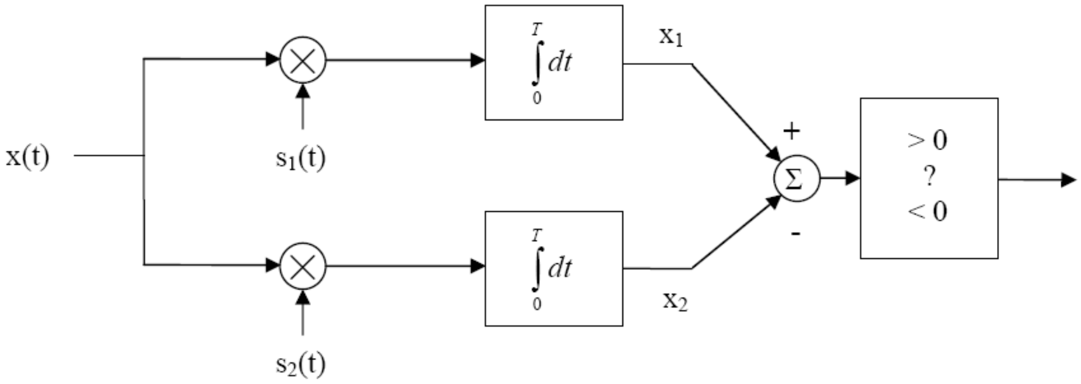
Solutions Homework #9
Introduction to Communication Systems – E3701

Problem 1

Part a



If we examine our two possible message signals, $s_1(t)$ and $s_2(t)$, we see that they both have amplitude $\sqrt{E}$ and thus energy E. Each has a corresponding basis function of amplitude $\sqrt{(1/T)}$ and energy 1. We will call these basis functions $\varphi_1(t)$ and $\varphi_2(t)$. The input to the receiver is

$$x(t) = s_i(t) + n(t), \quad i = 1,2$$

where n(t) is a noise signal. The output of either integrator looks as follows

$$x_1 = \int_0^T x(t)s_1(t)dt = \int_0^T [s_i(t)s_1(t) + n(t)s_1(t)]dt = E\int_0^T \phi_i(t)\phi_1(t)dt + \sqrt{E}\int_0^T n(t)\phi_1(t)dt$$

$$x_1 = \begin{cases} E + \sqrt{E}n_1 & if \quad i = 1 \\ \sqrt{E}n_1 & if \quad i = 2 \end{cases}$$

$$x_2 = \begin{cases} \sqrt{E}n_2 & if \quad i = 1 \\ E + \sqrt{E}n_2 & if \quad i = 2 \end{cases}$$

where n_1 and n_2 are zero mean, Gaussian random variables produced by the additive white Gaussian noise signal. The final step is to subtract these two signals and look at the value of the result. Because n_1 and n_2 are zero mean with variance $N_0/2$ on average they will be zero and can be ignored.

$$x_1 - x_2 = \begin{cases} > 0 & if \quad i = 1 \\ < 0 & if \quad i = 2 \end{cases}$$

Part b

There are two signals that can be transmitted, thus we only require a single bit. Ea
signal has energy E, so average energy is (E + E) / 2 = E.

$$\boxed{E_{av,bit} = E}$$

Part c

Signals are orthogonal if they satisfy the constraint

$$\int_0^T s_1(t)s_2(t)dt = 0$$

Multiplying our two signals together results in the first signal with its amplitude sc

$$\int_0^T s_1(t)s_2(t)dt = \frac{E}{T}\left(\frac{T}{2}\right) - \frac{E}{T}\left(\frac{T}{2}\right) = 0$$

Thus, these two signals are orthogonal.

Part d

By definition, the probability of error when noise power spectral density equals N

$$\Pr\{\varepsilon\} = Q\left(\frac{d_{1,2}}{\sqrt{2N_0}}\right)$$

We now need to find $d_{1,2}$. Again, by definition

$$d_{1,2}{}^2 = E_2 + E_1 - 2\rho\sqrt{E_2 E_1} = 2E(1-\rho) = 2E$$

where we used the fact that for both of our basis functions the signal energies were
to E and since the signals are orthogonal, the correlation coefficient ρ = 0. Thus o
probability of bit error is

$$\boxed{\Pr\{\varepsilon\} = Q\left(\sqrt{\frac{E}{N_0}}\right)}$$

Part e

Now, $s_2(t) = 0$

Since one of our signals we can now use a single branch receive

The energy of the second signal is now zero, so the average ene
(E + 0) / 2 = E / 2.

$$\boxed{E_{av,bit} = \frac{E}{2}}$$

The signals are indeed still orthogonal:

$$\int_0^T s_1(t)s_2(t)dt = \int_0^T 0dt = 0$$

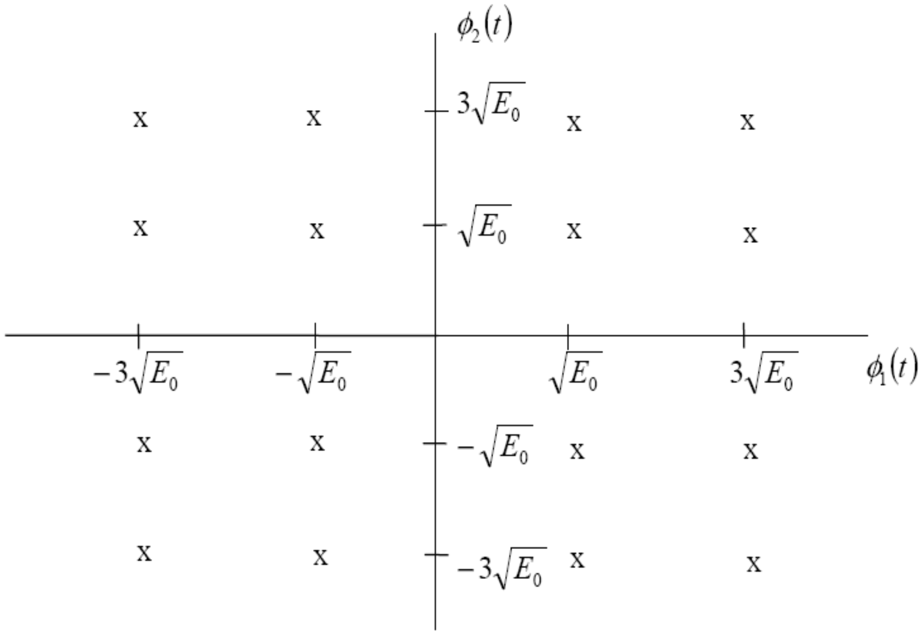
Since our second signal energy is zero, our distance drops to $d_{1,}$

$$\boxed{\Pr\{\varepsilon\} = Q\left(\sqrt{\frac{E}{2N_0}}\right)}$$

Problem 2

Part a

Our square QAM constellation is shown below. For simplicity we examine the M = 16 (L = 4) case, but we generalize our equations.



A QAM signal can be written as $x(t) = a_1\phi_1(t) - a_2\phi_2(t)$ where

$$\phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_0 t)$$

$$\phi_2(t) = \sqrt{\frac{2}{T}}\sin(2\pi f_0 t)$$

Examining the plot we see that $a_i = \pm(2i-1)\sqrt{E_0}$, $i = 1,2,\ldots,L/2$. The a_i are the amplitudes of each signal. Since the basis functions are orthonormal the energy of each signal is simply the sum of the squares of the amplitudes:

$$E_i = \int_0^T (a_1\phi_1(t) - a_2\phi_2(t))^2\,dt = a_1^{\,2}\int_0^T \phi_1^{\,2}(t)dt + a_2^{\,2}\int_0^T \phi_2^{\,2}(t)dt - 2a_1a_2\int_0^T \phi_1(t)\phi_2(t)dt$$

$$= a_1^{\,2} + a_2^{\,2} = 2a_1^{\,2} = 2(2i-1)^2 E_0$$

Here, we are focusing on a single quadrant and using the fact that due to the nature of QAM, $a_1 = a_2$ when we sum over all possible outcomes. Also, we need to multiply by the factor L/2 when summing over an entire quadrant. (try writing out this sum for two simple cases of L = 4 or L = 8 to see why this works) Our single quadrant sum is

$$\left(\frac{L}{2}\right)2E_0\sum_{i=1}^{L/2}(2i-1)^2 = LE_0\sum_{i=1}^{L/2}(2i-1)^2$$

To find the average energy we need to sum the energy of all signals and divide by the total number of signals (assuming that each signal is equally probable). This requires multiplying our quadrant sum by 4 to account for all of the quadrants and dividing by our total number of signals, $M = L^2$.

$$E_{av} = \frac{4}{L^2}\begin{pmatrix} single \\ quadrant \\ sum \end{pmatrix} = \frac{4E_0}{L}\sum_{i=1}^{L/2}(2i-1)^2 = \frac{4E_0}{L}\left[4\sum_{i=1}^{L/2}i^2 - 4\sum_{i=1}^{L/2}i + \sum_{i=1}^{L/2}(1)\right]$$

Next, use these identities to remove the summations:

$$\sum_{i=1}^{n}i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n}i = \frac{n(n+1)}{2}$$

$$\begin{aligned} E_{av} &= \frac{4E_0}{L}\left[\left(\frac{L}{3}\right)\left(\frac{L}{2}+1\right)(L+1) - L\left(\frac{L}{2}+1\right) + \frac{L}{2}\right] \\ &= E_0\left[\left(\frac{2}{3}\right)(L+2)(L+1) - 2(L+2) + 2\right] \\ &= E_0\frac{2}{3}\left(L^2-1\right) \end{aligned}$$

Finally, recall that the minimum distance between two orthonormal signals is $d_{\min} = 2\sqrt{E_0}$ and thus $E_0 = \frac{d_{\min}{}^2}{4}$. Substitute this, along with $L^2 = M$, into E_{av}.

$$E_{av} = 2\left[\frac{d_{\min}{}^2(M-1)}{12}\right]$$

$$\boxed{E_{av} = \left[\frac{(M-1)}{6}\right]d_{\min}{}^2}$$

Part b

The strongest signal occurs when the two orthonormal signals are farthest apa happens at the extreme corners of the QAM constellation.

$$E_{\max} = 2\left(2\left(\frac{L}{2}\right)-1\right)^2 E_0 = 2\left(\sqrt{M}-1\right)^2 \frac{d_{\min}{}^2}{4} = \left(\sqrt{M}-1\right)^2 \frac{d_{\min}{}^2}{2}$$

$$\frac{E_{\max}}{E_{av}} = \frac{\left[\left(\sqrt{M}-1\right)^2\right] d_{\min}{}^2 \big/ 2}{\left[(M-1)/6\right] d_{\min}{}^2} = 3\frac{\left(\sqrt{M}-1\right)^2}{(M-1)} = 3\frac{(L-1)^2}{\left(L^2-1\right)} = 3\frac{L-1}{L+1}$$

$$\boxed{\frac{E_{\max}}{E_{av}} = 3\frac{L-1}{L+1}}$$

Part c

$$\lim_{M\to\infty}\left[3\frac{\left(M-2\sqrt{M}+1\right)}{(M-1)}\right] = 3\lim_{M\to\infty}\left[\frac{1-1/\sqrt{M}}{1}\right] = 3$$

where we used L'Hôpital's rule after the first equality.

$$\boxed{\frac{E_{\max}}{E_{av}} = 3 \qquad \mathit{for} \quad M \;\; \mathit{very} \;\; \mathit{large}}$$

Problem 3

Part a

We can use the equation for average energy that we derived in problem 1 to find the average energy of each of the M = 16 constellations.

$$E_{av} = \left[\frac{(M-1)}{6}\right] d_{\min}{}^2 = \frac{15}{6} d_{\min}{}^2 = \frac{5}{2} d_{\min}{}^2$$

There are two constellations, each with equal average energies, so the overall system average is the same.

$$\boxed{E_{av} = \frac{5}{2} d_{\min}{}^2}$$

Part b

The system is composed of two constellations, each of which have two dimensions. Thus the total number of dimensions in the system is four.

$$\boxed{\text{dimensions} = 4}$$

Part c

Since both of the systems are the QAM system shown on the last page, we know that within each system the minimum distance between two signals is d. What about between the two systems? Since we are given that $f_0T = 1, 2, 3, \ldots$ we know that all basis functions within the two systems are orthogonal to one another. Thus, comparing two points at the same location in the two different systems, say d/2, produces this same minimum distance.

$$E_{\min} = \int_0^T (s_1(t) - s_2(t))^2 \, dt = \left(\frac{d}{2}\right)^2 \int_0^T (\phi_1(t) + \phi_2(t) - \phi_3(t) - \phi_4(t))^2 \, dt = \left(\frac{d}{2}\right)^2 \cdot 4 = d^2$$

$$d_{\min} = \sqrt{E_{\min}} = d$$

$$\boxed{d_{\min} = d}$$

Part d

Within a single 16 QAM system, the largest distance between two signals occurs when the signals are located at extreme opposite corners of the constellation. This distance is equal to twice the distance of a single signal from the origin.

$$d_{\max} = 2\left[\sqrt{\left(\frac{3d}{2}\right)^2 + \left(\frac{3d}{2}\right)^2}\right] = 3\sqrt{2}d$$

We can also examine this case between the two separate systems. The energy between two signals at extreme opposite corners (one in each system) is

$$E_{\max} = \int_0^T (s_1(t) - s_2(t))^2\, dt = \left(\frac{3d}{2}\right)^2 \int_0^T (\phi_1(t) + \phi_2(t) - (-\phi_3(t) - \phi_4(t)))^2\, dt$$

$$= \left(\frac{3d}{2}\right)^2 \int_0^T (\phi_1(t) + \phi_2(t) + \phi_3(t) + \phi_4(t))^2\, dt = \left(\frac{3d}{2}\right)^2 \cdot 4 = 9d^2$$

$$d_{\max} = \sqrt{E_{\max}} = 3d$$

The first value is larger, so it is the maximum distance.

$$\boxed{d_{\max} = 3\sqrt{2}d}$$

Part e

Nothing changes. All four of the basis functions are still orthogonal.