$a_0 = \frac{1}{2}$

$a_n = \begin{cases} & n = \dots \\ = & n = . \end{cases}$

$b_n = \phi$

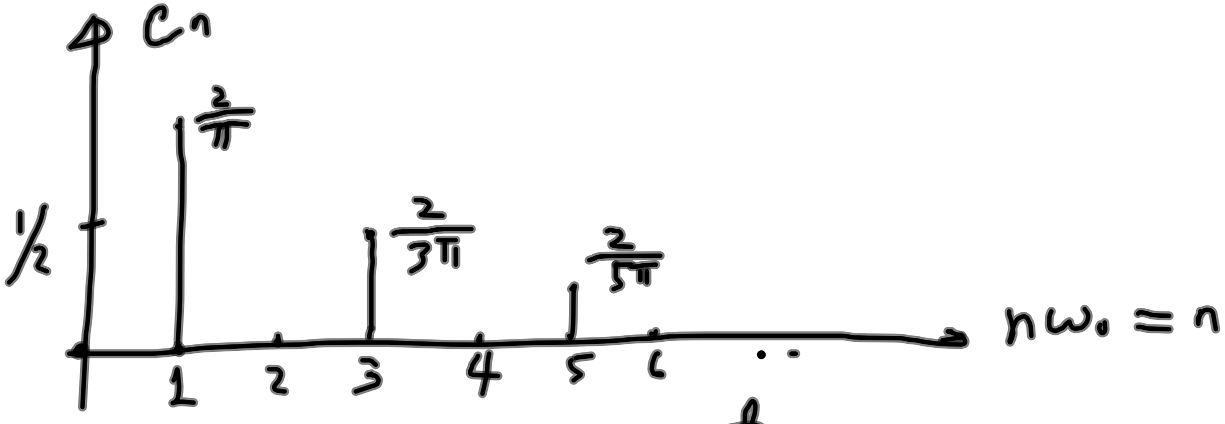
$c_0 = a_0 = \frac{1}{2}$

$c_n = \sqrt{a_n^2 + b_n^2}$

$= \begin{cases} 0 & n \text{ even} \\ \frac{2}{n\pi} & n \text{ odd} \end{cases}$

$\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right)$

$= \begin{cases} 0, & \text{o/w} \\ \pi, & k = 4k-1 \end{cases}$

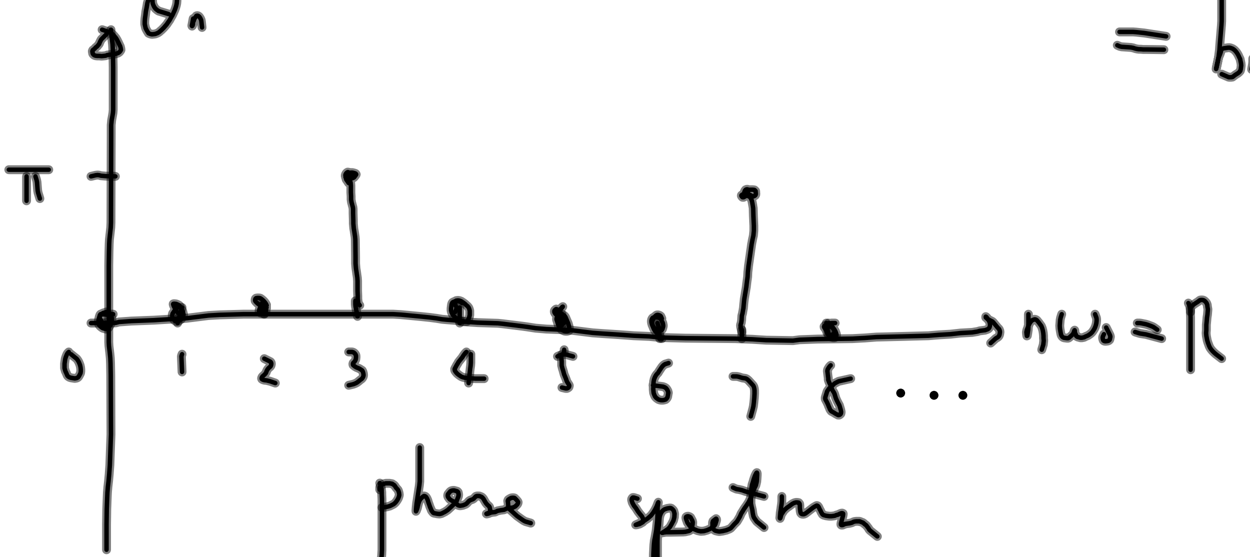


magnitude spectrum

$$b_n \sin n\omega_0 t = b_n \cos\left(n\omega_0 t - \frac{\pi}{2}\right)$$

$b_n > 0$. $\theta_n = -\frac{\pi}{2}$

$b_n < 0$. $\theta_n = \frac{\pi}{2}$.



phase spectrum

§2.3. Exponential Fourier Series

① complex exponential set

$$\left\{ e^{jn\omega_0 t}, \quad n = 0, \pm 1, \pm 2, \cdots \right\}$$

<u>FACT:</u> The above signal set is <u>orthogonal</u> over any interval of duration $T_0 = \frac{2\pi}{\omega_0}$

$$\int_{T_0} e^{jm\omega_0 t} \left(e^{jn\omega_0 t} \right)^* dt$$

$$= \int_{T_0} e^{j(m-n)\omega_0 t}\, dt$$

$m \neq n.$

$$= \frac{1}{(m-n)\omega_0}\, e^{j(m-n)\omega_0 t}\Big|_{t_0}^{t_0+T}$$

$$= \frac{1}{(m-n)\omega_0}\left[\underbrace{e^{j(m-n)\omega_0(t_0+T)}}_{e^{j(m-n)\omega_0 t_0}\,\cdot\,\underbrace{e^{j(m-n)\underbrace{\omega_0 T}_{2\pi}}}_{1}} - e^{j(m-n)\omega_0 t_0}\right]$$

$$= \phi$$

$m = n \longrightarrow \int_{T_0} 1\, dt = T_0$

$$\therefore \int e^{jm\omega_0 t}\left(e^{jn\omega_0 t}\right)^* dt = \begin{cases} 0 & m \neq n \\ T_0 & m = n. \end{cases}$$

→ orthogonal set.

Moreover, the set is complete.

② Exponential Fourier Series of periodic signals

$$\boxed{f(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}}$$

periodic with period $T = \frac{2\pi}{\omega_0}$

$$D_n \triangleq \frac{1}{\mathcal{E}_n} \int_{T_0} f(t) \left(e^{jn\omega_0 t}\right)^* dt$$

$$\boxed{D_n = \frac{1}{T_0} \int_{T_0} f(t) e^{-jn\omega_0 t} dt}$$

③ Relationship between trigonometric & exponential Fourier series

when $f(t)$ is real

$$C_n \cos(n\omega_0 t + \theta_n)$$ ← trigonometric form

exponential form

$$= \frac{C_n}{2}\left[e^{j(n\omega_0 t + \theta_n)} + e^{-j(n\omega_0 t + \theta_n)} \right]$$

$$= \underbrace{\left(\frac{C_n}{2} e^{j\theta_n}\right)}_{D_n} e^{jn\omega_0 t} + \underbrace{\left(\frac{C_n}{2} e^{-j\theta_n}\right)}_{D_{-n}} e^{-jn\omega_0 t}$$

$$\therefore f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

$$= D_0 + \sum_{n=1}^{\infty}\left(D_n e^{jn\omega_0 t} + D_{-n} e^{-jn\omega_0 t}\right)$$

More compact form $\longrightarrow$ $$= \sum_{n=-\infty}^{+\infty} D_n e^{jn\omega_0 t}$$

$\therefore$ The two representations are equivalent.

$$\begin{cases} D_0 = C_0 \\ D_n = \frac{1}{2} C_n e^{j\theta_n} & n = 1, 2, \cdots \\ D_{-n} = \frac{1}{2} C_n e^{-j\theta_n} \end{cases}$$

$$\therefore \quad D_{-n} = D_n^* \quad (\text{for real } f(t)).$$

$$\begin{cases} D_n = \frac{1}{2} (a_n - j b_n) \\ D_{-n} = \frac{1}{2} (a_n + j b_n) \end{cases}$$

④ Exponential Fourier Spectrum

$$D_0 = C_0 = a_0$$

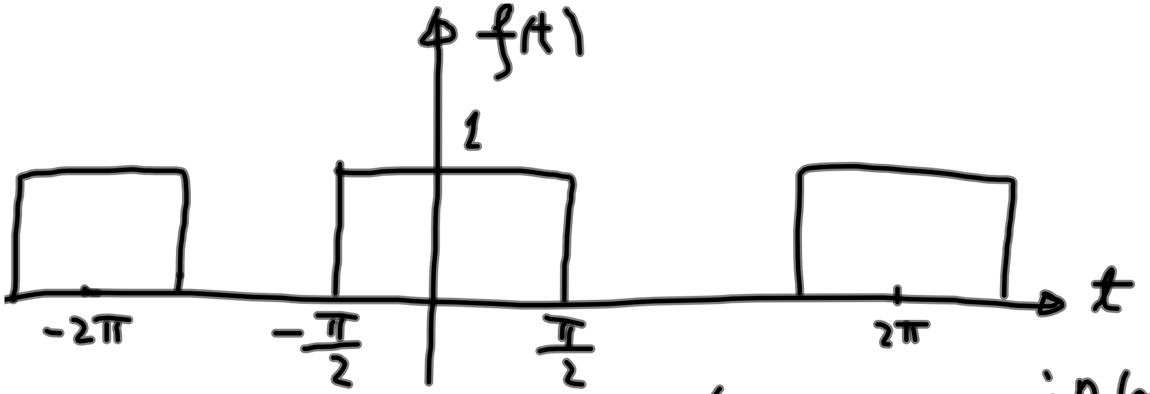
$$|D_n| = |D_{-n}| = \frac{1}{2} C_n, \quad n \neq 0.$$

$$\angle D_n = \theta_n, \quad \angle D_{-n} = -\theta_n$$

magnitude spectrum: $|D_n| \sim \omega = n\omega_0$ even

angle spectrum: $\angle D_n \sim \omega = n\omega_0$ odd

(for real $f(t)$)



$$D_n = \frac{1}{T_0}\int_{T_0} f(t)\, e^{-jn\omega_0 t}\, dt.$$

$$= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1\cdot e^{-jnt}\, dt$$

$n=0$: $D_0 = \frac{1}{2}$

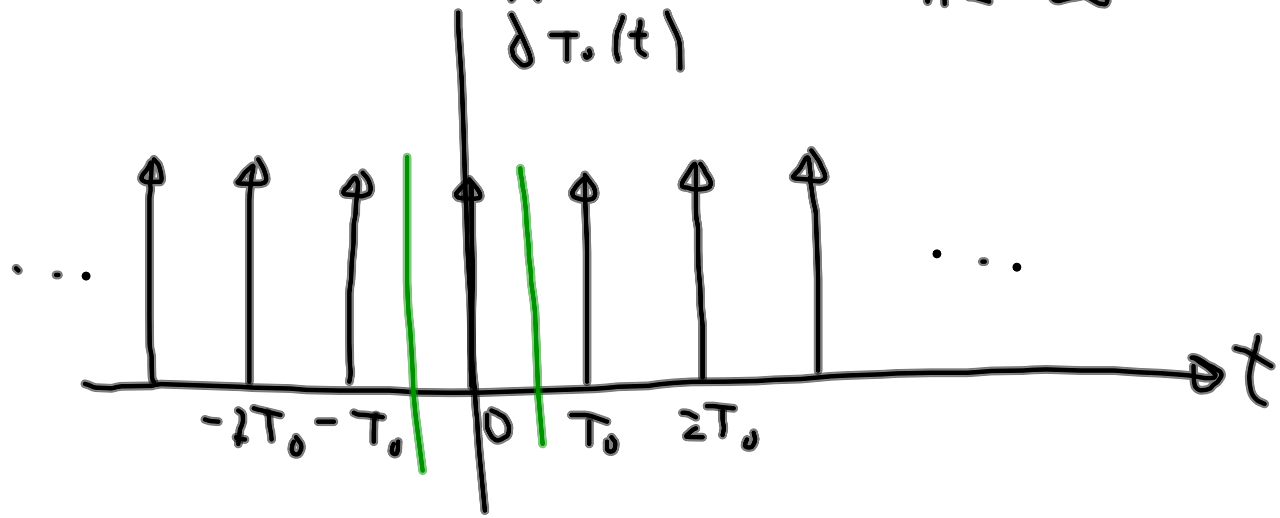
$n \neq 0$

$$= \frac{1}{2\pi}\cdot\frac{1}{-jn}\, e^{-jnt}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{n\cdot\pi}\cdot \underbrace{\frac{e^{jn\frac{\pi}{2}} - e^{-jn\frac{\pi}{2}}}{2j}}_{\sin\left(n\frac{\pi}{2}\right)}$$

$$D_n = \begin{cases} \frac{1}{2} & n = 0 \\ \frac{1}{n\pi} \sin \frac{\pi}{2} n, & n \neq 0. \end{cases}$$

__Ex:__

$$f(t) = \delta_{T_0}(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_0)$$



$\omega_0 = \frac{2\pi}{T_0}$

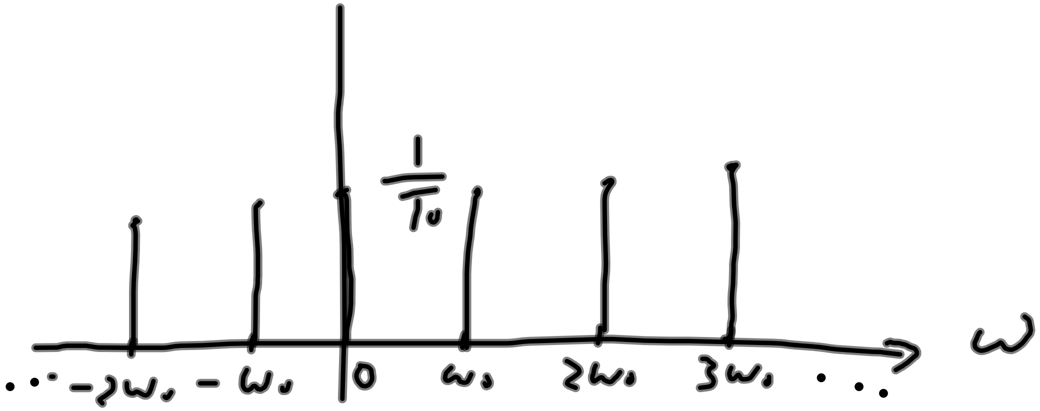
$$D_n = \frac{1}{T_0} \int_{T_0} \delta_{T_0}(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T_0} \underbrace{\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \delta(t)\, e^{-jn\omega_0 t}\, dt}_{e^{-jn\omega_0 t}\big|_{t=0}}$$

$$= \frac{1}{T_0}$$

$$\delta_{T_0}(t) = \frac{1}{T_0} \sum_{n=-\infty}^{+\infty} e^{jn\omega_0 t}$$

$=1$



spectrum

$$
\begin{cases}
c_0 = D_0 = \dfrac{1}{T_0} \\
C_n = 2\cdot|D_n| = \dfrac{2}{T_0}, \quad n = 1, 2, \ldots \\
\theta_n = 0
\end{cases}
$$

§ 2.4. Properties of Fourier Series

short-hand notation.

$$x(t) \xleftrightarrow{\mathcal{F}} X_n.$$

$x(t)$: periodic with period T_0.

X_n: Fourier series.

$$\begin{cases} x(t) = \sum_{n=-\infty}^{+\infty} X_n e^{jn\omega_0 t} \\ X_n = \int_{T_0} x(t) e^{-jn\omega_0 t} \end{cases}$$

① linearity

if $x(t) \longleftrightarrow X_n$.

$y(t) \longleftrightarrow Y_n$.

then

$$a\,x(t) + b\,y(t) \longleftrightarrow a \cdot X_n + b \cdot Y_n.$$

② Time shifting

if $x(t) \longleftrightarrow X_n$.

then $x(t-t_0) \longleftrightarrow e^{-jn(\omega_0 t_0)} \cdot X_n$

$\therefore$ a time-shift

$\longleftrightarrow$ linear phase shift in freq.