② Time shifting

$$x(t) \longleftrightarrow X_n$$

$$\Rightarrow x(t-t_0) \longleftrightarrow \underline{\underline{e^{-jn\omega_0 t_0} X_n}}$$

Proof: let $z(t) = x(t-t_0)$

then $$Z_n = \frac{1}{T_0}\int_{T_0} z(t)e^{-jn\omega_0 t}\,dt$$

$$= \frac{1}{T_0}\int_{T_0} x(t-t_0)e^{-jn\omega_0 t}\,dt$$

$$\overset{t-t_0=\tau}{=\!=\!=} \frac{1}{T_0}\int_{T_0} x(\tau)e^{-jn\omega_0(\tau+t_0)}\,d\tau$$

$$= e^{-jn\omega_0 t_0}\cdot \underbrace{\frac{1}{T_0}\int_{T_0} x(\tau)e^{-jn\omega_0\tau}\,d\tau}_{X_n}$$

$$x(t - t_0) \longleftrightarrow \begin{array}{l} \text{magnitude: } |X_n| \\ \text{phase: } \measuredangle X_n \underbrace{- n\omega_0 t_0}_{\text{linear phase shift}} \end{array}$$

time shift $\longleftrightarrow$ linear phase shift in freq

③ Time reversal

$$x(t) \longleftrightarrow X_n$$

$$\Longrightarrow x(-t) \longleftrightarrow X_{\boxed{-n}}$$

time reversal $\longleftrightarrow$ frq. reversal.

<u>Proof:</u> Let $z(t) = x(-t)$

then
$$Z_n = \frac{1}{T_0} \int_{T_0} z(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(-t)\, e^{-jn\omega_0 t}\, dt$$

$$\overset{\tau = -t}{=\!=} \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x(\tau)\, e^{-j(-n)\omega_0 \tau}\, d(+\tau)$$

$$= X_{-n}$$

$x(t)$ is even : $x(-t) = x(t)$

$\Longleftrightarrow X_n$ is even $\qquad X_{-n} = X_n$

but $z(t) = x(-t)$

$x(t)$ is odd

$\Longleftrightarrow X_n$ is odd

<u>Ex:</u> $x(t) \longleftrightarrow X_n$.

$T_0 = 4$

$\omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$. $\quad ? \longleftrightarrow \underbrace{e^{-j\frac{\pi}{2}n}}_{e^{-j\omega_0 t_0 n}} \underbrace{X_{-n}}_{Z_n} \longleftrightarrow z(t-1)$

$\therefore t_0 = 1$

$= x(-t+1)$

④ Time scaling

If $x(t)$ is periodic w/ period $T_0 \leftrightarrow \omega_0$

then $x(\alpha t)$ is periodic w/ period $T_0/\alpha \leftrightarrow \alpha \cdot \omega_0$

$$\Longrightarrow \text{If } x(t) = \sum_{n=-\infty}^{+\infty} X_n e^{jn\omega_0 t}$$

$$\text{then } x(\alpha t) = \sum_{n=-\infty}^{+\infty} X_n e^{jn(\omega_0 \alpha)t}$$

$\therefore$ Fourier coeff's remain the same.

But Fourier series representation has changed.
due to the change of fundamental freq.

5 Multiplication

both w/ period T_0
$$x(t) \longleftrightarrow X_n$$
$$y(t) \longleftrightarrow Y_n$$

$$z(t) = x(t)\,y(t)$$

$$\longleftrightarrow Z_n = \underbrace{\sum_{\ell=-\infty}^{+\infty} X_\ell \, Y_{n-\ell}}_{\text{discrete convolution}}$$

$$X_n * Y_n$$

Proof:

$$Z_n = \frac{1}{T_0}\int_{T_0} z(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T_0}\int_{T_0} x(t)\, y(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T_0}\int_{T_0} \sum_{l=-\infty}^{+\infty} X_l\, e^{jl\omega_0 t} \sum_{k=-\infty}^{+\infty} Y_k\, e^{jk\omega_0 t}\, e^{-jn\omega_0 t}\, dt$$

$$= \sum_{l=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} X_l\, Y_k \cdot \underbrace{\frac{1}{T_0}\int_{T_0} e^{j\underline{(l+k-n)}\omega_0 t}\, dt}_{= \begin{cases} 1 & \text{if } l+k=n \\ 0 & \text{o/w} \end{cases}}$$

$$= \sum_{l=-\infty}^{+\infty} X_l\, Y_{n-l}$$

⑥ Conjugation

$$x(t) \longleftrightarrow X_n$$

$$\longleftrightarrow \quad x(t)^* \longleftrightarrow X_{-n}^*$$

Proof:

Let $z(t) = x(t)^*$

$$Z_n = \frac{1}{T_0}\int_{T_0} z(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T_0}\int_{T_0} x(t)^*\, e^{-jn\omega_0 t}\, dt$$

$$= \left[\frac{1}{T_0}\int_{T_0} x(t)\, e^{-j(-n)\omega_0 t}\right]^* = X_{-n}^*$$

If $x(t)$ is real.

$$x(t) = x(t)^*$$

$$\updownarrow \qquad \updownarrow$$

$$X_n = X_{-n}^*$$

$$\Longrightarrow \boxed{X_{-n} = X_n^*}$$

magnitude is even

$$|X_{-n}| = |X_n|$$

$$\Longrightarrow |X_{-n}| = \underbrace{|X_n^*|}_{|X_n|}$$

phase is odd

$$\measuredangle X_{-n} = -\measuredangle X_n$$

$$\measuredangle X_{-n} = \measuredangle X_n^* = -\measuredangle X_n$$

7 Parseval's Theorem

$$\underbrace{\frac{1}{T_0}\int_{T_0} |x(t)|^2\,dt}_{\text{power of } x(t)} = \sum_{n=-\infty}^{+\infty} |X_n|^2$$

$\therefore$ power of a periodic signal = sum of powers in all its harmonics

$$x(t) = \sum_{n=-\infty}^{+\infty} X_n e^{jn\omega_0 t}\,dt$$

$\longrightarrow$ power of the n^{th} harmonic: $\frac{1}{T_0}\int_{T_0} \left|X_n e^{jn\omega_0 t}\right|^2 dt$

$= |X_n|^2$

Proof:

$$\frac{1}{T_0}\int_{T_0} \underbrace{|x(t)|^2}_{x(t)\;\;x(t)^*} dt$$

$$= \frac{1}{T_0}\int_{T_0} \left(\sum_{n=-\infty}^{+\infty} X_n e^{jn\omega_0 t}\right)\left(\sum_{m=-\infty}^{+\infty} X_m^* e^{-jm\omega_0 t}\right)$$

$$= \sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} X_n X_m^* \underbrace{\frac{1}{T_0}\int_{T_0} e^{j(n-m)\omega_0 t}\, dt}_{\begin{cases} 1 & m=n \\ 0 & \text{o/w} \end{cases}}$$

dt

$$= \sum_{n=-\infty}^{+\infty} |X_n|^2$$

(8) Symmetries

$x(t)$ is $\boxed{\underline{\underline{\text{real}}}}$ & periodic. $\longleftrightarrow X_n$.

<u>Recall:</u> $\quad \hookrightarrow X_{-n} = X_n^*$

$\Longrightarrow X_0 = X_0^* \Longrightarrow X_0$ is real

(1) If $x(t)$ is even $\Longrightarrow$ X_n is real & even

(2) If $x(t)$ is odd $\Longrightarrow$ $\begin{cases} X_n \text{ is imag \& odd} \\ X_0 = 0. \end{cases}$

(3) $x_e(t) \longleftrightarrow \mathrm{Re}\{X_n\}$

(4) $x_o(t) \longleftrightarrow j\,\mathrm{Im}\{X_n\}$

$\underline{\text{Proof:}}$

① $x(t)$ is even $\Longleftrightarrow$ $x(-t) = x(t)$

$$\underbrace{X_n^* = X_{-n}}_{x(t)\text{ is real}} = X_n$$

$X_n = X_n^* \Rightarrow X_n$ is real

$X_{-n} = X_n \Rightarrow X_n$ is even ✓

③ $x_e(t) = \frac{1}{2}\left[x(t) + x(-t)\right]$

$$\longleftrightarrow \frac{1}{2}\Big[X_n + \underbrace{X_{-n}}_{X_n^*}\Big] = Re\{X_n\}$$