$$\frac{d\,x(t)}{dt} \longleftrightarrow \underline{\underline{(j\omega)}}\, X(\omega)$$

⑨ Time Integration

If $x(t) \longleftrightarrow X(\omega)$

then $$\boxed{\int_{-\infty}^{t} x(\tau)\,d\tau} \longleftrightarrow \frac{X(\omega)}{j\omega} + \pi \cdot X(0) \cdot \delta(\omega)$$

Proof: Note that

$$x(t) * u(t) = \int_{-\infty}^{+\infty} x(\tau) u(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{t} x(\tau)\, d\tau$$

$$\therefore \mathcal{F}\left\{ \int_{-\infty}^{t} x(\tau)\, d\tau \right\}$$

$$= \mathcal{F}\left\{ x(t) * u(t) \right\}$$

$$= X(\omega) \cdot \underbrace{U(\omega)}_{\pi \delta(\omega) + \frac{1}{j\omega}} = \frac{X(\omega)}{j\omega} + \pi \cdot X(0) \cdot \delta(\omega)$$

⑩ Conjugation

for Fourier series

$\rightarrow \quad x(t)^* \longleftrightarrow X_{-n}^*$

If $\quad x(t) \longleftrightarrow X(\omega)$

then $\quad x(t)^* \longleftrightarrow X^*(-\omega)$

Proof: $\quad \mathcal{F}\left\{x(t)^*\right\} = \int_{-\infty}^{+\infty} x(t)^* e^{-j\omega t}\,dt$

$$= \left(\int_{-\infty}^{+\infty} x(t)\, e^{-j\underbrace{(-\omega)}t}\,dt\right)^* = X(-\omega)^*$$

11 Symmetry Properties

a) If $x(t)$ is real, then

$$\underline{X(-\omega) = X^*(\omega)}$$

$$\Longrightarrow \begin{cases} |X(-\omega)| = |X(\omega)|, & \text{magnitude is even} \\ \measuredangle X(-\omega) = -\measuredangle X(\omega). & \text{phase is odd.} \end{cases}$$

b) $x(t)$ is $\underline{\underline{\text{real}}}$ & even,

$\Longrightarrow X(\omega)$ is real & even

<u>Proof:</u> $x(t) = x^*(t) \Longrightarrow X(\omega) = X^*(-\omega)$

$$\Longrightarrow \underline{\underline{X(-\omega) = X^*(\omega)}}$$

$$x(-t) = x(t) \Longrightarrow X(-\omega) = X(\omega)$$

$$\therefore \underbrace{X(\omega) = X^*(\omega)}_{\text{real}} \underbrace{= X(-\omega)}_{\text{even}}$$

c) $x(t)$ is real & odd

$\Longrightarrow X(\omega)$ is imaginary & odd.

d) $$\mathcal{F}\{x_e(t)\} = Re\{X(\omega)\}$$

$$x_e(t) = \frac{1}{2}\left[x(t) + x(-t)\right]$$

$X(-\omega) = X(\omega)^*$

$$\leftrightarrow \frac{1}{2}\left[X(\omega) + X(\omega)^*\right]$$

$$= Re\{X(\omega)\}$$

$$\mathcal{F}\{x_o(t)\} = j \cdot Im\{X(\omega)\}$$

(12) Parseval's relation

for Fourier series:

$$\underbrace{\frac{1}{T}\int_T |x(t)|^2 dt}_{P_x} = \sum_{k=-\infty}^{+\infty} |X_k|^2$$

$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(\omega)|^2 d\omega$$

Proof:

$$\int_{-\infty}^{+\infty} |x(t)|^2 \, dt$$

$$= \int_{-\infty}^{+\infty} x(t) \underbrace{x(t)^*}_{\mathcal{F}^{-1}\{X(-\omega)^*\}} \, dt$$

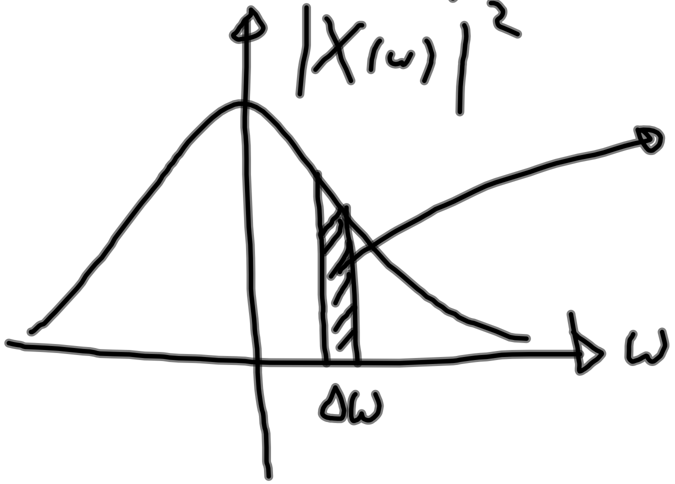
$$= \int_{-\infty}^{+\infty} x(t) \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} X^*(-\omega) e^{j\omega t} \, d\omega \right] dt$$

$$= \int_{-\infty}^{+\infty} x(t) \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} X^*(\omega) e^{-j\omega t} \, d\omega \right] dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(\omega)\underbrace{\left[\int_{-\infty}^{+\infty} x(t)\,e^{-j\omega t}\,dt\right]}_{X(\omega)} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \underbrace{|X(\omega)|^2}_{} d\omega$$

Energy spectral density: $|X(\omega)|^2$



$$\Delta E_x = \frac{1}{2\pi}\cdot |X(\omega)|^2 \cdot \Delta\omega$$

$$= |X(\omega)|^2 \cdot \underbrace{\frac{\Delta\omega}{2\pi}}_{\delta f \text{ Hz}}$$

$$E_x = \frac{1}{2\pi} \cdot \text{total area under } |X(\omega)|^2$$

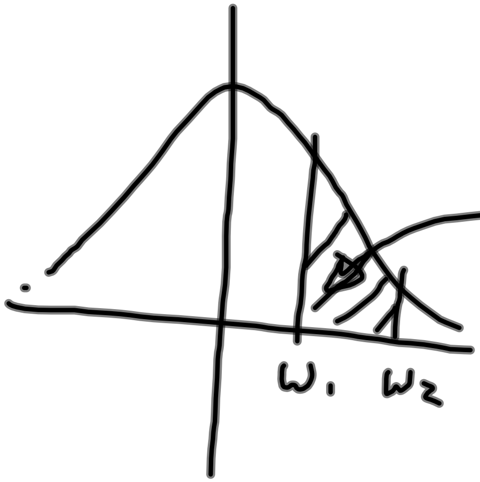
for real valued signals:

$$|X(\omega)|^2 = X(\omega) \cdot \underbrace{X(\omega)^*}_{X(-\omega)}$$

$$\Longrightarrow |X(-\omega)|^2 = X(-\omega) \cdot X(\omega)$$

$$= |X(\omega)|^2$$

$\therefore |X(\omega)|^2$ is even $\Longrightarrow E_x = \frac{1}{\pi}\int_0^{\infty} |X(\omega)|^2 d\omega$



Energy between ω_1 & ω_2 is

$$\frac{1}{\pi}\int_{\omega_1}^{\omega_2} |X(\omega)|^2 \, d\omega$$

<u>Essential bandwidth B</u>: The band that contains certain percentage of the signal energy.

e.g., 95%

Ex:

$$x(t) = e^{-at} u(t)$$

$$E_x = \int_{-\infty}^{+\infty} x(t)^2 \, dt$$

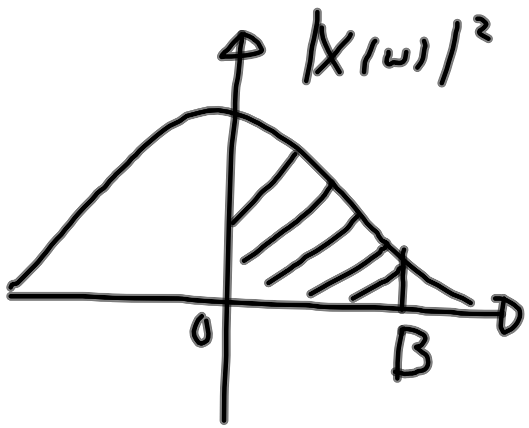
$$= \int_{0}^{\infty} e^{-2at} \, dt = \frac{1}{2a}$$

$$X(\omega) = \frac{1}{a+j\omega} \, , \quad |X(\omega)|^2 = \frac{1}{\omega^2 + a^2}$$

or. $$E_x = \underbrace{\frac{1}{\pi} \int_{0}^{\infty} \frac{1}{\omega^2 + a^2} \, d\omega}_{} = \frac{1}{\pi \cdot a} \tan^{-1}\left(\frac{\omega}{a}\right)\Bigg|_{0}^{\infty}$$

$$= \frac{1}{2a}$$

Bandwidth with 95% energy criterion:



$$\frac{1}{2a} \times 0.95 = \frac{1}{\pi}\int_0^B \frac{1}{\omega^2+a^2}\,d\omega$$

$$= \frac{1}{\pi \cdot a} \tan^{-1}\left(\frac{B}{a}\right)$$

$$B = 12.706\, a \quad \underbrace{\text{rad/s}}$$

$$|X(\omega)|^2 = X(\omega)\cdot X^*(\omega)$$

$$\begin{array}{cc} X(\omega) & X^*(\omega) \\ \updownarrow & \updownarrow \\ x(t) & x(-t)^* \end{array}$$

$$\therefore\ \mathcal{F}\Big\{\underbrace{x(t) * x(-t)^*}_{\text{auto correlation of } x(t)}\Big\} = |X(\omega)|^2$$

$$\psi_x(t) = x(t) * x^*(-t)$$

$$= \int_{-\infty}^{+\infty} x(\tau)\, x(\tau - t)^*\, d\tau$$

Ex: $\mathcal{F}\left\{ f(at-b) \right\}$

Method 1: $\begin{cases} g(t) = f(t-b) \\ \phi(t) = g(at) \end{cases}$

$$\Rightarrow \begin{cases} G(\omega) = F(\omega)\cdot e^{-j\omega b} \\ \Phi(\omega) = \frac{1}{|a|}\cdot G\left(\frac{\omega}{a}\right) \end{cases}$$

$$= \frac{1}{|a|}\cdot F\left(\frac{\omega}{a}\right)\cdot e^{-j\omega\left(\frac{b}{a}\right)}$$

Method 2:

$$\begin{cases} h(t) = f(at) \\ \phi(t) = h\left(t - \frac{b}{a}\right) \end{cases}$$

$$\Rightarrow \begin{cases} H(\omega) = \frac{1}{|a|} F\left(\frac{\omega}{a}\right) \\ \Phi(\omega) = H(\omega)\, e^{-j\omega\left(\frac{b}{a}\right)} \end{cases}$$

$$= \frac{1}{|a|} \cdot F\left(\frac{\omega}{a}\right) e^{-j\omega\left(\frac{b}{a}\right)}$$

Fourier transform for periodic signals

<u>Recall:</u> Fourier series for periodic signal:

$$f(t) = \sum_{n=-\infty}^{+\infty} D_n e^{jn\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}.$$

$$\Longrightarrow F(\omega) = \sum_{n=-\infty}^{+\infty} D_n \underbrace{\mathcal{F}\left\{ e^{jn\omega_0 t} \right\}}_{\mathcal{F}\left\{ 1 \cdot e^{jn\omega_0 t} \right\}}$$

$$2\pi \cdot \delta(\omega - n\omega_0)$$

$$1 \leftarrow 2\pi \cdot \delta(\omega)$$

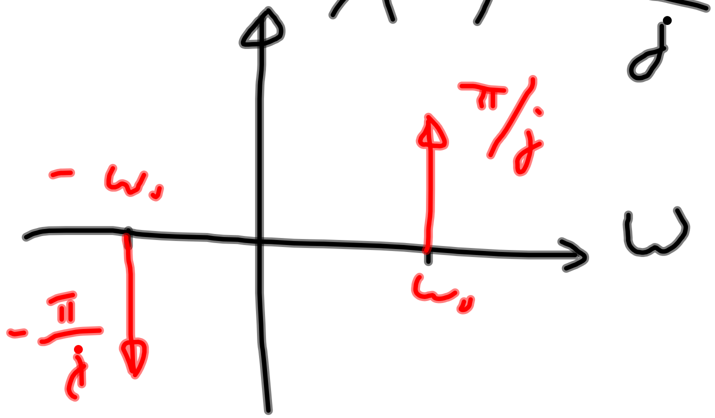
$$\therefore \quad F(\omega) = 2\pi \cdot \sum_{n=-\infty}^{+\infty} D_n \, \delta(\omega - n\omega_0)$$

Ex:

$$x(t) = \sin \omega_0 t$$

$$= \frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)$$

$$\longleftrightarrow \quad X(\omega) = \frac{\pi}{j}\left(\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]$$

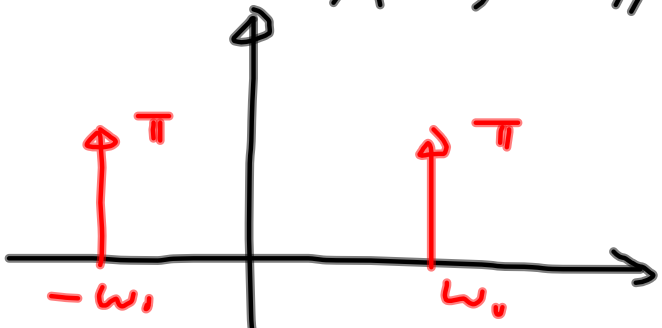


Ex:

$$x(t) = \cos\omega_0 t$$

$$= \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

$$\longleftrightarrow X(\omega) = \pi\cdot\left(\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right).$$



$$y(t) = \cos\omega_1 t \cos\omega_2 t$$

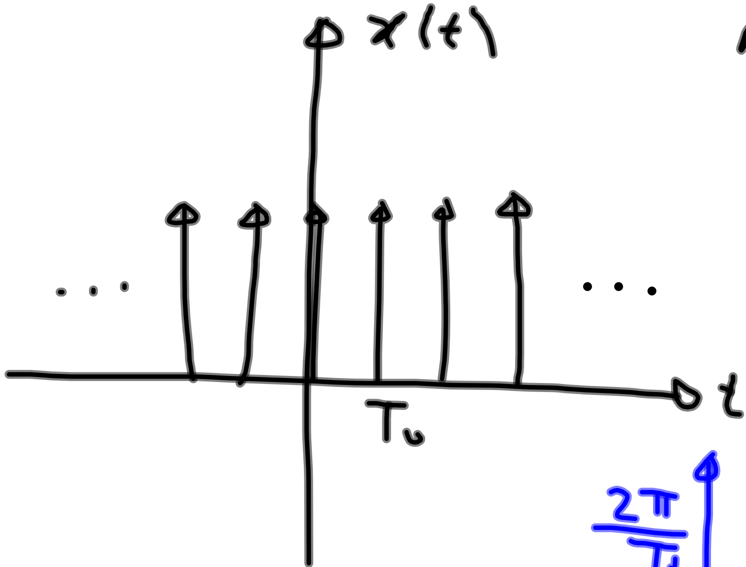
$$\frac{1}{2}\left[e^{j\omega_1 t} + e^{-j\omega_1 t}\right]\cdot$$

$$\frac{1}{2}\left[e^{j\omega_2 t} + e^{-j\omega_1 t}\right]$$

$$= \frac{1}{4}\left[e^{j(\omega_1+\omega_2)t} + e^{-j(\omega_1+\omega_2)t} + e^{j(\omega_1-\omega_2)t} + e^{-j(\omega_1-\omega_2)t}\right]$$

Ex:

$$x(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT_0)$$



Recall: $D_n = \frac{1}{T_0}$

$$X(\omega) = \frac{2\pi}{T_0} \sum_{n=-\infty}^{+\infty} \delta(\omega - n\omega_0)$$