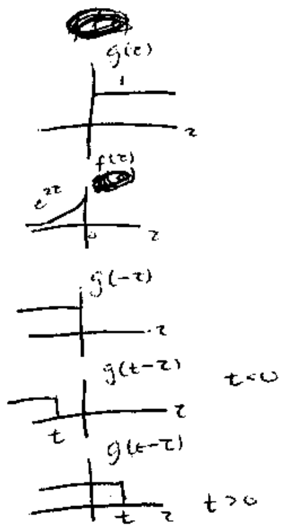
Problem I [10 pts]
Calculate the following convolution:

$f(t) * g(t)$

$$y(t) = e^{2t}u(-t) * u(t).$$

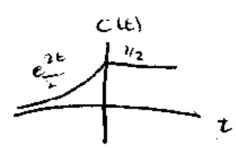
$c(t) = \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau$



For $t<0$: $c(t) = \int_{-\infty}^{t} e^{2\tau}\, d\tau = \frac{e^{2t}}{2} \qquad t<0$

For $t \geq 0$: $c(t) = \int_{-\infty}^{0} e^{2\tau}\, d\tau = \frac{1}{2} \qquad t \geq 0$

$$\Rightarrow \quad c(t) = \begin{cases} \frac{e^{2t}}{2} & t<0 \\ \frac{1}{2} & t \geq 0 \end{cases}$$



Problem II [20 pts]

1. Find the exponential Fourier series of the following signal

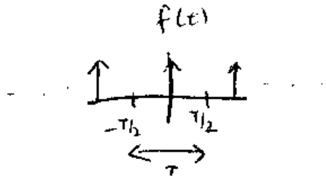
$$f(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$$

2. Using your answer above and the property of Fourier series, find the exponential Fourier series of the following signal

$$g(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0 - T_0/2)$$

3. Find the Fourier transform of $g(t)$.

(1)



$$a_n = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{-jn\omega_0 t} = \frac{1}{T}(1) = \frac{1}{T}$$

$$\boxed{f(t) = \sum_n \frac{1}{T} e^{jn\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}}$$

(2) $g(t) = f(t - \frac{nT_0}{2}) = \sum \frac{1}{T} e^{-jn\omega_0 T_0/2} e^{jn\omega_0 t} = \boxed{\frac{1}{T}\sum_n (-1)^n e^{jn\omega_0 t}}$

Since $\omega_0 T_0 = 2\pi$
$e^{-jn\pi} = (-1)^n$

(3) $e^{jn\omega_0 t} \xrightarrow{F} 2\pi\delta(\omega - n\omega_0)$

$\Rightarrow g(t) \xrightarrow{F} \frac{1}{T}\sum_n (-1)^n (2\pi)\,\delta(\omega - n\omega_0) = \omega_0 \sum_n (-1)^n \delta(\omega - n\omega_0)$

Problem III [10 pts]
Suppose $f(t) \xleftrightarrow{\mathcal{F}} F(\omega)$. Find the Fourier transform of the signal

$$f(at+b)$$

with a and b being some real-valued constants.

$$f(t+b) \xrightarrow{\mathcal{F}} F(\omega) e^{j\omega b}$$

$$f(at+b) \xrightarrow{\mathcal{F}} \frac{1}{|a|} F\left(\frac{\omega}{a}\right) e^{\frac{j\omega b}{a}}$$

Problem IV [10 pts]

Define $\operatorname{rect}\left(\frac{t}{\tau}\right) = \begin{cases} 1, & |t| \le \frac{\tau}{2}, \\ 0, & |t| > \frac{\tau}{2}. \end{cases}$ Given the Fourier transform $\operatorname{rect}\left(\frac{t}{\tau}\right) \xleftrightarrow{\mathcal{F}} \tau \operatorname{sinc}\left(\frac{\omega\tau}{2}\right)$, find the Fourier transform of the signal

$$f(t) = \operatorname{sinc}(Wt)$$

with W being some constant, by making use of the duality property of the Fourier transform.

$$f(t) = \operatorname{rect}\left(\frac{t}{\tau}\right) \xrightarrow{\mathcal{F}} \tau \operatorname{sinc}\left(\frac{\omega\tau}{2}\right) = F(\omega)$$

Duality: $F(t) = \tau \operatorname{sinc}\left(\frac{t\tau}{2}\right) \xrightarrow{\mathcal{F}} 2\pi f(-\omega) = 2\pi \operatorname{rect}\left(-\frac{\omega}{\tau}\right) = 2\pi \operatorname{rect}\left(\frac{\omega}{\tau}\right)$

Let $\frac{\tau}{2} = W$

$$2W \operatorname{sinc}(Wt) \xrightarrow{\mathcal{F}} 2\pi \operatorname{rect}\left(\frac{\omega}{2W}\right)$$

$$\Rightarrow \operatorname{sinc}(Wt) \xrightarrow{\mathcal{F}} \frac{\pi}{W} \operatorname{rect}\left(\frac{\omega}{2W}\right)$$

A

Problem V [10 pts]
An LTI system has an impulse response

$$h(t) = e^{-at}u(t)$$

with $a > 0$. The input signal is given by

$$f(t) = e^{-bt}u(t)$$

with $b > 0$. Find the magnitude spectrum and the phase spectrum of the output signal.

$$\int_0^\infty e^{-\alpha t} e^{-j\omega t}\, dt = \int_0^\infty e^{-(\alpha+j\omega)t}\, dt = \frac{e^{-(\alpha+j\omega)t}}{-(\alpha+j\omega)}\Big|_0^\infty = \frac{1}{\alpha+j\omega}$$

$$Y(\omega) = F(\omega)H(\omega) = \left(\frac{1}{a+j\omega}\right)\left(\frac{1}{b+j\omega}\right)$$

$$|Y(\omega)| = \sqrt{\left(\frac{1}{a^2+\omega^2}\right)\left(\frac{1}{b^2+\omega^2}\right)} = \frac{1}{\sqrt{a^2b^2+a^2\omega^2+b^2\omega^2+\omega^4}}$$

$$\measuredangle Y(\omega) = \measuredangle F(\omega) + \measuredangle H(\omega)$$

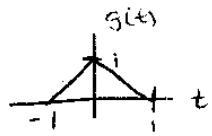
$$\measuredangle Y(\omega) = -\left(\tan^{-1}\left(\frac{\omega}{a}\right) + \tan^{-1}\left(\frac{\omega}{b}\right)\right)$$

Problem VI [20 pts]
Find the Fourier transforms of the following signals:
(a) $f(t) = \sin(\omega_0 t)$;
(b) $g(t) = (1+t)[u(t+1) - u(t)] + (1-t)[u(t) - u(t-1)]$.

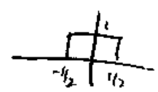
(a) $\sin(\omega_0 t) = \dfrac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} \xrightarrow{\mathcal{F}} \dfrac{1}{2j}\left(2\pi\delta(\omega-\omega_0) - 2\pi\delta(\omega+\omega_0)\right)$

$F(\omega) = j\pi\left(\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right)$

(b)



Let $f(t) = \text{rect}(t)$



then $g(t) = f(t) * f(t)$
$G(\omega) = F(\omega)\,F(\omega)$

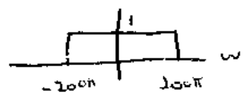
From Prob IV with $\tau = 1$, $f(t) \xrightarrow{\mathcal{F}} \text{sinc}\left(\frac{\omega}{2}\right)$

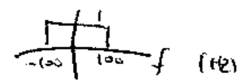
$\boxed{G(\omega) = \text{sinc}^2\left(\frac{\omega}{2}\right)} = \dfrac{\sin^2\left(\frac{\omega}{2}\right)}{\frac{\omega^2}{4}} = \left(\frac{4}{\omega^2}\right)\left(\frac{1}{2}\right)\left(1 - \cos\left(2\,\frac{\omega}{2}\right)\right)$

$G(\omega) = \boxed{\dfrac{2\left(1-\cos(\omega)\right)}{\omega^2}}$

Problem VII [20 pts] A signal $f(t)$ has the spectrum $F(\omega) = u(\omega + 200\pi) - u(\omega - 200\pi)$.

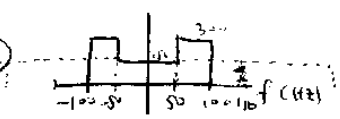
1. What is the Nyquist sampling rate of $f(t)$?
2. It is sampled at a rate of (a) 150Hz (b) 200 Hz (b) 300Hz. For each of the three cases: If the sampled signal is passed through an ideal lowpass filter of bandwidth 100Hz and unit gain, sketch the spectrum of the output signal.

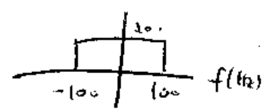
① 

or in Hz 

So Nys. rate = $\boxed{200 \text{ Hz}}$

② $\bar{F}(\omega) = \frac{1}{T}\sum F(\omega - n\omega_s) = f_s \sum F(\omega - n2\pi f_s)$

(a) 

(b) 

(c)