(8) Invertible vs. Non-Invertible

Invertible System: Can get $f(t)$ from $y(t)$

Ex: $y(t) = af(t) + b$

Non-Invertible System: Not Invertible

Ex: $y(t) = |f(t)|$

9 Stable vs. Non-Stable Systems

Stable System: bounded Input $\longrightarrow$ bounded output

- unstable System: bounded input $\Longrightarrow$ unbounded output

  Ex: $y(t) = 1/x$

1.6 The Convolution Integral:

① Definition:

Convolution of two function $f_1(t)$ & $f_2(t)$

$$f_1(t) * f_2(t) \triangleq \int_{-\infty}^{+\infty} f_1(\tau) f_2(t-\tau)\, d\tau$$

② Properties:

- Commutativity: $f_1(t) * f_2(t) = f_2(t) * f_1(t)$

Proof: $f_2(t) * f_1(t) = \int_{-\infty}^{+\infty} f_2(\tau) f_1(t-\tau)\, d\tau$

$u \triangleq t-\tau \quad = \int_{\infty}^{-\infty} f_1(u) f_2(t-u)\, d(-u)$

$= \int_{-\infty}^{\infty} f_1(u) f_2(t-u)\, du = f_1(t) * f_2(t)$

· Distributivity

$$f_1(t) * \left[f_2(t) + f_3(t) \right] = f_1(t) * f_2(t) + f_1(t) * f_3(t)$$

Proof:

$$f_1(t) * \left[\underbrace{f_2(t) + f_3(t)}_{g(t)} \right] = \int f_1(\tau)\, g(t-\tau)\, d\tau = \int f_1(t) f_2(t-\tau) + f_1(t) f_3(t-\tau)\, d\tau$$

$$= f_1 * f_2 + f_1 * f_3$$

· Associativity:

$$f_1(t) * \left[f_2(t) * f_3(t)\right] = \left[f_1(t) * f_2(t)\right] * f_3(t)$$

Proof: $g(t) \triangleq f_2(t) * f_3(t) = f_3(t) * f_2(t) = \int f_3(\tau_1) f_2(t-\tau_1)\, d\tau_1$

$$f_1(t) * g(t) = \int f_1(\tau_2)\, g(t-\tau_2)\, d\tau_2$$

$$= \iint f_1(\tau_2)\, f_2(t-\tau_1-\tau_2)\, f_3(\tau_1)\, d\tau_2\, d\tau_1$$

$$= \int \underbrace{\left[\int f_1(\tau_2) f_2(t-\tau_1-\tau_2)\, d\tau_2 \right]}_{(f_1 * f_2)(t-\tau_1) \,\triangleq\, h(t-\tau_1)} f_3(\tau_1)\, d\tau_1$$

$$= \int h(t-\tau_1) f_3(\tau_1)\, d\tau_1 = h(t) * f_3(t)$$

$$= \left[f_1(t) * f_2(t) \right] * f_3(t)$$

· Shift Property:

$$f_1(t) * f_2(t) = c(t)$$

$$f_1(t) * f_2(t-T) = c(t-T)$$

$$f_1(t-T) * f_2(t) = c(t-T)$$

$$f_1(t-T_1) * f_2(t-T_2) = c(t-T_1-T_2)$$

Proof:

$$\underbrace{f_1(t-T_1)}_{g_1(t)} * \underbrace{f_2(t-T_2)}_{g_2(t)}$$

$$= \int g_1(\tau)\, g_2(t-\tau)\, d\tau$$

$$= \int f_1(\tau-T_1)\, f_2(t-T_2-\tau)\, d\tau$$

$$\overset{u \triangleq \tau - T_1}{=} \int f_1(u)\, f_2(t-T_1-T_2-u)\, du$$

$$= c(t-T_1-T_2)$$

· Convolution with an impulse

$$f(t) * \delta(t) = f(t)$$

<u>Proof:</u> Recall def of $\delta(t)$: $\int f(t)\,\delta(t-t_0)\,dt = f(t_0)$

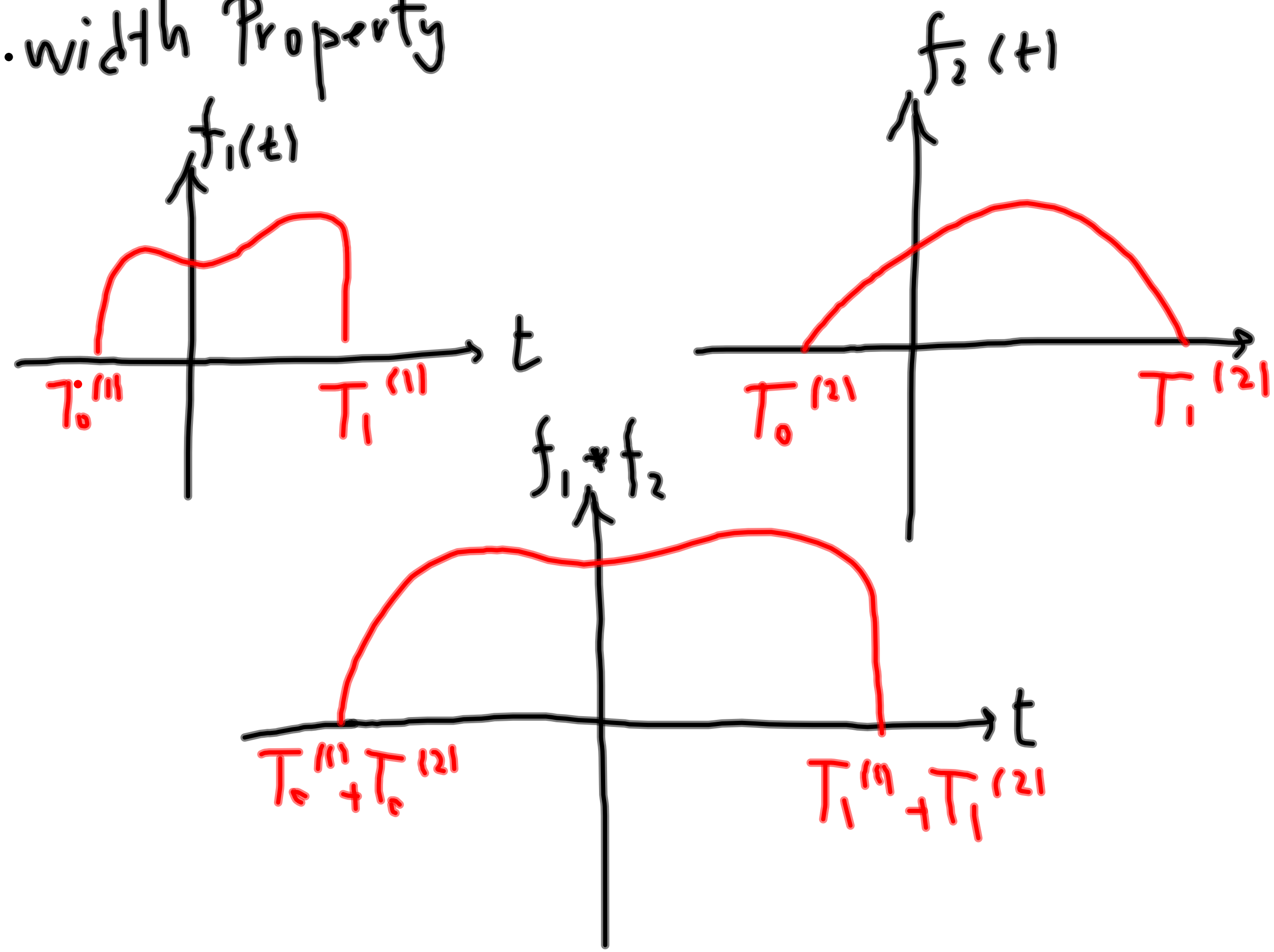
$t \rightarrow \tau$

$t_0 \rightarrow t$

$$\int f(\tau)\,\underset{\delta(t-\tau)}{\delta(\tau-t)}\,d\tau = f(t) * \delta(t)$$

$\because \delta(t-\tau) = \delta(\tau-t)$

· width Property



· Convolution of two causal Systems :

$f_1(t) = f_1(t)\, u(t)$

$f(t) = 0$ for $t < 0$

$f_2(t) = f_2(t)\, u(t)$

$f(\tau) = 0$ for $\tau < 0$

$f_1 \& f_2 = 0$ for $t < 0$

$$\Rightarrow f_1(t) * f_2(t) = \int_0^t f_1(\tau) f_2(t-\tau)\, d\tau$$

$f_2(t-\tau)$: $\tau > t$
$= 0$

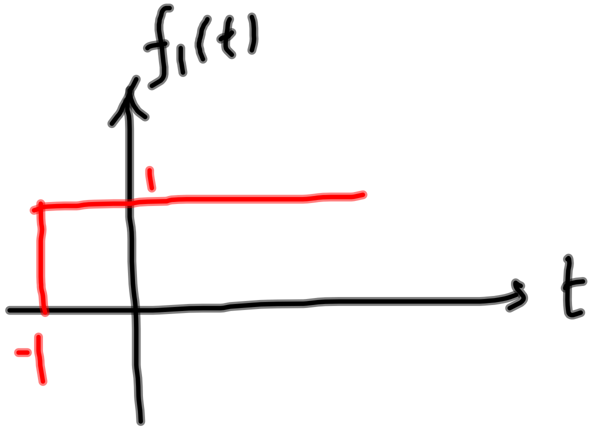
$$\int_{-\infty}^{+\infty}$$

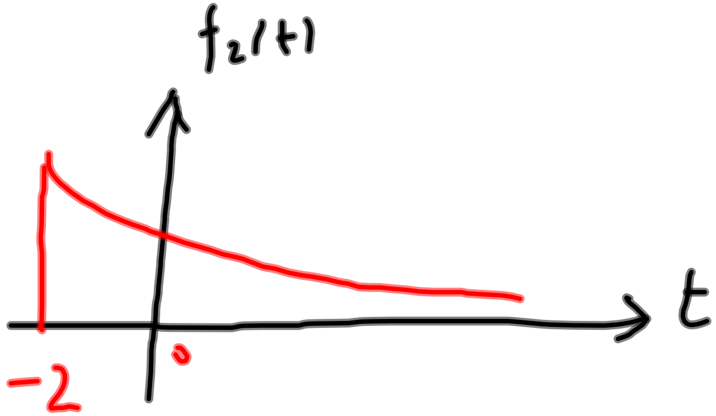
③ Graphical Interpretation of Convolution :

$$c(t) = \int_{-\infty}^{+\infty} \underline{f_1(\tau) f_2(t-\tau)}\, d\tau$$

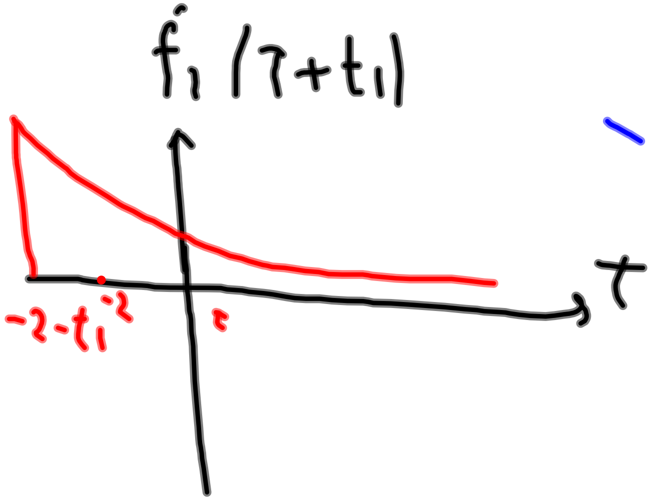
· Sketch $f_1(\tau)$ and $f_2(t-\tau)$ as a function of τ.

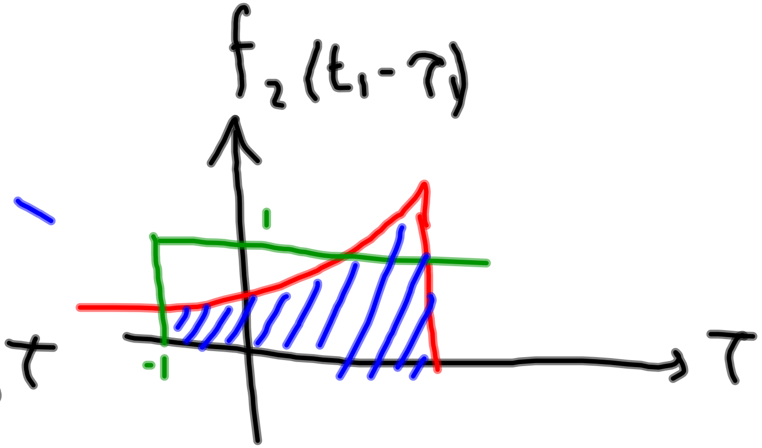
t : is a constant value



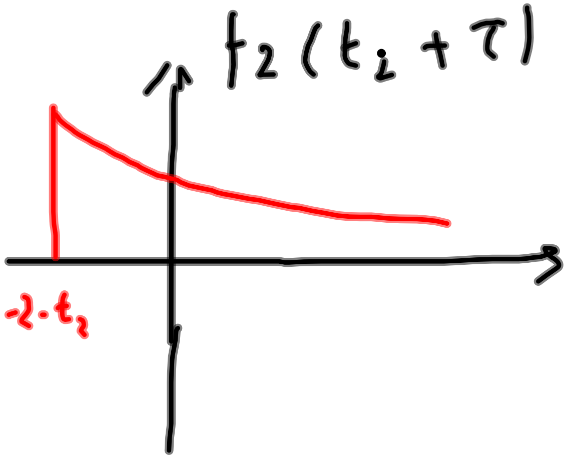


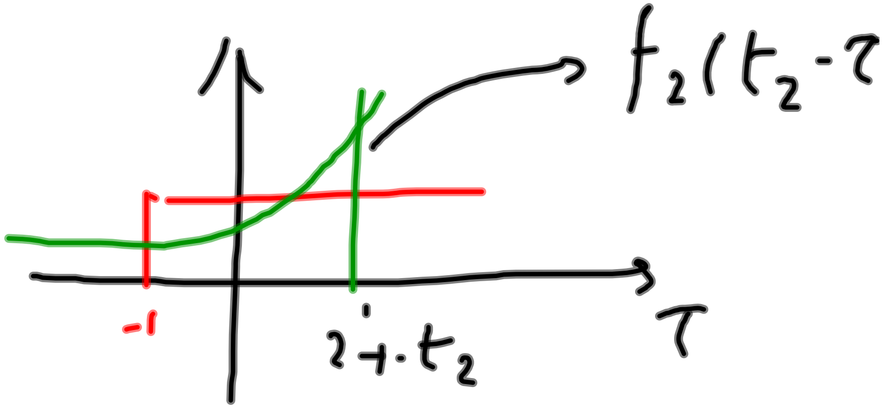
$t_1 > 0$:





$t_2 < 0$





$t_3 < -3$