Chapt 1 Introduction

§1.0 Background

① Complex numbers

- imaginary unit number j

$$j^2 = -1$$

or $\sqrt{-1} = \pm j$

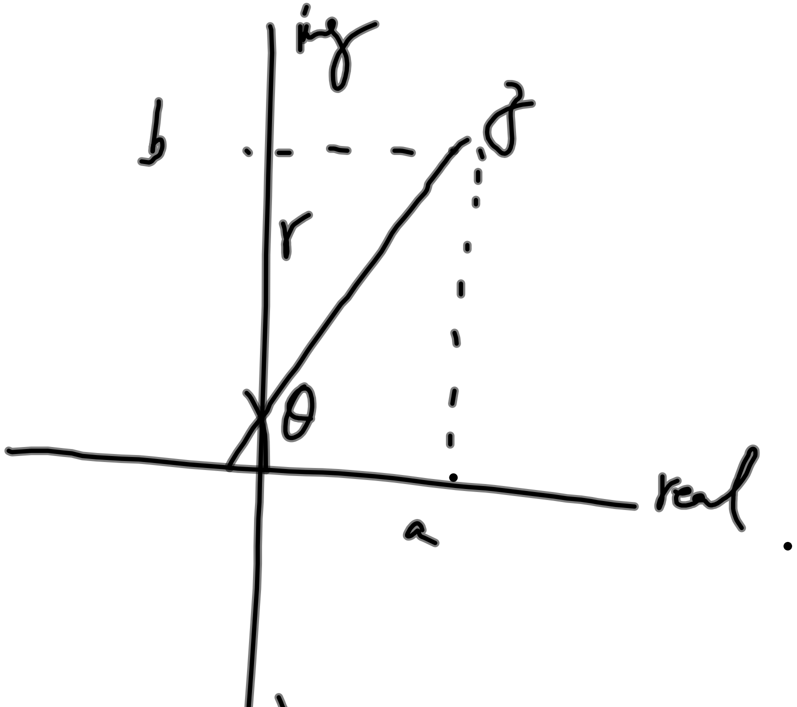
- A complex number

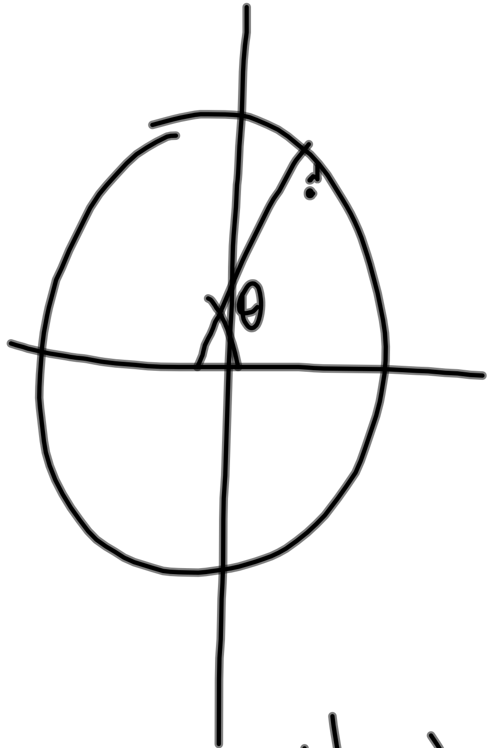
$$z = a + jb \quad \text{(cartesian)}$$

a: real part, b: imaginary part

$$\mathrm{Re}(z) = a$$
$$\mathrm{Im}(z) = b$$

- graphical representation on complex plane





polar form $\begin{cases} a = r \cdot \cos\theta. \\ b = r \cdot \sin\theta \end{cases}$

$z = a + jb$

$= r(\cos\theta + j\sin\theta)$

$= r \cdot e^{j\theta}$ polar form

Euler formula $e^{j\theta} = \cos\theta + j\sin\theta$

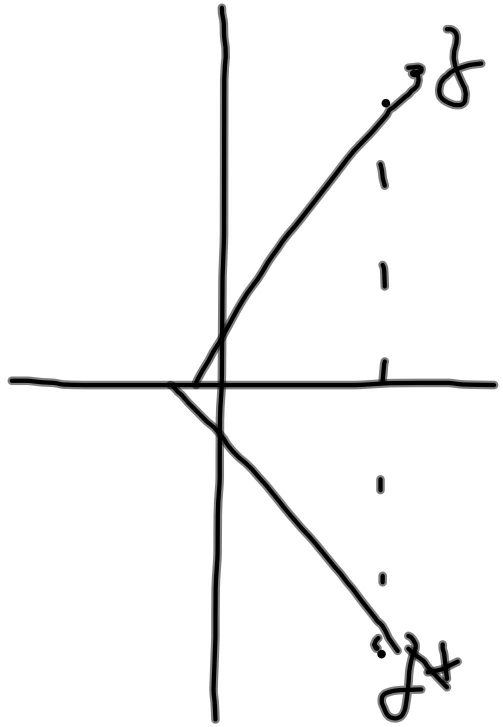
$r = \sqrt{a^2 + b^2}$ (magnitude) $|z| = r$

$\theta = \tan^{-1}\left(\frac{b}{a}\right)$. (angle) $\measuredangle z = \theta$ $\measuredangle z$

- conjugate of a complex number

$z = a + jb = |z| \cdot e^{j\measuredangle z}$

$z^* = a - jb = |z| \cdot e^{-j\measuredangle z}$



properties:

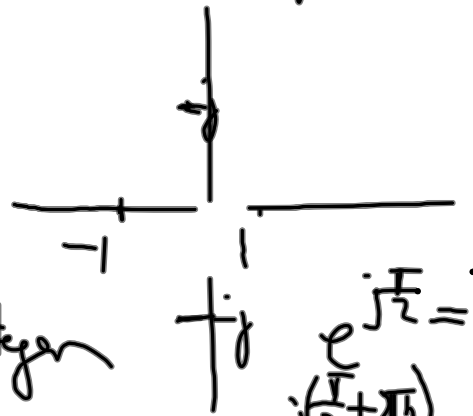
① $z + z^* = 2\,\mathrm{Re}(z) = (a+jb)+(a-jb) = 2a$.

② $z\cdot z^* = (a+jb)(a-jb) = a^2-(jb)^2 = a^2+b^2$

$= r^2 = |z|^2$

$= r\cdot e^{j\theta}\cdot r\cdot e^{-j\theta} = r^2$

· Useful identities:



$e^{j0} = 1$

$e^{j2\pi n} = 1$. n is any integer

$e^{\pm j\pi} = -1$

$e^{\pm j(2n+1)\pi} = -1$

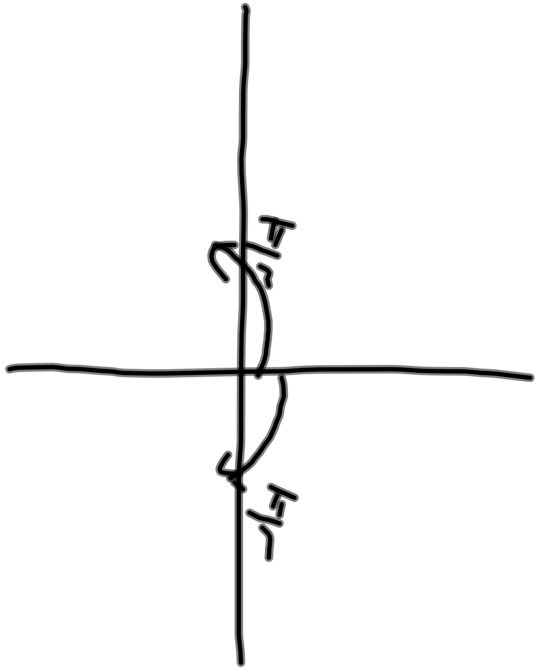
$e^{j\frac{\pi}{2}} = j$

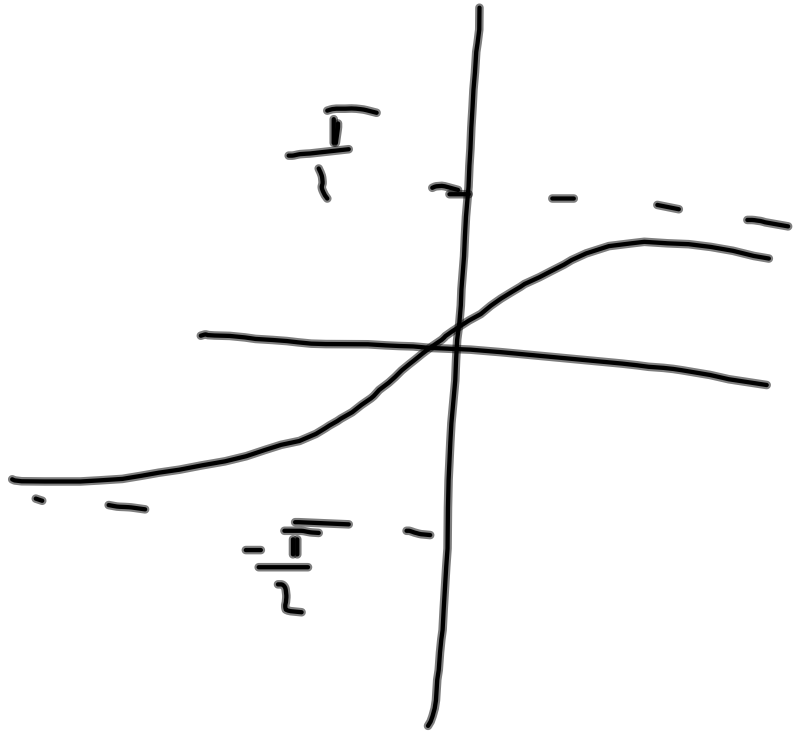
$e^{j(\frac{\pi}{2}+2\pi n)} = j$

$e^{-j\frac{\pi}{2}} = -j$

$e^{j(-\frac{\pi}{2}+2\pi n)} = -j$

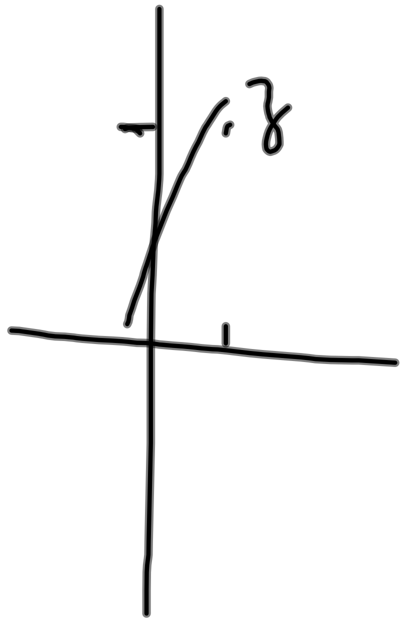
$$\begin{cases} r = \sqrt{a^2 + b^2}. \\ \theta = \tan^{-1}\left(\frac{b}{a}\right) \end{cases}$$





In using $\theta = \tan^{-1}\left(\frac{b}{a}\right)$, we must first decide the quadrant in which z lies

$$-\frac{\pi}{2} \le \tan^{-1}(\;) \le \frac{\pi}{2}.$$



Ex: $z = 2 + j3$

$$|z| = \sqrt{2^2 + 3^2} = \sqrt{13}$$

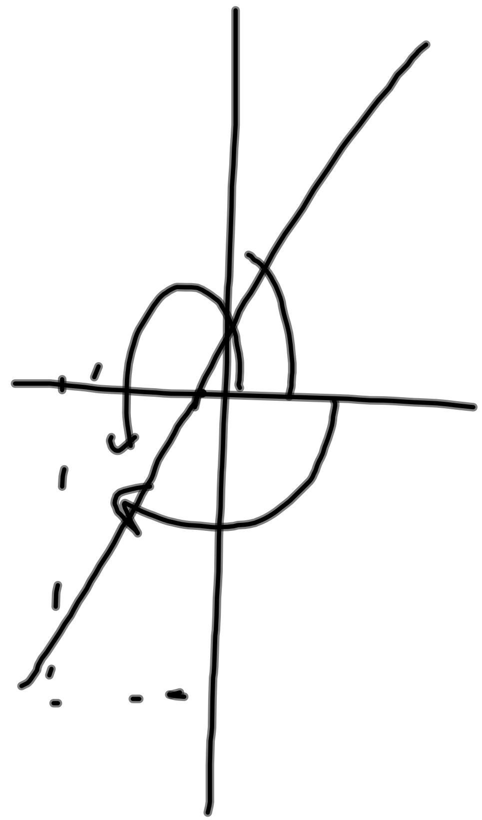
$$\angle z = \tan^{-1}\left(\frac{3}{2}\right)$$

Ex: $z = -2 - j3$

$$|z| = \sqrt{(-2)^2 + (-3)^2} = \sqrt{13}$$

$$\angle z = \tan^{-1}\left(\frac{-3}{-2}\right) \neq \tan^{-1}\left(\frac{3}{2}\right)$$

$$= \tan^{-1}\left(\frac{3}{2}\right) \pm \pi$$



Ex: $z = 2 - j3$

$|z| = \sqrt{13}.$

$\angle z = \tan^{-1}\left(\frac{-3}{2}\right) = -\tan^{-1}\left(\frac{3}{2}\right)$

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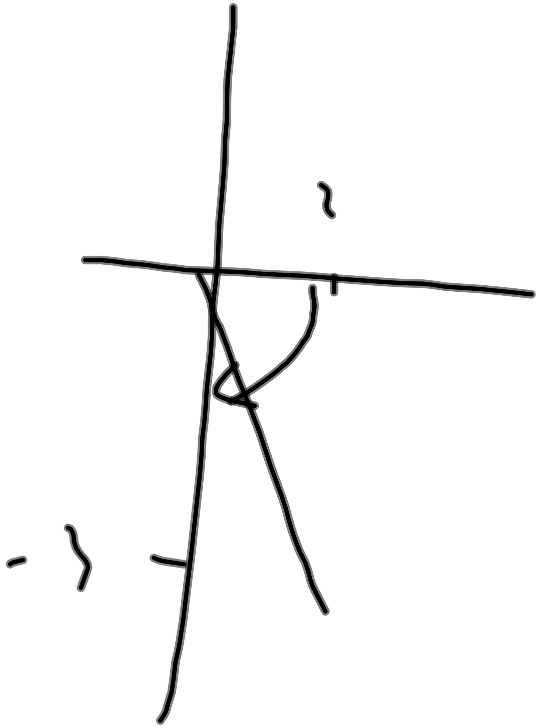
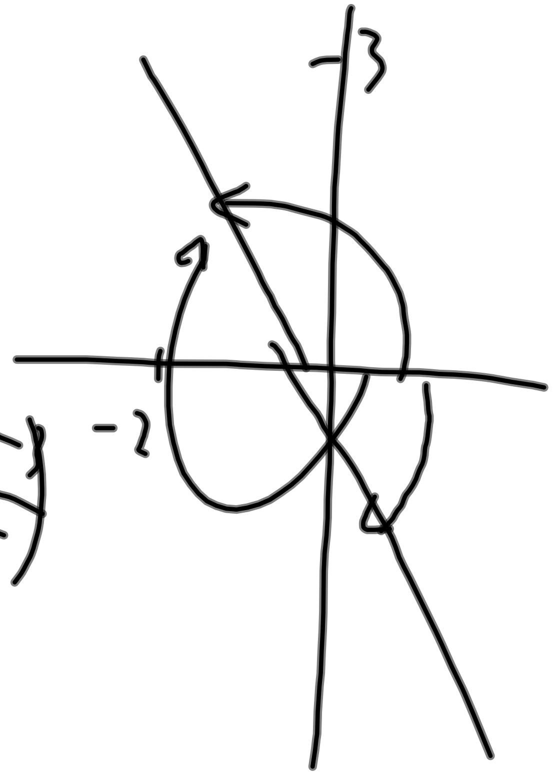


Ex: $z = -2 + j3$

$$|z| = \sqrt{13}$$

$$\angle z = \tan^{-1}\left(\frac{3}{-2}\right) \neq \tan^{-1}\left(\frac{-3}{2}\right)$$

$$= -\tan^{-1}\left(\frac{3}{2}\right) + \pi.$$



Summary: ① If z is in 1st or 4th quad.

$$\measuredangle z = \tan^{-1}\left(\frac{b}{a}\right)$$

② If z is in 2nd or 3rd quad

$$\measuredangle z = \tan^{-1}\left(\frac{b}{a}\right) \pm \pi.$$

- Addition & subtraction — use Cartesian form

$$z_1 = a_1 + j b_1$$

$$z_2 = a_2 + j b_2 .$$

$$z_1 \pm z_2 = (a_1 \pm a_2) + j(b_1 \pm b_2).$$

- multiplication & division

$$z_1 z_2 = (a_1 + jb_1) \times (a_2 + jb_2)$$

$$= (a_1 a_2 - b_1 b_2) + j(a_1 b_2 + a_2 b_1).$$

$$z_1 \cdot z_2 = r_1 e^{j\theta_1} \cdot r_2 e^{j\theta_2}$$

$$= r_1 r_2 e^{j(\theta_1 + \theta_2)}$$

$$\frac{z_1}{z_2} = \frac{a_1 + jb_1}{a_2 + jb_2} = \frac{(a_1 + jb_1)(a_2 - jb_2)}{(a_2 + jb_2)(a_2 - jb_2)}$$

$$= \frac{\underbrace{(a_1a_2 + b_1b_2)}_{Re} + j\underbrace{(b_1a_2 - a_1b_2)}_{Im}}{a_2^2 + b_2^2}$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \left(r_1/r_2\right) e^{j(\theta_1 - \theta_2)}$$

- power & root — polar form

$$z^n = (re^{j\theta})^n = r^n \cdot e^{jn\theta}$$

$$z^{\frac{1}{n}} = (re^{j\theta})^{\frac{1}{n}} = r^{\frac{1}{n}} e^{j\frac{\theta}{n}}.$$

② Sinusoid.

$$f(t) = C \cdot \cos(2\pi F_0 t + \theta)$$

C — amplitude $\quad \theta$: phase.

F_0 — frequency (in Hz)

$T_0 = \frac{1}{F_0}$ period (in Sec)

$$\omega_0 = 2\pi F_0 = \frac{2\pi}{T_0}$$ — radian freq.

· Addition of Sinusoids

$$C\cos(\omega_0 t+\theta) = C\cdot\cos(\omega_0 t)\cos\theta - C\sin\omega_0 t\sin\theta$$

$$= a\cdot\cos(\omega_0 t) + b\sin\omega_0 t$$

$$\begin{cases} a = C\cos\theta \\ b = -C\sin\theta \end{cases} \Rightarrow \quad C=\sqrt{a^2+b^2}$$

$$\theta = \tan^{-1}\left(\frac{-b}{a}\right)$$

Again e.g. $a>0,\ b>0$

$$\tan^{-1}\left(\frac{b}{-a}\right) = -\tan^{-1}\left(\frac{b}{a}\right) \pm \pi.$$

$$\tan^{-1}\left(\frac{-b}{-a}\right) = \tan^{-1}\left(\frac{b}{a}\right) \pm \pi.$$

$$f(t) = \cos\omega_0 t - \sqrt{3}\sin\omega_0 t$$

$$\begin{cases} a = 1 \\ b = -\sqrt{3} \end{cases}$$

$$c = \sqrt{1+3} = 2.$$

$$\theta = \tan^{-1}\left(\frac{-b}{a}\right) = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}.$$

$$\therefore f(t) = 2\cos\left(\omega_0 t + \frac{\pi}{3}\right).$$

$$f(t) = -3\cos\omega_0 t + 4\sin\omega_0 t$$

$$a = -3$$

$$b = 4$$

$$c = \sqrt{3^2 + 4^2} = 5$$

$$\theta = \tan^{-1}\left(\frac{-b}{a}\right) = \tan^{-1}\left(\frac{-4}{-3}\right)$$

$$= \tan^{-1}\left(\frac{4}{3}\right) \pm \pi.$$