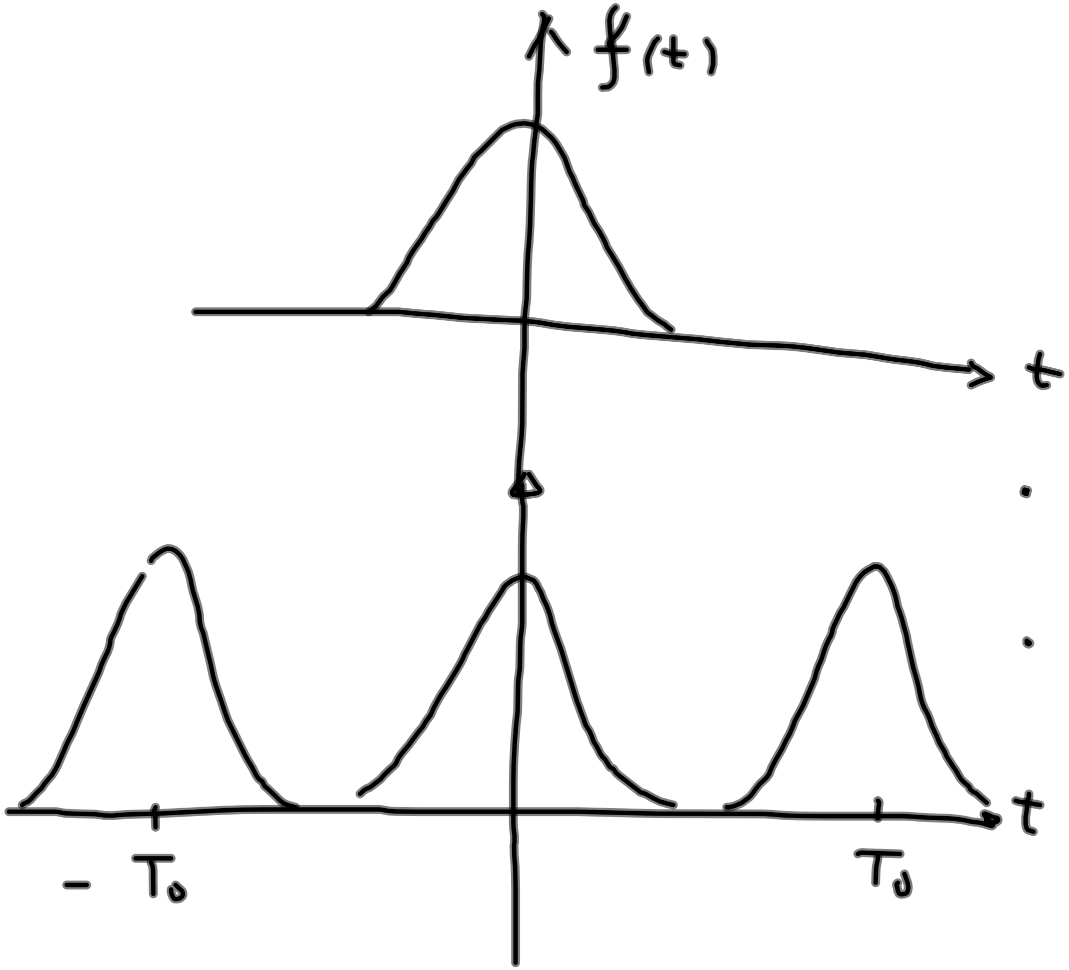
Chapter 3. Fourier Transform & Frequency-domain Analysis

§ 3.1. Fourier Transform

① Aperiodic Signal Representation by Fourier Integral



- Aperiodic signal $f(t)$
- Construct a periodic signal $f_{T_0}(t)$ by repeating $f(t)$ every T_0
- Let $T_0 \to \infty$

$$\lim_{T_0 \to \infty} f_{T_0}(t) = f(t)$$

Fourier Series of $f_{T_0}(t)$ ω

$$f_{T_0}(t) = \sum_{n=-\infty}^{+\infty} D_n \underbrace{e^{j n\omega_0 t}}, \quad \omega_0 = \frac{2\pi}{T_0}$$

$$D_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f_{T_0}(t)\, e^{-jn\omega_0 t}\, dt$$

as $T_0 \to \infty$:

$$\omega_0 = \frac{2\pi}{T_0} \longrightarrow 0 \qquad \omega_0 \Leftarrow \Delta\omega$$

$$\therefore \int_{-T_0/2}^{T_0/2} f_{T_0}(t)\, e^{-jn\omega_0 t}\, dt$$

$$\longrightarrow \int_{-\infty}^{\infty} f(t)\, e^{-j\boxed{n\Delta\omega}t}\, dt = F(n\Delta\omega)$$

where $$F(\omega) \stackrel{def}{=} \int_{-\infty}^{+\infty} f(t)\, e^{-j\omega t}\, dt$$

$$\Rightarrow \quad D_n = \frac{1}{T_0} F(n \cdot \Delta\omega) = \frac{\Delta\omega}{2\pi} \cdot F(n \cdot \Delta\omega)$$

$$\therefore f_{T_0}(t) = \sum_{n=-\infty}^{+\infty} \underbrace{\frac{F(n\cdot\Delta\omega)\cdot\Delta\omega}{2\pi}}_{D_n} e^{j n \cdot \Delta\omega \cdot t}$$

$$\Rightarrow f(t) = \lim_{T_0 \to \infty} f_{T_0}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega)\, \boxed{e^{j\omega t}}\, d\omega$$

Fourier transform:

$$F(\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{+\infty} f(t)\, e^{-j\omega t}\, dt$$

Inverse Fourier transform

$$f(t) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega)\, e^{j\omega t}\, d\omega$$

② Sufficient conditions for existence of Fourier transform

- Weak condition:

let $\hat{f}(t) \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) e^{j\omega t} \, d\omega$

$$e(t) = f(t) - \hat{f}(t)$$

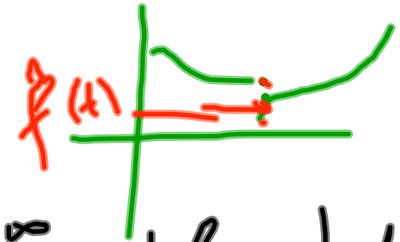
Theorem 1: If $f(t)$ has finite energy

i.e. $\int_{-\infty}^{+\infty} |f(t)|^2 dt < \infty$

then $F(\omega)$ exists & $\int_{-\infty}^{+\infty} |e(t)|^2 dt = 0$.

$\longrightarrow$ $f(t)$ & $\hat{f}(t)$ may differ at discrete values of t. But the energy difference is 0.

• Strong condition



Theorem 2: If

a) $\int_{-\infty}^{+\infty} |f(t)|\, dt < \infty$

b) $f(t)$ has a finite # of max & min within any finite interval.

c) $f(t)$ has a finite # of discontinuities within any interval; each of these discontinuities are finite

Then $f(t) = \hat{f}(t)$ for every t, except at a discontinuity, where $\hat{f}(t)$ equal to the avg of the values on two sides of discontinuity.

§ 3.2. Some useful Transforms

① $\delta(t) \xleftrightarrow{\mathcal{F}} 1$

$$\mathcal{F}\{\delta(t)\} = \int_{-\infty}^{+\infty} \delta(t) e^{-j\omega t} dt = 1.$$

② $1 \xleftrightarrow{\mathcal{F}} 2\pi\delta(\omega)$

$$\mathcal{F}^{-1}\{2\pi \cdot \delta(\omega)\} = 2\pi \cdot \frac{1}{2\pi} \int_{-\infty}^{+\infty} \delta(\omega) e^{j\omega t} d\omega = 1.$$

③ $e^{-at}u(t)$, $a>0$.

$$\xleftrightarrow{\mathcal{F}} \frac{1}{a+j\omega}$$

$$\mathcal{F}\left\{e^{-at}u(t)\right\} = \int_{-\infty}^{+\infty} e^{-at}\underline{u(t)}\,e^{-j\omega t}\,dt$$

$$= \int_{0}^{\infty} e^{-(a+j\omega)t}\,dt$$

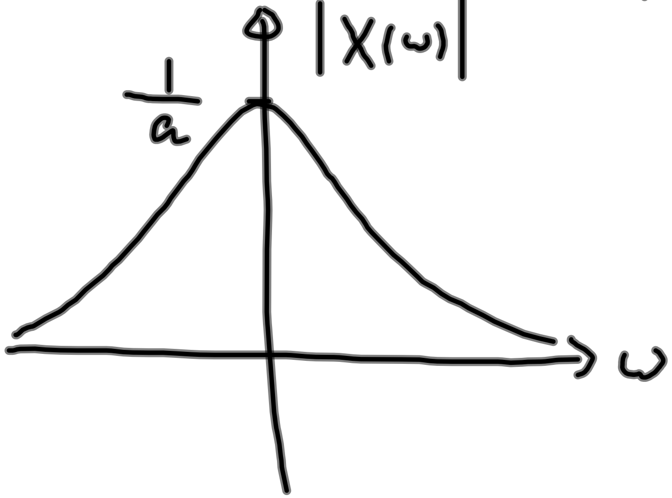
$$= -\frac{1}{a+j\omega}\,e^{-(a+j\omega)t}\Big|_{0}^{\infty}$$

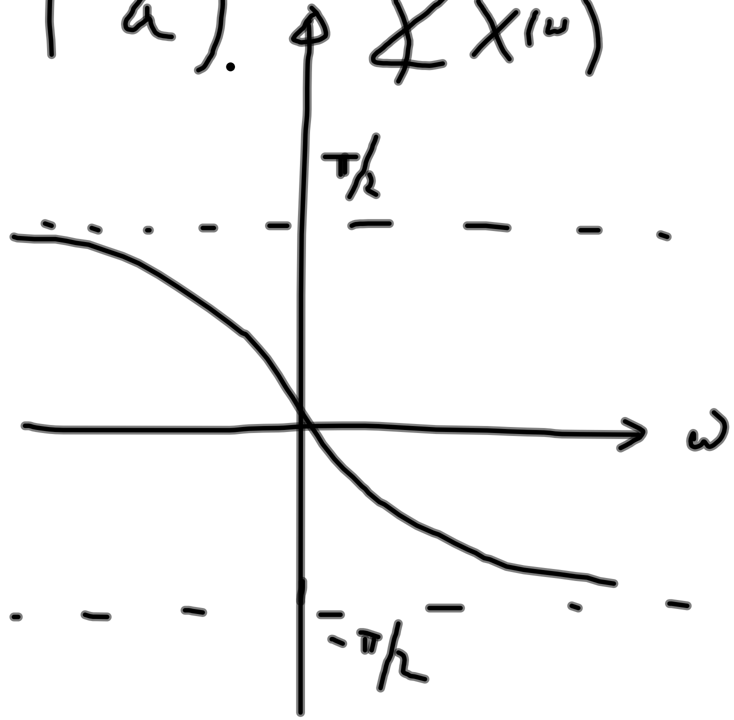
$$= \frac{1}{a+j\omega}$$

$$X(\omega) = \frac{1}{a+j\omega} = \frac{a-j\omega}{a^2+\omega^2}$$

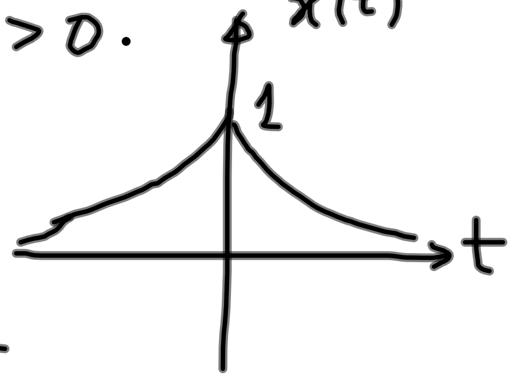
$$|X(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}.$$

$$\measuredangle X(\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right).$$





④ $x(t) = e^{-a|t|}$ $\quad a > 0.$



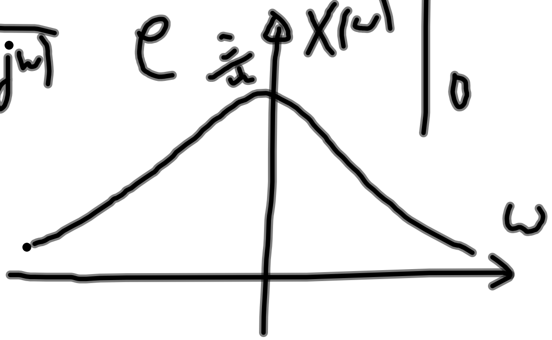
$$\overset{\mathcal{F}}{\longleftrightarrow} \frac{2a}{\omega^2+a^2}$$

$$X(\omega) = \int_{-\infty}^{+\infty} e^{-a|t|} e^{-j\omega t}\, dt$$

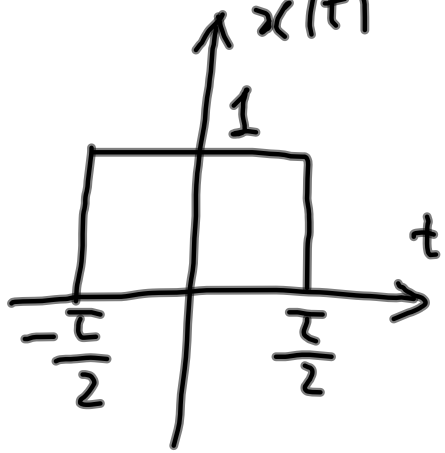
$$= \int_{-\infty}^{0} e^{(a-j\omega)t}\, dt + \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$

$$= \frac{1}{a-j\omega} e^{(a-j\omega)t}\Big|_{-\infty}^{0} + \frac{1}{-(a+j\omega)} e^{-(a+j\omega)t}\Big|_{0}^{\infty}$$

$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{2a}{\omega^2+a^2}$$



(5) $$x(t) = \begin{cases} 1 & |t| < \frac{\tau}{2} \\ 0, & |t| > \frac{\tau}{2} \end{cases}$$



$$\xleftrightarrow{\mathcal{F}} \tau \cdot \mathrm{sinc}\left(\frac{\omega\tau}{2}\right).$$

$$X(\omega) = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{-j\omega t}\, dt = \frac{1}{-j\omega}\, e^{-j\omega t}\Big|_{-\tau/2}^{\tau/2}$$

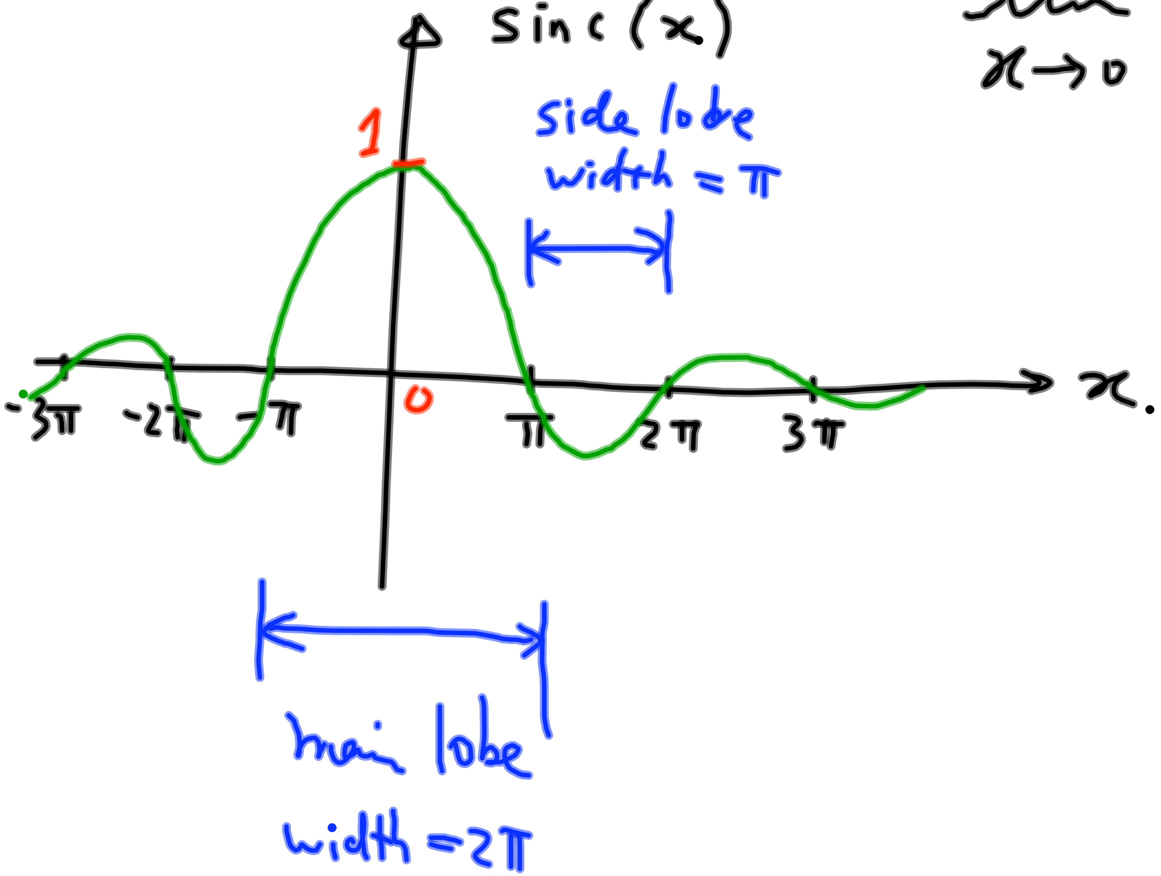
$$= \frac{e^{j\frac{\omega\tau}{2}} - e^{-j\frac{\omega\tau}{2}}}{j\omega} = \frac{2\cancel{j}\sin\left(\frac{\omega\tau}{2}\right)}{\cancel{j}\omega} = \tau \cdot \boxed{\frac{\sin\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)}}$$

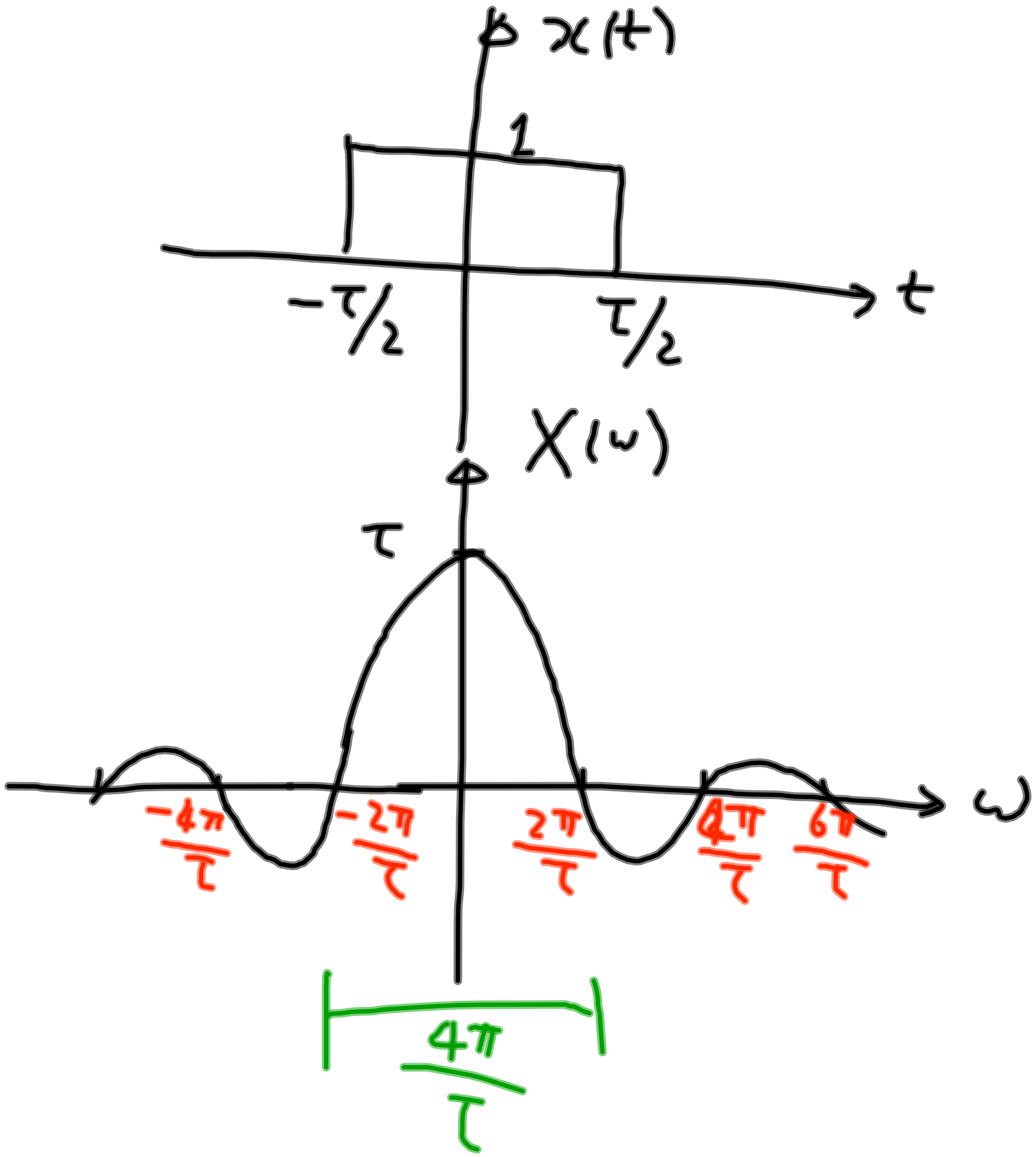
$$= \tau \cdot \mathrm{sinc}\left(\frac{\omega\tau}{2}\right)$$

$$\mathrm{sinc}(x) = \frac{\sin x}{x}.$$

$$\lim_{x \to 0} \frac{\sin x}{x} = \left.\frac{\cos x}{1}\right|_{x=0}$$

$$= 1.$$





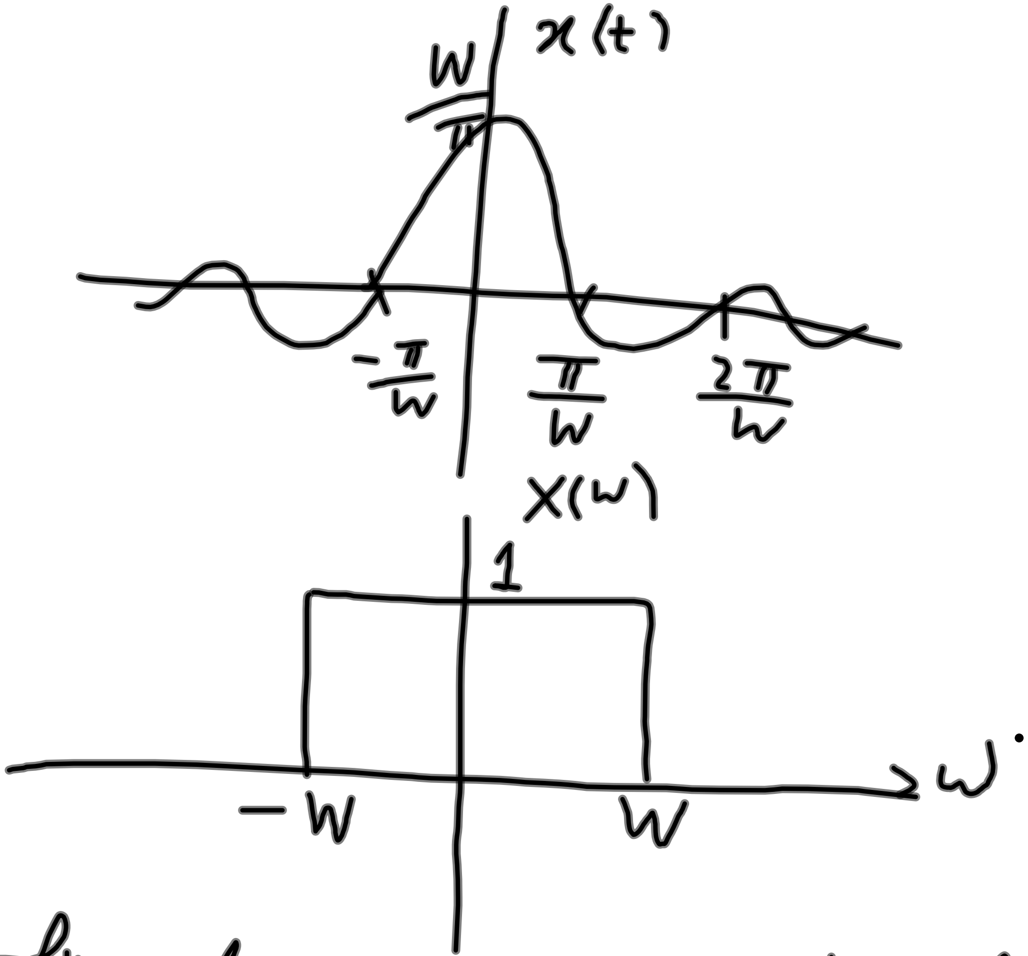
x(t)
1
−τ/2
τ/2
t
X(ω)
τ
ω
−4π/τ
−2π/τ
2π/τ
4π/τ
6π/τ
4π/τ

(6) $$x(t) = \frac{W}{\pi}\,\mathrm{sinc}(Wt)$$

$$\overset{\mathcal{F}}{\longleftrightarrow}\ X(\omega) = \begin{cases} 1 & |\omega| < W \\ 0 & |\omega| > W \end{cases}$$

$$x(t) = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\, d\omega$$

$$= \frac{\sin Wt}{\pi t} = \frac{W}{\pi}\,\mathrm{sinc}(Wt).$$



freq-domain wider $\longleftrightarrow$ time-domain narrower