Ex. We know the following about $x(t)$.

① $x(t)$ is real

② $x(t)$ is periodic w/ $T = 4$

③ $X_n = 0$ for $|n| > 1$

④ The signal $y(t)$ with Fourier coefficients

$$Y_n = e^{-j\frac{\pi}{2}n} X_{-n} \quad \text{is odd}$$

⑤ $\frac{1}{4}\int_4 |x(t)|^2 \, dt = \frac{1}{2}$

Find $x(t)$

② $\Rightarrow \omega_0 = \frac{2\pi}{T} = \frac{\pi}{2}$

③ $\Rightarrow \quad x(t) = X_0 + X_1 e^{j\frac{\pi}{2}t} + X_{-1} e^{-j\frac{\pi}{2}t}$

① $X_{-1} = X_1^* \Rightarrow x(t) = X_0 + 2\,\mathrm{Re}\left\{ X_1 e^{j\frac{\pi}{2}t} \right\}.$

④ $\quad Y_n = e^{-j\frac{\pi}{2}n} \underbrace{X_{-n}}_{x(-t)} \Rightarrow \underline{X_{-n} = Y_n e^{j\frac{\pi}{2}n}}$

$$y(t) = x(-(t-1)) = x(-t+1)$$

$\Rightarrow$ Y_n is imaginary & odd

is odd & real

$$\Rightarrow \begin{cases} Y_0 = 0 \\ Y_{-1} = -Y_1 \end{cases}$$

(5)

$$\frac{1}{4}\int_4 |x(t)|^2\,dt = 1/2$$

$$\Rightarrow \underbrace{\frac{1}{4}\int_4 |y(t)|^2\,dt}_{\parallel} = 1/2$$

$$\underbrace{|Y_0|^2}_{0} + |Y_1|^2 + |Y_{-1}|^2 = 1/2.$$

$Y_{-1} = -Y_1$. Y_1 is imaginary

$$2|Y_1|^2 = 1/2$$

$$\Rightarrow |Y_1|^2 = 1/4 \Rightarrow |Y_1| = \frac{1}{2}$$

$$\Rightarrow \begin{cases} Y_1 = \frac{j}{2} \\ Y_{-1} = -\frac{j}{2} \end{cases}$$

$$\begin{cases} Y_1 = -\frac{j}{2} \\ Y_{-1} = \frac{j}{2} \end{cases}$$

$$\therefore \quad X_{-n} = Y_n \cdot e^{j\frac{\pi}{2}n} \Rightarrow X_n = Y_{-n} e^{-j\frac{\pi}{2}n}$$

$$\Rightarrow X_0 = Y_0 = 0$$

(A) $\begin{cases} X_1 = e^{-j\frac{\pi}{2}1} Y_{-1} = -1/2 \\ X_{-1} = e^{+j\frac{\pi}{2}} Y_1 = -1/2 \end{cases} \Rightarrow x(t) = -\frac{1}{2}\left[e^{j\frac{\pi}{2}t} + e^{-j\frac{\pi}{2}t}\right)$

$$= -\cos\left(\frac{\pi}{2}t\right)$$

(B) $\Rightarrow x(t) = \cos\left(\frac{\pi}{2}t\right).$

Fourier Transform:

$$\mathcal{F}\{u(t)\} = \pi \cdot \delta(\omega) + \frac{1}{j\omega}$$

§ 3.3. Properties of Fourier Transform

① linearity:

$$x(t) \longleftrightarrow X(\omega)$$
$$y(t) \longleftrightarrow Y(\omega)$$

$$\Longrightarrow a \cdot x(t) + b \cdot y(t) \longleftrightarrow a \cdot X(\omega) + b \cdot Y(\omega)$$

② Time-shifting

<u>Recall:</u> for Fourier series.

$$x(t-t_0) \longleftrightarrow X_n \cdot e^{-j\underbrace{n\omega_0}_{\omega}t_0}$$

$$x(t) \longleftrightarrow X(\omega)$$

$$x(t-t_0) \longleftrightarrow \underbrace{X(\omega)\cdot e^{-j\omega t_0}}_{|X(\omega)|\cdot e^{j(\measuredangle X(\omega)\underbrace{-\omega t_0}_{\text{linear phase shift}})}}$$

Proof:

$$\mathcal{F}\{x(t-t_0)\}$$

$$= \int_{-\infty}^{+\infty} x(t-t_0)\, e^{-j\omega t}\, dt$$

$$\underline{\underline{\tau = t - t_0}} \quad \int_{-\infty}^{+\infty} x(\tau)\, e^{-j\omega(\tau + t_0)}\, d\tau$$

$$= e^{-j\omega t_0} \underbrace{\int_{-\infty}^{+\infty} x(\tau)\, e^{-j\omega\tau}\, d\tau}_{X(\omega)}.$$

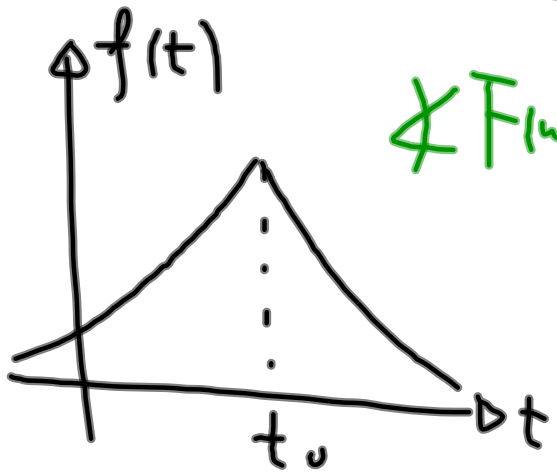
time-shift $\longleftrightarrow$ $\begin{cases} \text{magnitude unchanged} \\ \text{linear phase shift} \end{cases}$

Ex:

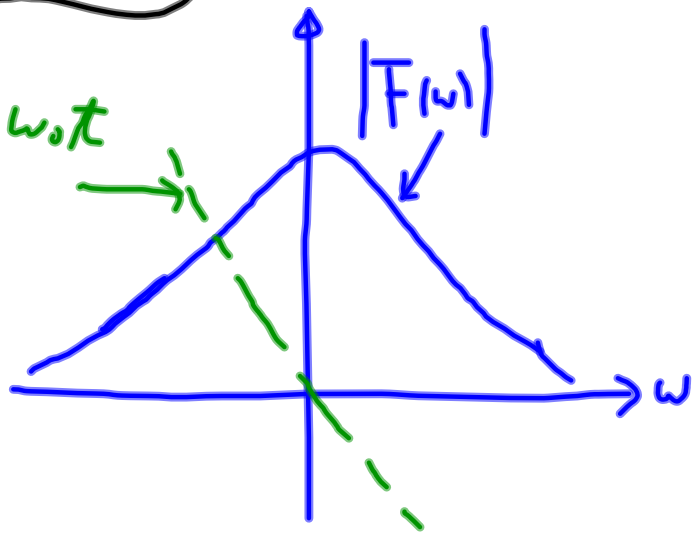
$$\mathcal{F}\left\{ e^{-a|t-t_0|} \right\}$$

$$\longleftrightarrow \frac{2a}{a^2+\omega^2} \cdot \underbrace{e^{-j\omega t_0}}$$

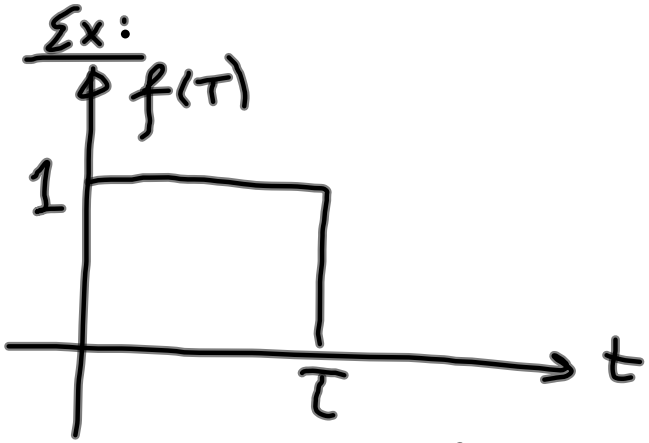
$$e^{-a|t|} \leftrightarrow \frac{2a}{a^2+\omega^2}$$

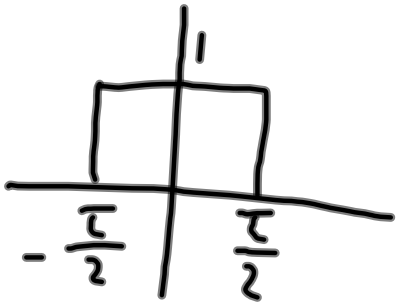


$\measuredangle F(\omega) = -\omega_0 t$



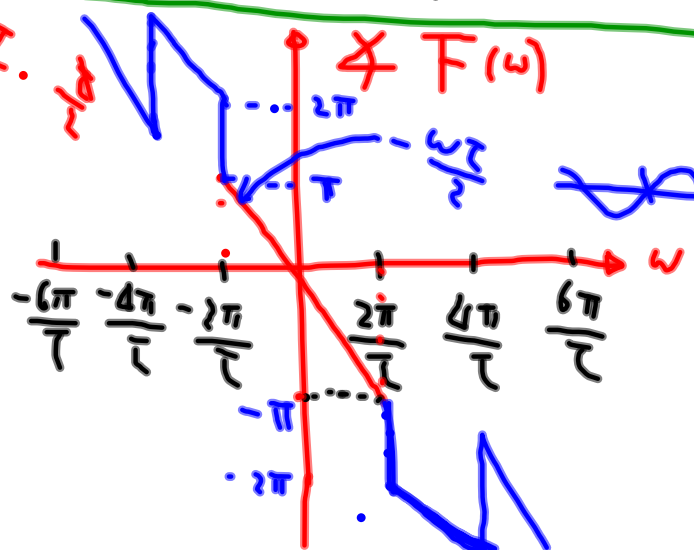
Ex:

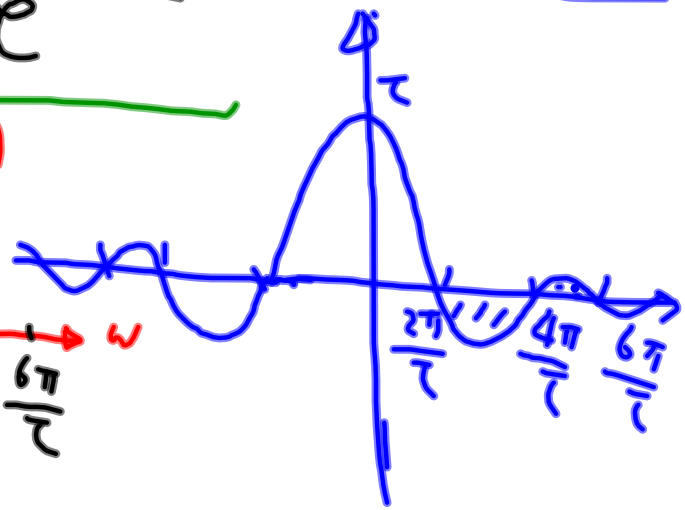




$$\Updownarrow \qquad\qquad \Updownarrow$$

$$F(\omega) = \tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2}\right)\cdot e^{-j\frac{\omega\tau}{2}} \qquad \tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2}\right)$$





③ Frequency-shifting

$$x(t) \longleftrightarrow X(\omega)$$

$$x(t)e^{j\omega_0 t} \longleftrightarrow X(\omega-\omega_0)$$

<u>Proof:</u>

$$\mathcal{F}\left\{x(t)e^{j\omega_0 t}\right\}$$

$$= \int_{-\infty}^{+\infty} x(t)e^{j\omega_0 t}\cdot e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{+\infty} x(t)e^{-j\underline{\underline{(\omega-\omega_0)t}}}\,dt$$

$$= X(\omega-\omega_0)$$

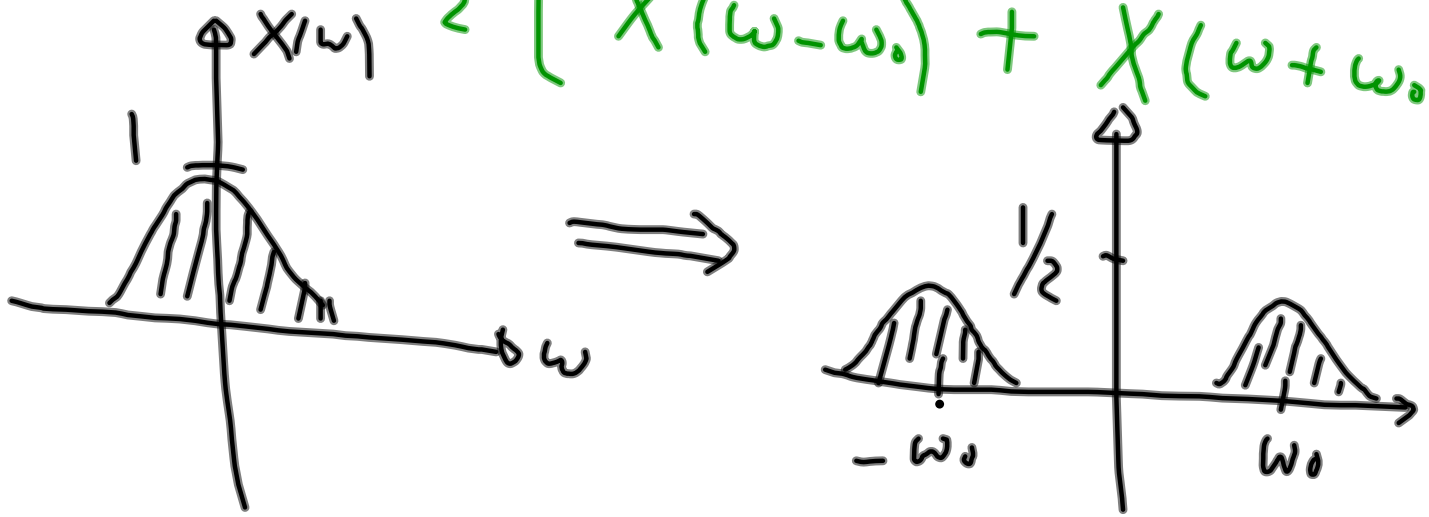
Ex: Amplitude Modulation (AM).

$$x(t)\cos\omega_0 t$$

$$= x(t)\cdot\frac{1}{2}\left[e^{j\omega_0 t}+e^{-j\omega_0 t}\right].$$

$$= \frac{1}{2}\left[x(t)e^{j\omega_0 t}+x(t)e^{-j\omega_0 t}\right]$$

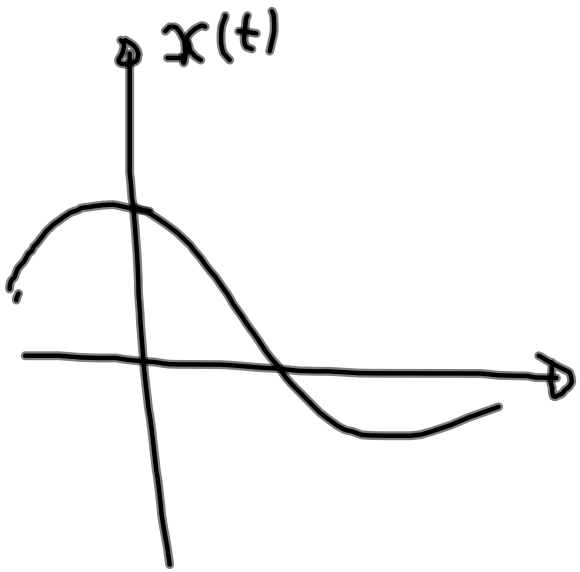
$$\longleftrightarrow \frac{1}{2}\left[X(\omega-\omega_0)+X(\omega+\omega_0)\right]$$

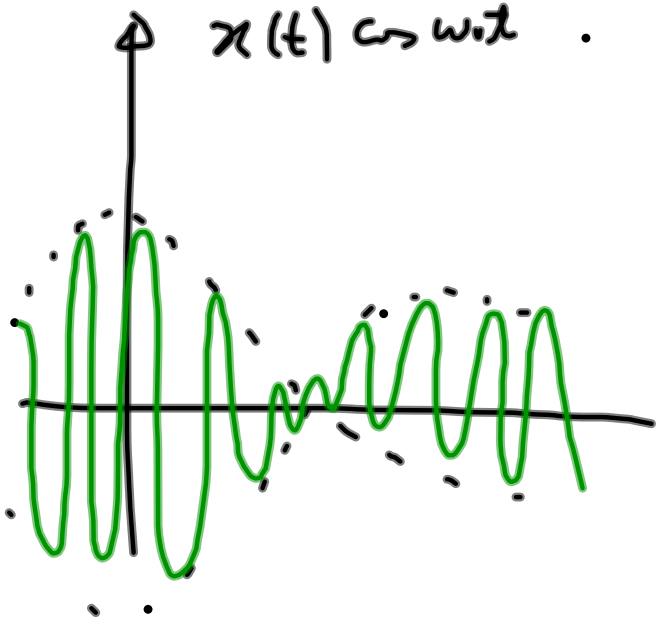


$x(t)$: modulating signal (message)

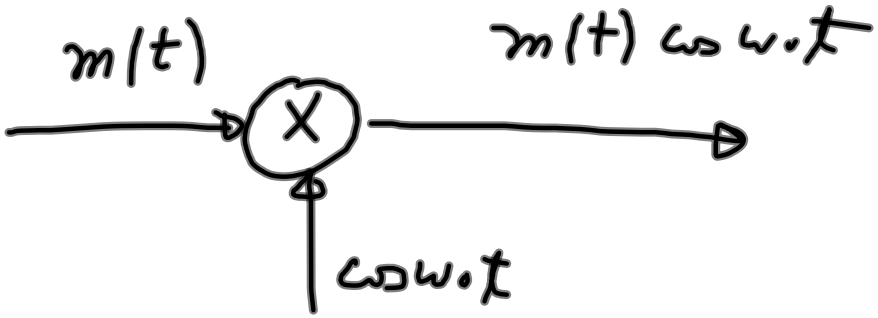
$\cos \omega_c t$: carrier

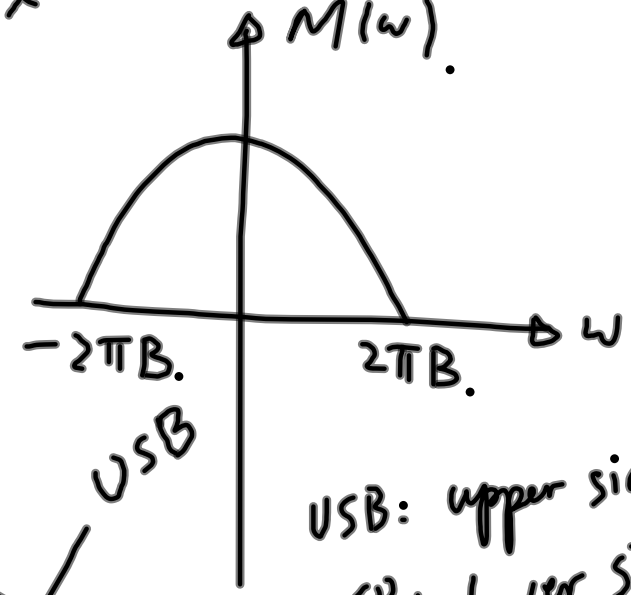
$x(t) \cdot \cos \omega_c t$: modulated signal

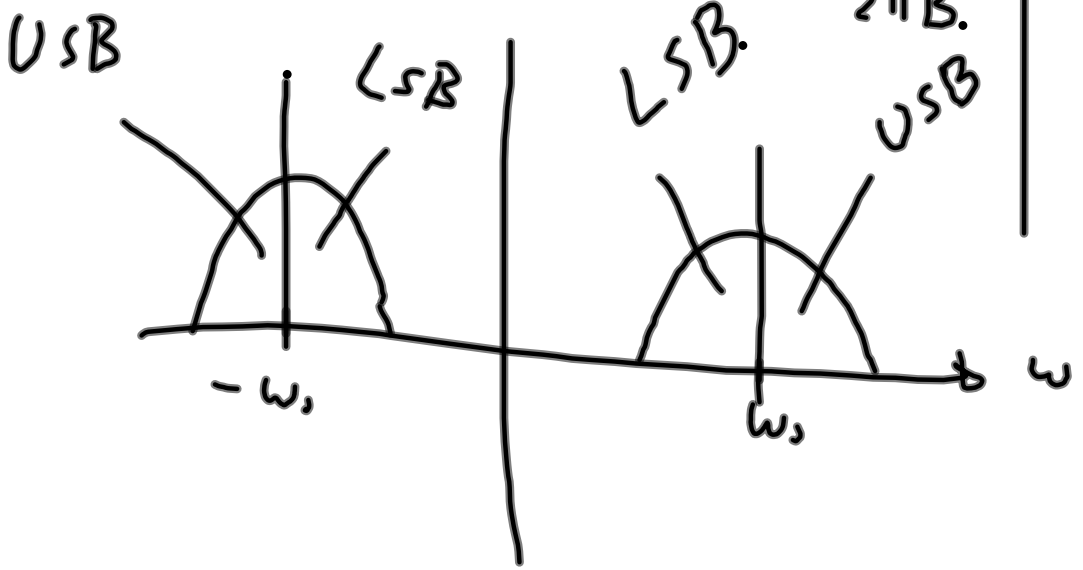




Double-sideband suppressed carrier (DSB-SC) modulation





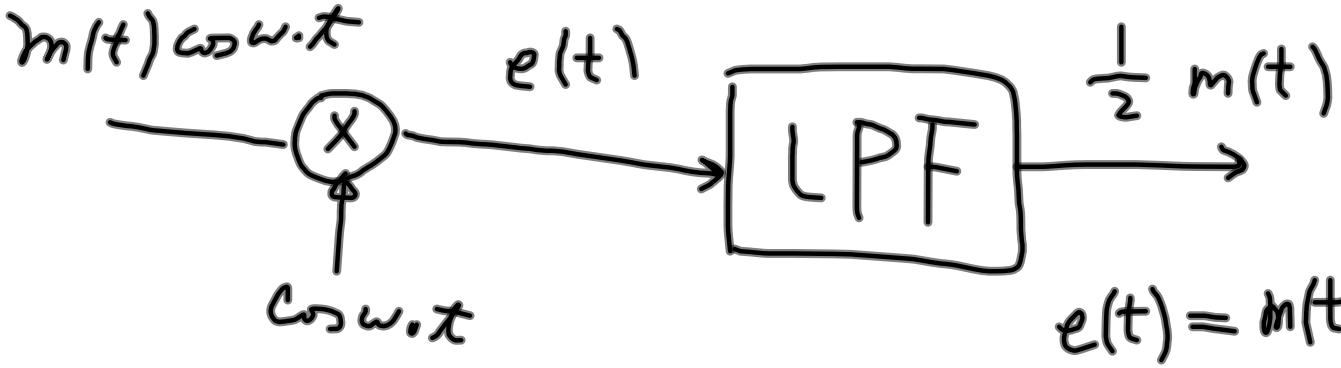


USB: upper sideband

LSB: lower sideband

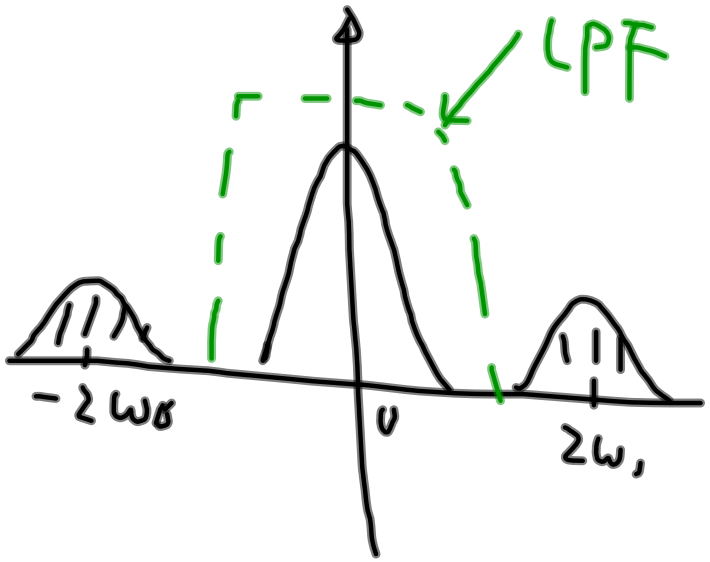
DSB.

demodulation:



$$e(t) = m(t)\cos^2 w_0 t$$

$$= m(t)\,\frac{1}{2}\left[1+\cos 2w_0 t\right]$$



$$E(w) = \frac{1}{2}M(w)$$

$$+\frac{1}{4}\left[M(w+2w_0)+M(w-2w_0)\right]$$