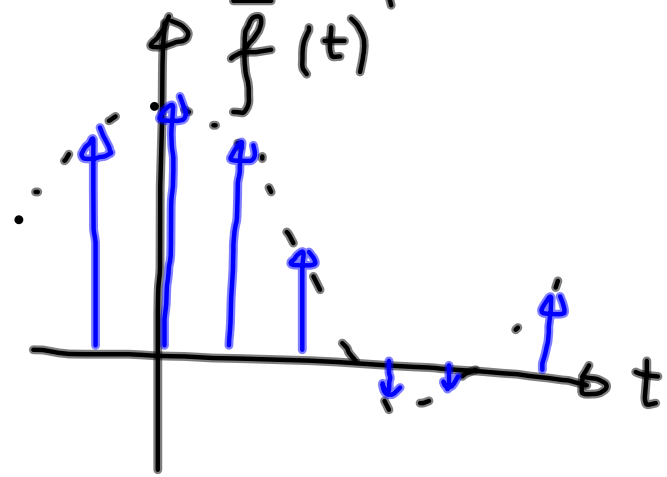


Proof:

- sampling $f(t)$ at rate F_s



$\Longrightarrow$



$$T_s = \frac{1}{F_s}$$

$$\bar{f}(t) = f(t) \cdot \delta_{T_s}(t)$$

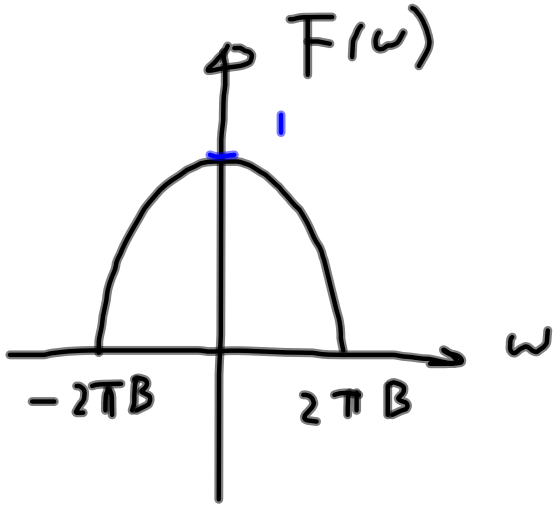
$$= f(t) \cdot \sum_{n=-\infty}^{+\infty} \delta(t - nT_s) = \sum f(nT_s) \cdot \delta(t - nT_s)$$

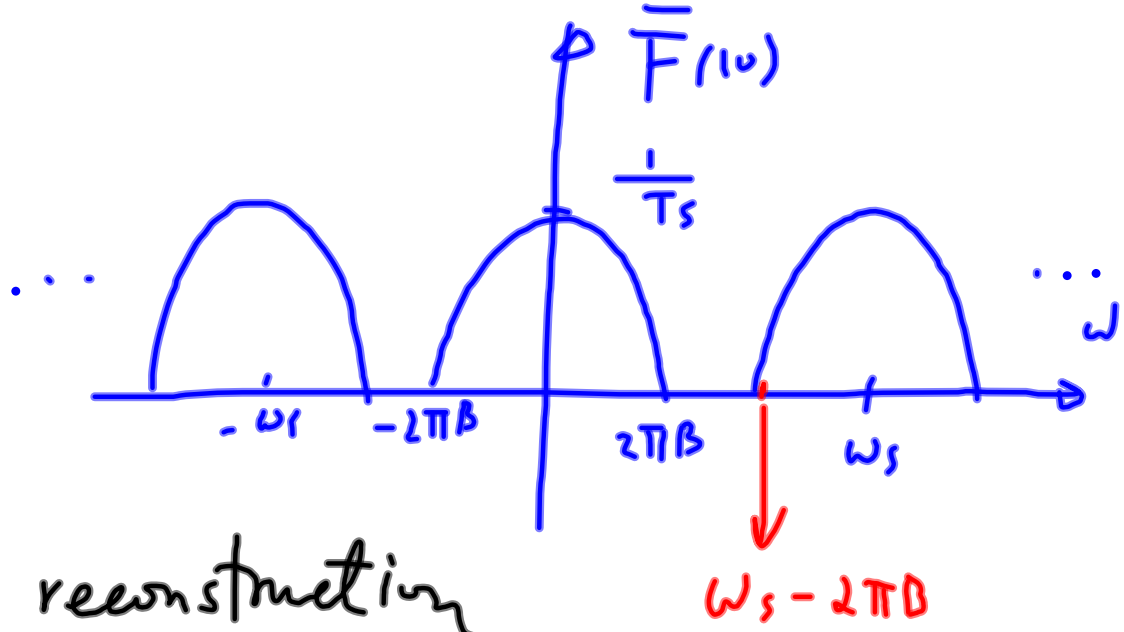
<u>Recall:</u> $\delta_{T_s}(t) = \frac{1}{T_s} \sum_{n=-\infty}^{+\infty} e^{jn\omega_s t}$

— Fourier series expansion

$$\therefore \bar{f}(t) = \frac{1}{T_s} \sum_{n=-\infty}^{+\infty} f(t) \cdot e^{jn\omega_s t}$$

$$\Rightarrow \bar{F}(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{+\infty} F(\omega - n\omega_s)$$





for perfect reconstruction

$\Longrightarrow$ can recover $F(\omega)$ from $\bar{F}(\omega)$

$\Longrightarrow$ There is no overlap between adjacent cycles of $\bar{F}(\omega)$

$$\Longrightarrow \quad \underbrace{\omega_s}_{2\pi \cdot F_s} - 2\pi B \geq 2\pi B \Longrightarrow \boxed{F_s \geq 2B}$$

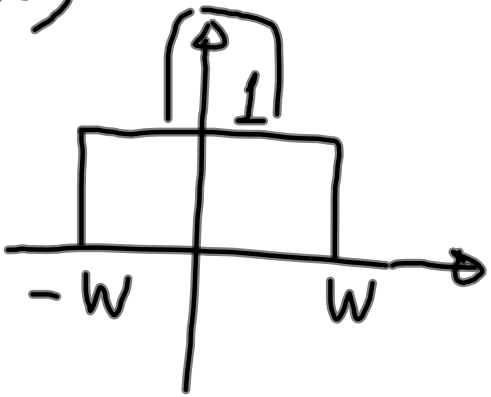
$$F_s \geq \underbrace{2B}_{\text{Nquist rate.}}$$

$$T_s \leq \underbrace{\frac{1}{2B}}_{\text{Nquist interval}}$$

Ex: Find the Nyquist rate & interval

① $f(t) = \text{sinc}(100\pi t)$

recall: $\frac{W}{\pi}\,\text{sinc}(Wt) \longleftrightarrow$



$W = 2\pi B = 100\pi \Longrightarrow B = 50\,Hz$

$F_N = 2B = 100\,Hz$

$T_N = \frac{1}{2B} = 0.01\ \text{sec}$

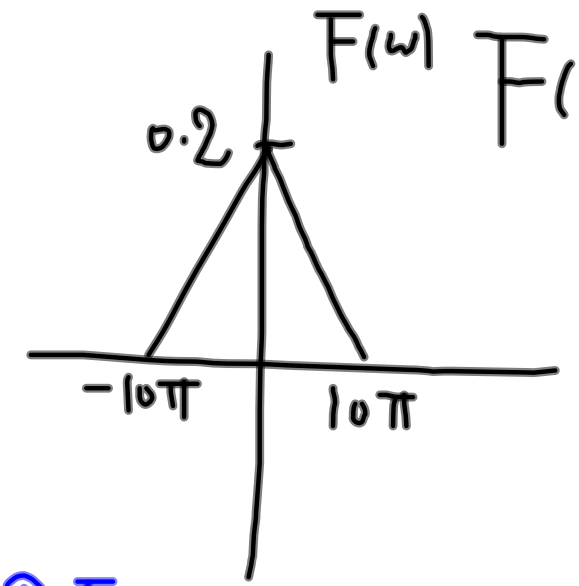
② $f(t) = \text{sinc}(100\pi t) + \text{sinc}(50\pi t)$

same!

Ex:

$$f(t) = \text{sinc}^2(5\pi t)$$

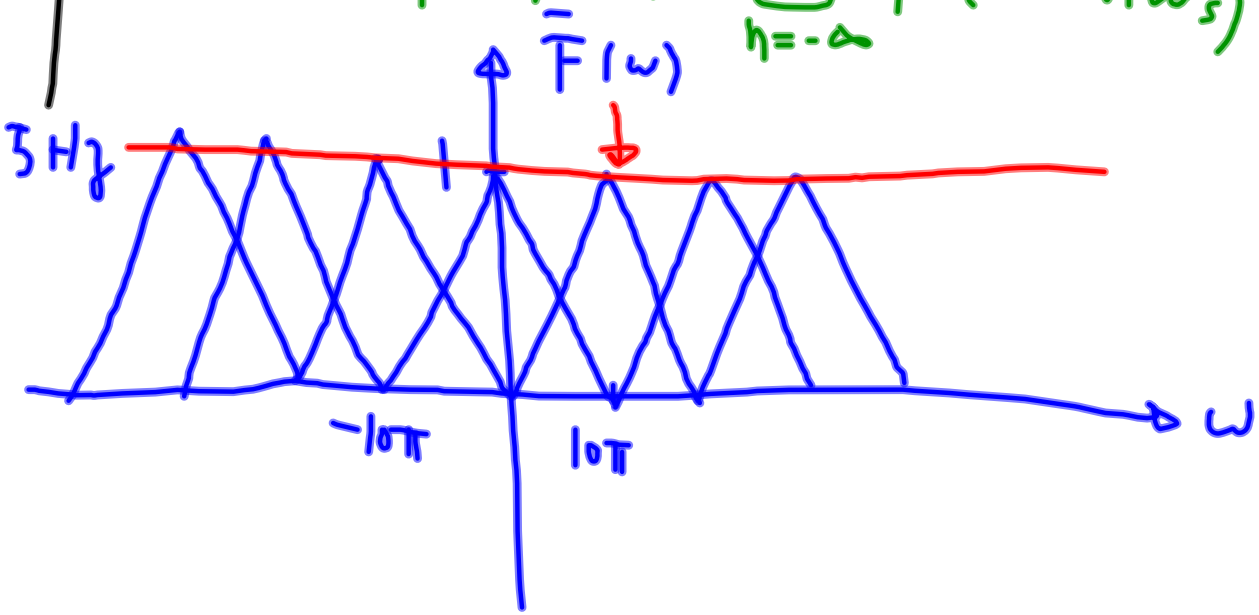
$$F(\omega) = 0.2\,\Delta\left(\frac{\omega}{20\pi}\right)$$



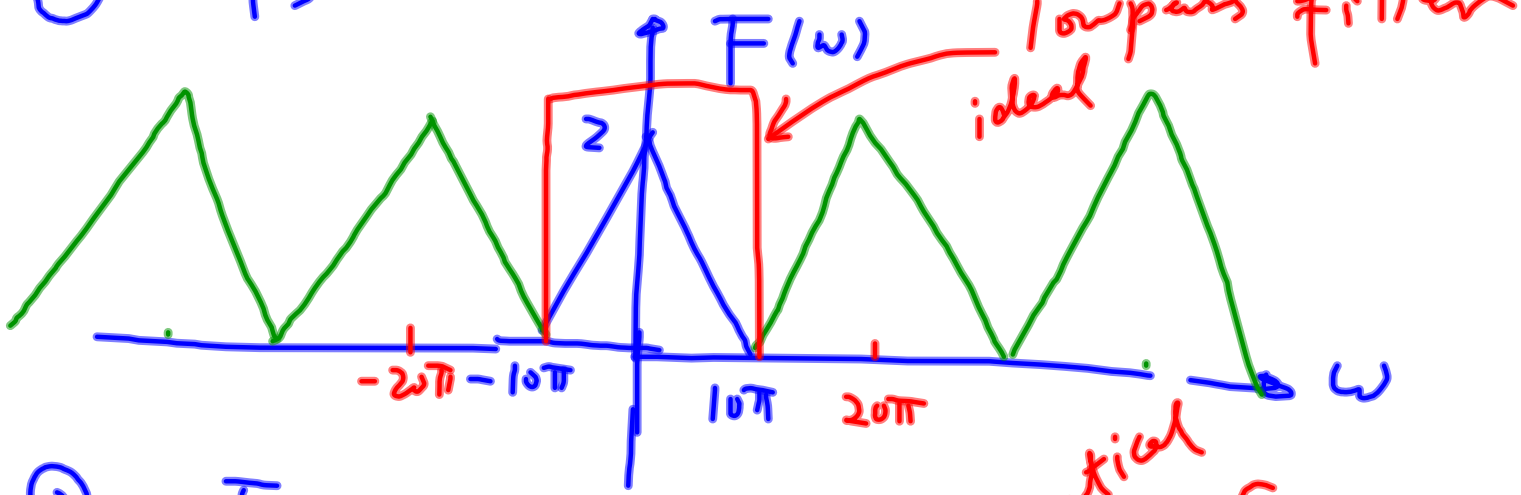
Find $\bar{F}(\omega)$ for $F_s = 5\,Hz$, $10\,Hz$, $20\,Hz$

$$\bar{F}(\omega) = F_s \cdot \sum_{n=-\infty}^{+\infty} F(\omega - n\omega_s)$$

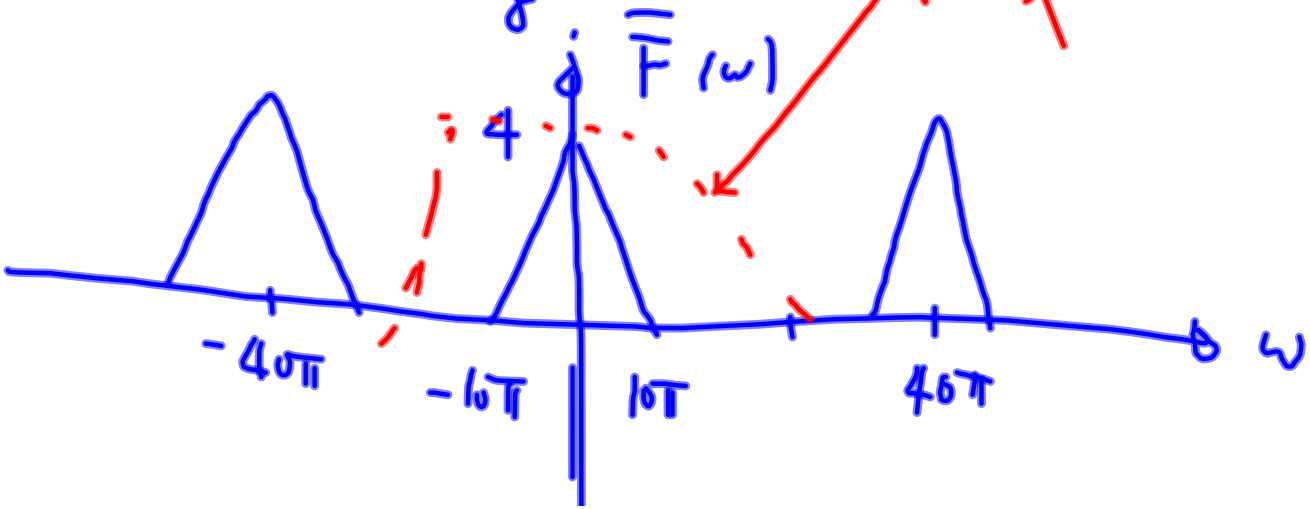
① $F_s = 5\,Hz$



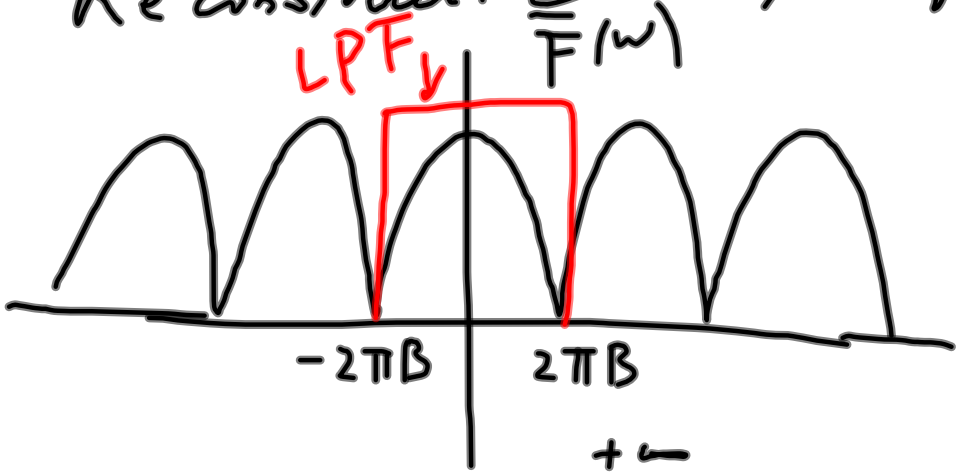
② $F_s = 10\,Hz$ (Nyquist rate)



③ $F_s = 20\,Hz$



② Reconstruction — Interpolation formula



$$\bar{F}(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{+\infty} F(\omega - n\omega_s)$$

$$\therefore \quad F(\omega) = T_s \cdot \bar{F}(\omega) \cdot \mathrm{rect}\left(\frac{\omega}{4\pi B}\right)$$

$$\therefore \quad f(t) = \bar{f}(t) \circledast \boxed{T_s \cdot 2B} \, \mathrm{sinc}(2\pi B t)$$

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$$\therefore \underline{\underline{f(t)}} = \bar{f}(t) \circledast \operatorname{sinc}(2\pi B t)$$

$$= \sum_{n=-\infty}^{+\infty} f(nT_s) \cdot \underbrace{\delta(t - nT_s) \circledast \operatorname{sinc}(2\pi B t)}$$

$$= \sum_{n=-\infty}^{+\infty} f(nT_s) \cdot \operatorname{sinc}\left(2\pi B (t - nT_s)\right)$$

$$= \sum_{n=-\infty}^{+\infty} \underline{\underline{f(nT_s)}} \cdot \operatorname{sinc}(2\pi B t - n\pi)$$

In general:

$$F(\omega) = \bar{F}(\omega) \cdot \underbrace{W(\omega)}_{\text{some filter}}$$

$$\Longrightarrow f(t) = \bar{f}(t) \circledast w(t)$$

$$= \sum_{n=-\infty}^{+\infty} f(nT_s) \cdot \delta(t - nT_s) \circledast w(t)$$

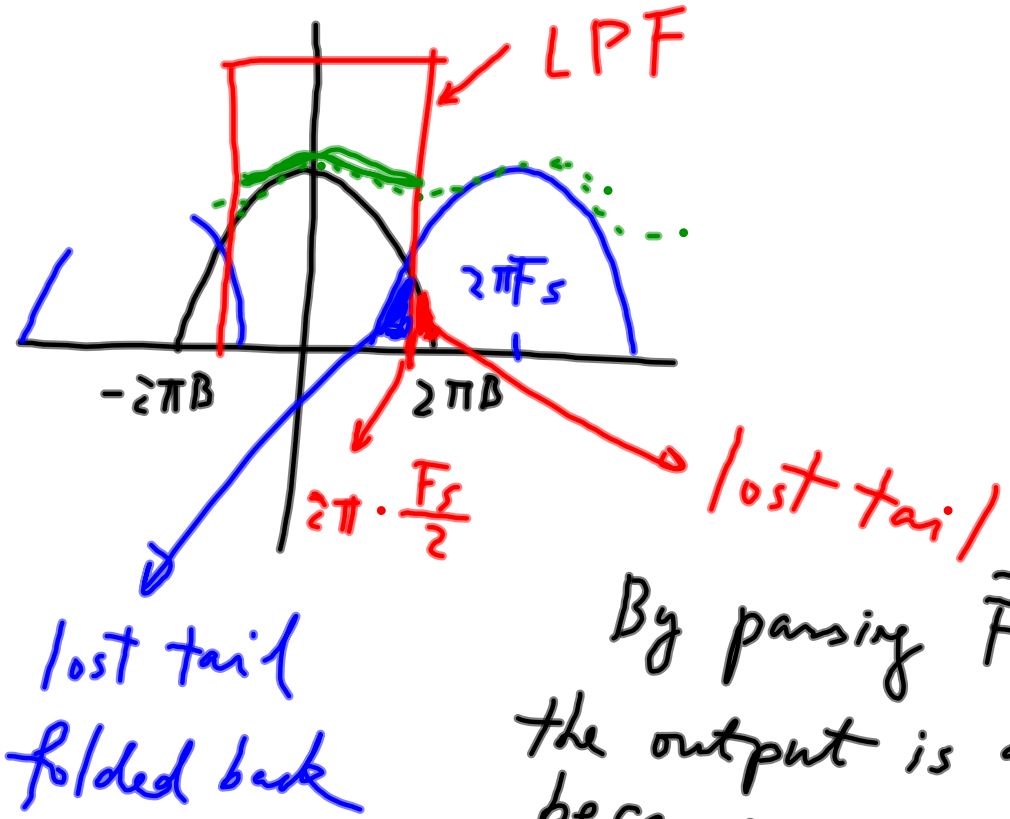
$$= \sum_{n=-\infty}^{+\infty} f(nT_s) \cdot w(t - nT_s)$$

Aliasing

- A signal can not be both time-limited & band-limited
- All practical signals are time-limited and therefore non-band-limited

$\Longrightarrow$ regardless of sampling rate, spectral overlap alway exists.

$\Longrightarrow$ impossible to recover $f(t)$ from $\bar{f}(t)$ perfectly



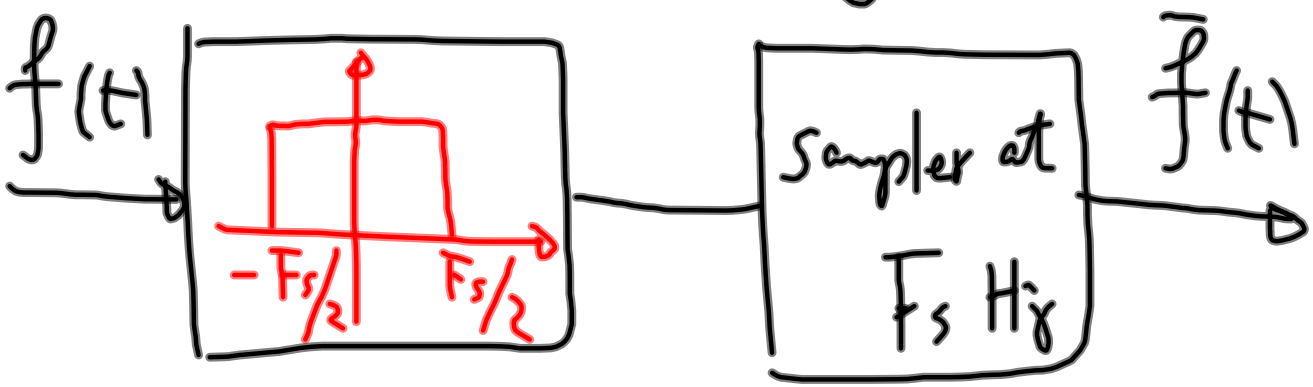
By passing $\bar{F}(\omega)$ through a LPF, the output is a distorted $F(\omega)$ because:

① The loss of the tail of $F(\omega)$ beyond $|F| > F_s/2$

② The reappearance of this tail folded into the spectrum — aliasing

- Anti-aliasing filter

  Apply a LPF with cutoff freq. $\frac{F_s}{2}$ to $f(t)$ <u>before</u> sampling

  — only lose components $> \frac{F_s}{2}$

  — no aliasing