$$f(t) = \sum_{n=1}^{\infty} c_n \, x_n(t), \quad t \in [t_1, t_2]$$

$$c_n = \frac{1}{\varepsilon_n} \int_{t_1}^{t_2} f(t)\, x_n(t)\, dt$$

$\Big\}$ real-valued signals

② Complex-valued signals

- $\{ x_n(t) \}$ is an orthogonal set if

$$\int_{t_1}^{t_2} x_m(t)\, \underline{\underline{x_n(t)^*}}\, dt = \begin{cases} 0, & m \neq n \\ \varepsilon_n, & m = n \end{cases}$$

- If the set is <u>complete</u>, then

$$f(t) = \sum_{n=1}^{\infty} c_n x_n(t), \quad t \in [t_1, t_2]$$

$$c_n = \frac{1}{\varepsilon_n} \int_{t_1}^{t_2} f(t)\, \underline{\underline{x_n(t)}}^* \, dt$$

§ 2.2. Trigonometric Fourier Series.

① Trigonometric Set

$$\left\{ 1, \begin{array}{l} \cos\omega_0 t, \ \cos 2\omega_0 t, \ \cdots, \ \cos n\omega_0 t, \cdots, \\ \sin\omega_0 t, \ \sin 2\omega_0 t, \ \cdots, \ \sin n\omega_0 t, \cdots \end{array} \right\}$$

<u>FACT 1:</u> This set is <u>orthogonal</u> over any interval of duration $T_0 = \frac{2\pi}{\omega_0}$.

ω_0 is called the fundamental freq.

$$\int_{T_0} \cos n\omega_0 t \cdot \cos m\omega_0 t = \begin{cases} 0, & m \neq n \\ \frac{T_0}{2} & m = n \neq 0 \end{cases}$$

energy

$$\int_{T_0} \sin n\omega_0 t \cdot \sin m\omega_0 t = \begin{cases} 0, & m \neq n \\ \frac{T_0}{2}, & m = n \neq 0 \end{cases}$$

$$\int_{T_0} \sin n\omega_0 t \cdot \cos m\omega_0 t = 0$$

$\forall\, m, n.$

$$\boxed{\int_{T_0} \equiv \int_{t_1}^{t_1+T_0} \quad \text{for any } t_1}$$

② Trignometric Fourier Series

<u>FACT 2:</u> Trignometric set is <u>complete</u>.

$$\therefore\ f(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right)$$

$$t_1 \le t \le t_1 + T_0$$

$$a_0 = \frac{1}{T_0}\int_{T_0} f(t)\,dt \qquad T_0 = \frac{2\pi}{\omega_0}$$

$$a_n = \frac{2}{T_0}\int_{T_0} f(t)\cos n\omega_0 t\,dt$$

$$n = 1, 2, \ldots$$

$$b_n = \frac{2}{T_0} \int_{T_0} f(t) \sin n\omega_0 t \, dt$$

$$n = 1, 2, \cdots$$

③ Compact Trigonometric Fourier series

$$a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

$$= C_n \cos(n\omega_0 t + \theta_n)$$

where

$$\begin{cases} C_n = \sqrt{a_n^2 + b_n^2} \\ \theta_n = \tan^{-1}\left(\dfrac{-b_n}{a_n}\right) \\ C_0 = a_0 \end{cases}$$

$$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

$$t_1 \le t \le t_1 + T_0 .$$

④ Fourier series representation of periodic signals

Denote $\varphi(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$

$$-\infty < t < \infty$$

then $\varphi(t)$ is periodic w/ period T_0

- If $f(t)$ is also $\boxed{\text{periodic}}$ w/ period T_0,
  then $f(t)$ & $\varphi(t)$ are equal everywhere

$$\therefore \quad f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

- Fourier Spectrum: $\quad -\infty < t < \infty$

$$\left.\begin{array}{l} \text{plot } C_n - \omega = n\omega_0 \longrightarrow \text{amplitude spectrum} \\ \text{plot } \theta_n - \omega = n\omega_0 \longrightarrow \text{phase spectrum} \end{array}\right\}$$

freq. description of $f(t)$

⑤ Existence conditions of Fourier Series

<u>Theorem:</u> If the periodic signal $f(t)$ satisfies

$$\int_{T_0} |f(t)|^2 dt < \infty$$

then the Fourier series coefficients $\{C_n\}$ are finite, and the approximation error signal's energy $\underline{\underline{\mathcal{E}_e \to 0}}$ as $N \to \infty$

But $f(t)$ & $\varphi(t)$ may NOT be equal at every t.

⑥ Determining the fundamental frequency/period.

FACT: A sum of sinusoids of some frq's is periodic iff the ratio of any two frq's is a rational number. (i.e., the ratio of two integers)

When the ratio of two frq's is a rational number, they are said to be harmonically related.

Ex: $$f(t) = 2 + 7\cos\left(\frac{1}{2}t + \theta_1\right) + 3\cos\left(\frac{2}{3}t + \theta_2\right) + 5\cos\left(\frac{7}{6}t + \theta_3\right)$$

$$\omega_1 = \frac{1}{2} \Rightarrow T_1 = \frac{2\pi}{\omega_1} = 4\pi$$

$$\omega_2 = \frac{2}{3} \Rightarrow T_2 = \frac{2\pi}{\omega_2} = 3\pi$$

$$\omega_3 = \frac{7}{6} \Rightarrow T_3 = \frac{2\pi}{\omega_3} = \frac{12\pi}{7}$$

$\therefore$ periodic !

$\Longrightarrow T_0 =$ least common multiple (T_1, T_2, T_3)

$= 12\pi.$

$$\Longrightarrow \omega_0 = \frac{2\pi}{T_0} = \frac{1}{6}$$

$\underline{Ex:}$ $f(t) = 2\cos(2t+\theta_1) + 5\sin(\pi t + \theta_2)$

$\omega_1 = 2$

$\omega_2 = \pi$

$\therefore$ aperiodic !

$\underline{Ex:}$ $f(t) = 3\sin(3\sqrt{2}t + \theta_1) + 7\cos(6\sqrt{2}t + \theta_2)$

$\omega_1 = 3\sqrt{2}$

$\omega_2 = 6\sqrt{2}$

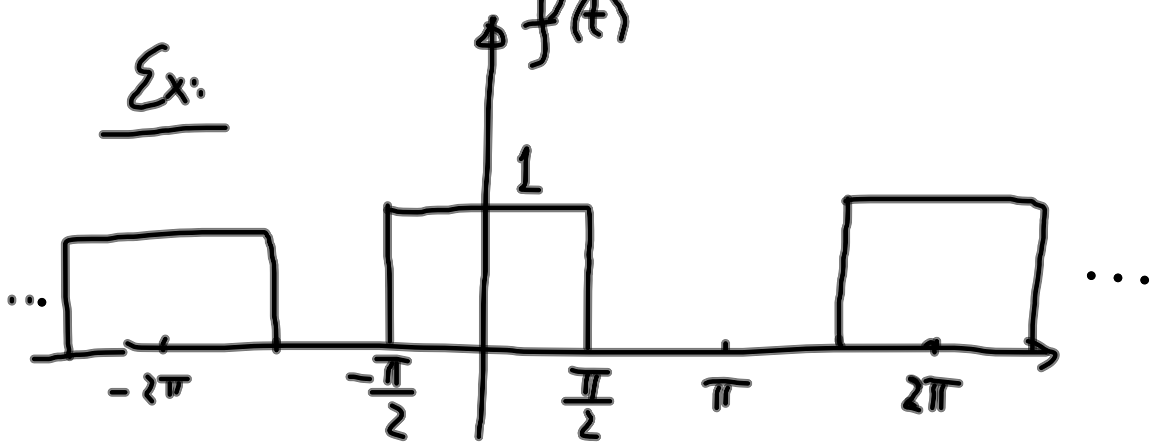
$T_1 = \frac{2\pi}{3\sqrt{2}}$

$T_2 = \frac{2\pi}{6\sqrt{2}}$

$\Longrightarrow T_0 = \frac{2\pi}{3\sqrt{2}}.$

$\omega_0 = 3\sqrt{2}.$

Ex:



$$T_0 = 2\pi$$

$$\omega_0 = \frac{2\pi}{T_0} = 1.$$

$$f(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nt + b_n \sin nt\right)$$

• The effect of Symmetry

$$a_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f(t)\,dt$$

$$a_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} f(t)\cos n\omega_0 t\,dt$$

$$b_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} f(t)\sin n\omega_0 t\,dt$$

If $f(t)$ is even:

$$\begin{cases} a_0 = \dfrac{2}{T_0}\displaystyle\int_0^{T_0/2} f(t)\,dt \\ a_n = \dfrac{4}{T_0}\displaystyle\int_0^{T_0/2} f(t)\cos n\omega_0 t\,dt \\ b_n = \emptyset \end{cases}$$

If $f(t)$ is odd:
$$\begin{cases} a_0 = a_n = 0 \\ b_n = \dfrac{4}{T_0}\displaystyle\int_0^{T_0/2} f(t)\sin n\omega_0 t\, dt \end{cases}$$

(cont) ✓ $b_n = 0, \quad \forall n.$

$$\checkmark\, a_0 = \frac{2}{T_0}\int_0^{T_0/2} f(t)\,dt$$

$$= \frac{2}{2\pi}\int_0^{\pi} f(t)\,dt = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} 1\cdot dt = 1/2.$$

$$a_n = \frac{4}{T_0}\int_0^{\pi} f(t)\cos n\omega_0 t\, dt$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos nt\, dt$$

$$= \frac{2}{\pi}\cdot\frac{1}{n}\sin nt\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{2}{n\cdot\pi}\sin\left(\frac{n\pi}{2}\right), \quad n = 1, 2, \cdots$$

$$= \begin{cases} 0, & n \text{ is even} \\ \frac{2}{n\pi}, & n = 1, 5, \cdots 4k+1 \\ -\frac{2}{n\pi}, & n = 3, 7, \cdots 4k+3 \end{cases}$$

$$\therefore f(t) = \frac{1}{2} + \frac{2}{\pi}\left(\cos t - \frac{1}{3}\cos 3t + \frac{1}{5}\cos 5t - \frac{1}{7}\cos 7t + \cdots \right).$$