# Properties of Fourier Transform

① $a\,x(t) + b\,y(t) \longleftrightarrow a\,X(\omega) + b\,Y(\omega)$

② $x(t - t_0) \longleftrightarrow X(\omega)\, e^{-j\omega t_0}$

③ $x(t)\, e^{j\omega_0 t} \longleftrightarrow X(\omega - \omega_0)$

$\longrightarrow x(t) \cos \omega_0 t \longleftrightarrow \frac{1}{2}\left[ X(\omega + \omega_0) + X(\omega - \omega_0) \right]$

## ④ Time-frequency duality

If $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$

then $X(t) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\, x(-\omega)$

<u>Proof:</u> $\because\ x(t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty} X(\lambda)\, e^{j\lambda t}\, d\lambda$

$$\Longrightarrow 2\pi \cdot x(-\omega) = \underbrace{\int_{-\infty}^{+\infty} \boxed{X(\lambda)}\, e^{-j\lambda\omega}\, d\lambda}_{\mathcal{F}\{X(t)\}}$$

Ex:

$$\mathcal{F}\left\{ e^{-at}u(t) \right\} = \frac{1}{a+j\omega}$$

$$\Rightarrow \mathcal{F}\left\{ \frac{1}{a+jt} \right\} = 2\pi \cdot e^{a\omega} u(-\omega)$$

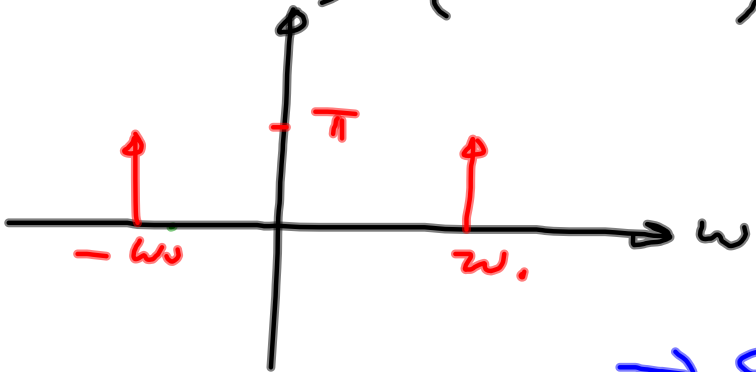
Ex:

$$\mathcal{F}\left\{ e^{-a|t|} \right\} = \frac{2a}{a^2+\omega^2}$$

$$\Rightarrow \mathcal{F}\left\{ \frac{2a}{a^2+t^2} \right\} = 2\pi \cdot e^{-a|\omega|}$$

Ex.: $\mathcal{F}\{\cos \omega_0 t\} = \mathcal{F}\{1 \cdot \cos \omega_0 t\}$

$= \pi \cdot [\delta(\omega+\omega_0) + \delta(\omega-\omega_0)]$




$\delta(t) \longleftrightarrow 1$

$1 \longleftrightarrow 2\pi \cdot \delta(\omega)$

$\Rightarrow \mathcal{F}\{\delta(t+t_0) + \delta(t-t_0)\}$

$= 2 \cos \omega t_0$

⑤ Scaling property

If $x(t) \longleftrightarrow X(\omega)$

then $x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$, a is real.

**Proof:** If $a > 0$

$$\mathcal{F}\{x(at)\} = \int_{-\infty}^{+\infty} x(at) e^{-j\omega t} dt$$

$$\overset{at=\tau}{=\!=\!=} \frac{1}{a} \underbrace{\int_{-\infty}^{\infty} x(\tau) e^{-j\frac{\omega\tau}{a}} d\tau}_{X\left(\frac{\omega}{a}\right)}$$

If $a < 0$

$$\mathcal{F}\{x(at)\} = \int_{-\infty}^{+\infty} x(at)\, e^{-j\omega t}\, dt$$

$$\underset{\displaystyle at = \tau}{=} \frac{1}{a}\int_{\infty}^{-\infty} x(\tau)\, e^{-j\frac{\omega}{a}\tau}\, d\tau$$

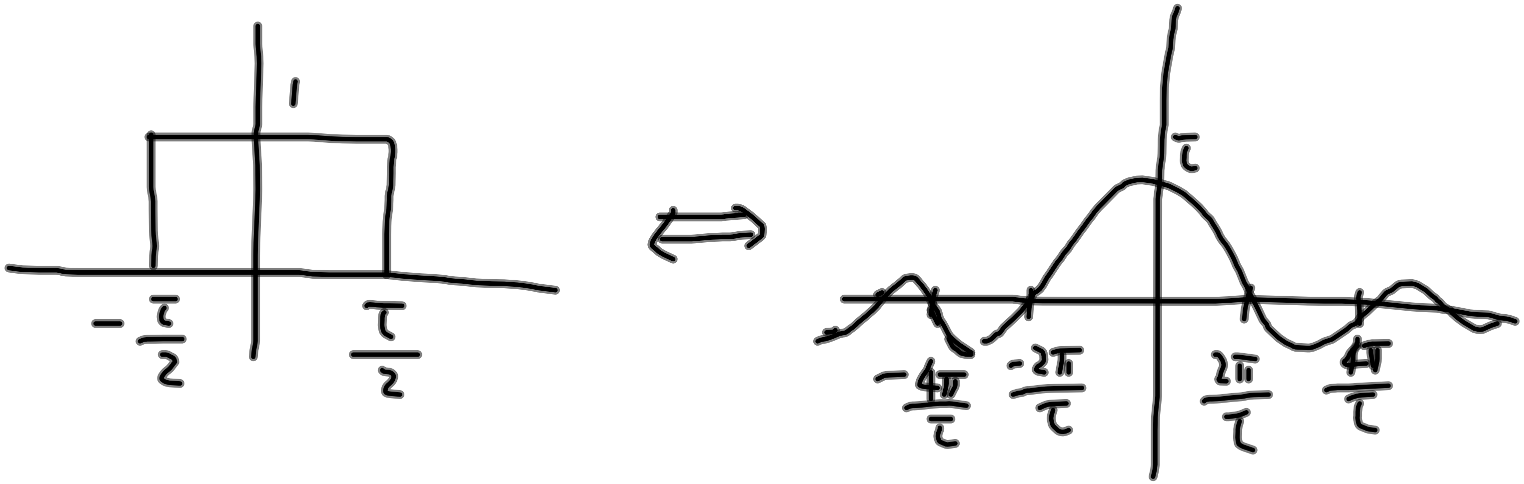
time compression

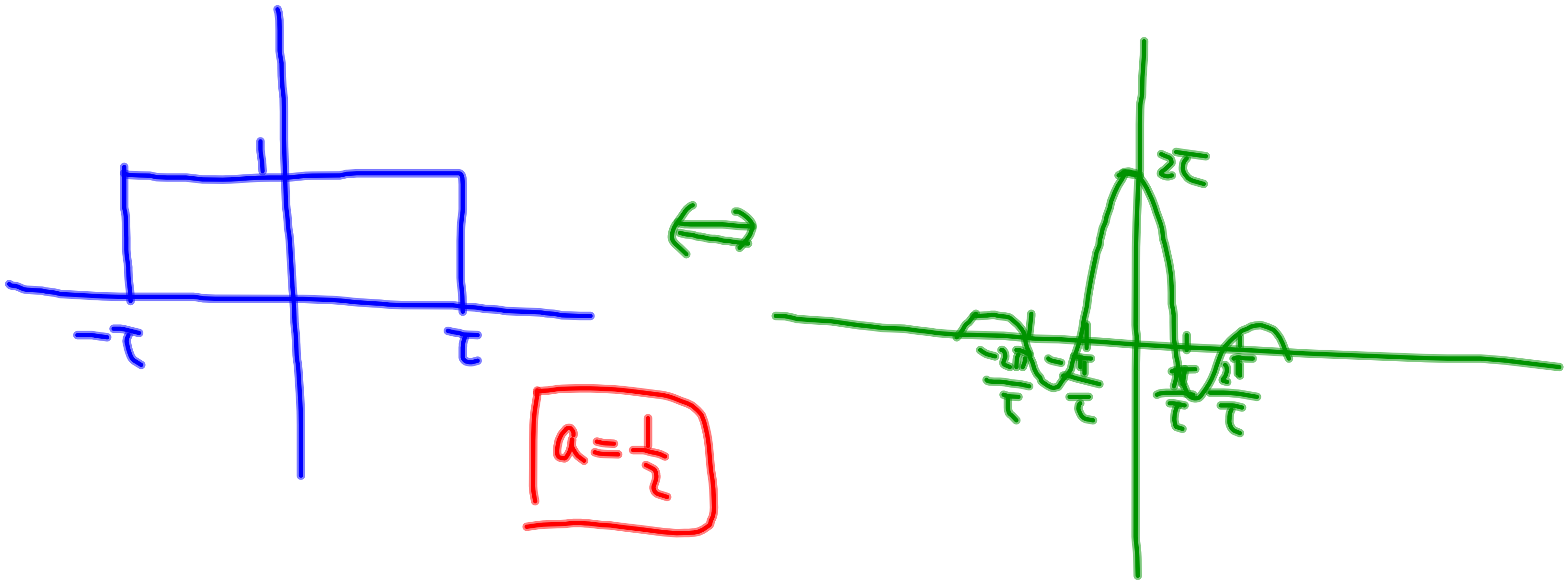
$\Leftrightarrow$ spectral expansion

time expansion

$\longleftrightarrow$ spectral compression

$$= \frac{1}{-a}\underbrace{\int_{-\infty}^{+\infty} x(\tau)\, e^{-j\frac{\omega}{a}\tau}\, d\tau}_{X\left(\frac{\omega}{a}\right)}$$


1
$-\frac{\tau}{2}$
$\frac{\tau}{2}$
$\tau$
$-\frac{4\pi}{\tau}$
$-\frac{2\pi}{\tau}$
$\frac{2\pi}{\tau}$
$\frac{4\pi}{\tau}$


1
$-\tau$
$\tau$
$2\tau$
$-\frac{2\pi}{\tau}$
$-\frac{\pi}{\tau}$
$\frac{\pi}{\tau}$
$\frac{2\pi}{\tau}$
$a = \frac{1}{2}$

## ⑥ Time / Frequency reversal

If $x(t) \longleftrightarrow X(\omega)$

then $x(-t) \longleftrightarrow X(-\omega)$

<u>Proof</u>: Let $a = -1$ in the scaling property.

recall for Fourier series:

$$x(-t) \longleftrightarrow X_{-n}$$

## ⑦ Convolution

Recall for Fourier series:

$$x_1(t) \cdot x_2(t) \longleftrightarrow \sum_{k=-\infty}^{+\infty} X_1(k) X_2(n-k)$$

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If $x_1(t) \longleftrightarrow X_1(\omega)$, $x_2(t) \longleftrightarrow X_2(\omega)$

<u>time convolution:</u> $x_1(t) * x_2(t) \longleftrightarrow X_1(\omega) \cdot X_2(\omega)$

<u>freq. convolution:</u> $x_1(t) \cdot X_2(t) \longleftrightarrow \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$

Proof:

$$\mathcal{F}\left\{ x_1(t) * x_2(t) \right\}$$

$$= \int_{-\infty}^{+\infty} e^{-j\omega t} \left[ \int_{-\infty}^{+\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau \right] dt$$

$$= \int_{-\infty}^{+\infty} x_1(\tau) \left[ \int_{-\infty}^{+\infty} x_2(t-\tau)\, e^{-j\omega t}\, dt \right] d\tau$$

$$\mathcal{F}\left\{ x_2(t-\tau) \right\} = \boxed{X_2(\omega)\, e^{-j\omega\tau}}$$

$$= X_2(\omega) \underbrace{\int_{-\infty}^{+\infty} \underline{x_1(\tau)}\, e^{-j\omega\tau} d\tau}_{\mathcal{F}\{x_1(\tau)\} = X_1(\omega)}$$

$$\mathcal{F}^{-1}\left\{\frac{1}{2\pi} X_1(\omega) * X_2(\omega)\right\}$$

$$= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{+\infty} e^{j\omega t} \left[\int_{-\infty}^{+\infty} X_1(\lambda) X_2(\omega-\lambda)\, d\lambda\right] d\omega$$

$$= \left(\frac{1}{2\pi}\right) \int_{-\infty}^{+\infty} X_1(\lambda) \left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} X_2(\omega-\lambda) e^{j\omega t}\, d\omega\right] d\lambda$$

$$\mathcal{F}^{-1}\{X_2(\omega-\lambda)\} = x_2(t)\, e^{j\lambda t}$$

$$= x_2(t) \cdot \underbrace{\frac{1}{2\pi} \int_{-\infty}^{+\infty} \underline{X_1(\lambda)}\, e^{j\lambda t}\, d\lambda}_{x_1(t)}$$

## ⑧ Time differentiation

If $x(t) \longleftrightarrow X(\omega)$

then $\dfrac{d\,x(t)}{dt} \longleftrightarrow (j\omega) \cdot X(\omega)$

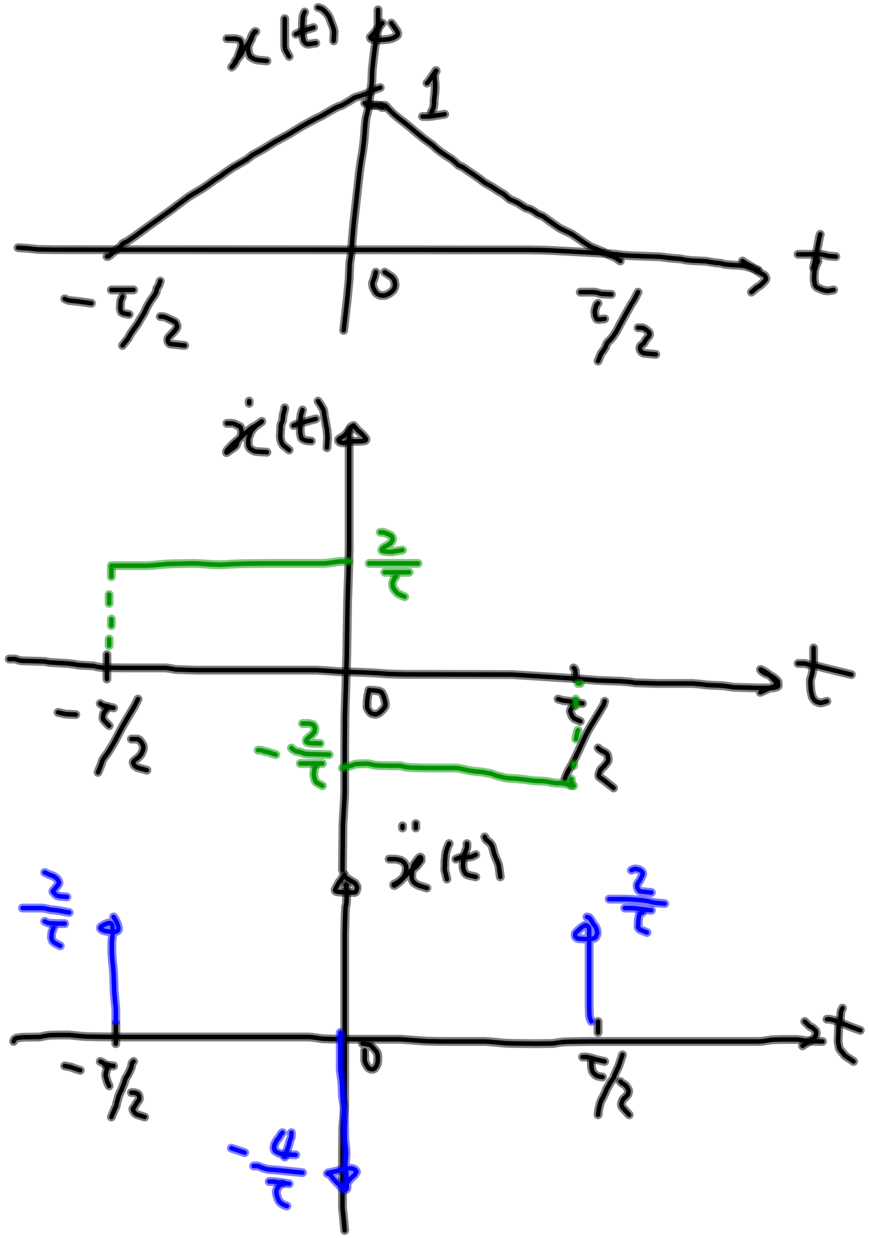
& $\dfrac{d^n x(t)}{dt^n} \longleftrightarrow (j\omega)^n \cdot X(\omega)$

Proof:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$\Longrightarrow \frac{d\,x(t)}{dt} = (j\omega)\cdot \underbrace{\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)\, e^{j\omega}\, d\omega}_{\mathcal{F}^{-1}\{x(t)\}}$$

**Ex:**



$$\ddot{x}(t) \longleftrightarrow (j\omega)^2 X(\omega) = -\omega^2 \cdot X(\omega)$$

But $\mathcal{F}\left\{ \ddot{x}(t) \right\}$

$$= \mathcal{F}\left\{ \frac{2}{\tau}\left[\delta(t+\tau/2) + \delta(t-\tau/2)\right] - \frac{4}{\tau}\delta(t) \right\}.$$

$$= \frac{2}{\tau}\underbrace{\left(e^{j\frac{\tau}{2}\omega} + e^{-j\frac{\tau}{2}\omega}\right)}_{2\cos\left(\frac{\tau\omega}{2}\right)} - \frac{4}{\tau}$$

$1 - \cos x$

$= 2\sin^2\left(\frac{x}{2}\right)$

$$= \frac{4}{\tau}\left(\cos\left(\frac{\omega\tau}{2}\right) - 1\right) = -\frac{8}{\tau}\sin^2\left(\frac{\omega\tau}{4}\right)$$

$$\Rightarrow X(\omega) = \frac{8}{\omega^2 \tau} \sin^2\left(\frac{\omega\tau}{4}\right)$$

$$= \frac{\tau}{2} \cdot \left( \frac{\sin\left(\frac{\omega\tau}{4}\right)}{\frac{\omega\tau}{4}} \right)^2$$

$$= \frac{\tau}{2} \operatorname{sinc}^2\left(\frac{\omega\tau}{4}\right).$$