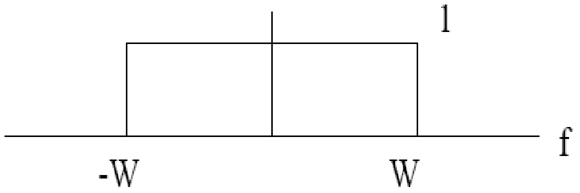
Solutions Homework #7
Introduction to Communication Systems – E3701

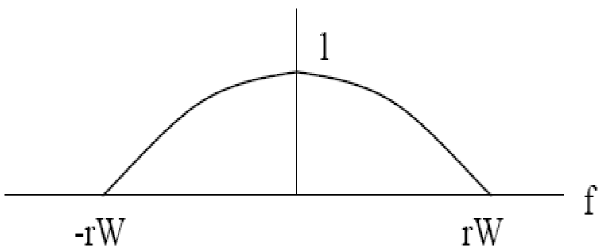
Problem 1

We are told that one of the two spectra is the ideal rectangular spectrum, shown below.



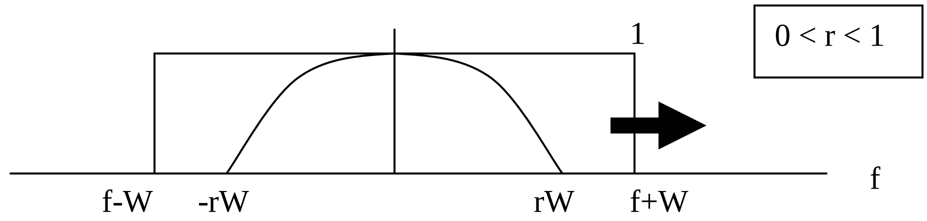
$$Y(f) = rect\left(\frac{f}{2W}\right)$$

By examining the given spectrum, S(f), we see that it start and ends at –(1+r)W and (1+r)W, respectively. Thus, we know the end points of our second spectrum must be ±rW. After some experimentation we can find that the required spectrum is a cosine function for frequencies between these two points.



$$X(f) = \cos\frac{2\pi f}{4rW} \quad , \quad |f| < rW$$

If we convolve these two functions in the frequency domain we can check that we obtain the correct spectrum. Please note that 0 < r < 1.



If we consider first the case when the cosine function is entirely inside of the rectangle during convolution we can use the following integral:

$$X(f)*Y(f)=\int_{-rW}^{rW}\cos\left(\frac{2\pi\phi}{4rW}\right)d\phi=\frac{4rW}{2\pi}\left[\sin\left(\frac{2\pi\phi}{4rW}\right)\Big|_{-rW}^{rW}\right]=\frac{4rW}{\pi}$$

As we expect, this area remains constant as long as f is in the range:

$$f+W>rW \quad \cap \quad f-W<-rW$$
$$f>-W(1-r) \quad \cap \quad f<W(1-r)$$
$$-W(1-r)<f<W(1-r)$$
$$0<|f|<W(1-r)$$

It is also obvious that if the rectangle and cosine do not overlap at all the convolution integral is zero. This occurs for f in the range:

$$f>W(1+r) \quad \cup \quad f<-W(1+r)$$
$$|f|>W(1+r)$$

The final case to examine is when only part of the rectangle overlaps the cosine function. Since this area will change with frequency, we expect our integral to result in a function of f.

$$X(f)*Y(f)=\int_{f-W}^{rW}\cos\left(\frac{2\pi\phi}{4rW}\right)d\phi=\frac{4rW}{2\pi}\left[\sin\left(\frac{2\pi\phi}{4rW}\right)\Big|_{f-W}^{rW}\right]=\frac{4rW}{2\pi}\left[1-\sin\left(\frac{2\pi(f-W)}{4rW}\right)\right]$$

As we expect, this convolution is a function of frequency, f. Note also that this integral was only calculated for the lagging edge of the rectangle function, f-W. However, since the cosine function is odd symmetric we can use the same function for the leading edge of the rectangle function by cleverly defining the frequency range. The frequency range for the lagging edge is:

$$-rW < f - W < rW$$
$$W(1-r) < f < W(1+r)$$

Due to the cosine symmetry we can use the same convolved formula above with the same frequency range by simply taking the absolute value of f.

The final step is to notice that we need to normalize our equations by $\frac{\pi}{4rW}$, as our spectrum has a maximum height of unity. Our spectrum equation is thus:

$$H(f) = \begin{cases} 1 & , \quad 0 < |f| < W(1-r) \\ \frac{1}{2}\left[1 - \sin\left(\frac{2\pi(|f| - W)}{4rW}\right)\right] & , \quad W(1-r) < |f| < W(1+r) \\ 0 & , \quad |f| > W(1+r) \end{cases}$$

The final step is to transform this to the time domain. We have a rectangle function convolved with a frequency limited cosine function. The limited cosine function can be created by multiplying the full cosine with an appropriate rectangle function, shown below, remembering to add in our normalizing factor.

$$H(f) = \left(\frac{\pi}{4rW}\right) rect\left(\frac{f}{2W}\right) * \left(rect\left(\frac{f}{2rW}\right)\cos\left(\frac{2\pi f}{4rW}\right)\right)$$

Our transform now consists of one multiplication and one convolution. The remaining steps are as follows.

$$h(t)=\Im\{H(f)\}=\left(\frac{\pi}{4rW}\right)\Im\left\{rect\left(\frac{f}{2W}\right)\right\}\left[\Im\left\{rect\left(\frac{f}{2rW}\right)\right\}*\Im\left\{\cos\left(\frac{2\pi f}{4rW}\right)\right\}\right]$$

$$=\left(\frac{\pi}{4rW}\right)2Wsinc(2Wt)\left[2rWsinc(2rWt)*\left(\frac{1}{2}\right)\left(\delta\left(t-\frac{1}{4rW}\right)+\delta\left(t+\frac{1}{4rW}\right)\right)\right]$$

$$=\left(\frac{\pi}{4}\right)2Wsinc(2Wt)\left[sinc\left(2rW\left(t-\frac{1}{4rW}\right)\right)+sinc\left(2rW\left(t+\frac{1}{4rW}\right)\right)\right]$$

$$=\left(\frac{\pi}{4}\right)2Wsinc(2Wt)\left[\frac{\sin\left(2\pi rWt-\frac{\pi}{2}\right)}{2\pi rWt-\frac{\pi}{2}}+\frac{\sin\left(2\pi rWt+\frac{\pi}{2}\right)}{2\pi rWt+\frac{\pi}{2}}\right]$$

$$=\left(\frac{\pi}{4}\right)2Wsinc(2Wt)\left[\frac{-\cos(2\pi rWt)}{2\pi rWt-\frac{\pi}{2}}+\frac{\cos(2\pi rWt)}{2\pi rWt+\frac{\pi}{2}}\right]$$

$$=\left(\frac{\pi}{4}\right)2Wsinc(2Wt)\cos(2\pi rWt)\left[\frac{-2\pi rWt-\frac{\pi}{2}+2\pi rWt-\frac{\pi}{2}}{(2\pi rWt)^2-\left(\frac{\pi}{2}\right)^2}\right]$$

$$=\left(\frac{\pi^2}{4}\right)2Wsinc(2Wt)\left[\frac{\cos(2\pi rWt)}{\left(\frac{\pi}{2}\right)^2-(2\pi rWt)^2}\right]$$

$$=\left(\frac{\pi^2}{4}\right)\left(\frac{2}{\pi}\right)^2 2Wsinc(2Wt)\left[\frac{\cos(2\pi rWt)}{1-(4rWt)^2}\right]$$

$$=2Wsinc(2Wt)\left[\frac{\cos(2\pi rWt)}{1-(4rWt)^2}\right]$$

We thus arrive at the final time domain solution.

$$\boxed{h(t)=2Wsinc(2Wt)\left[\frac{\cos(2\pi rWt)}{1-(4rWt)^2}\right]}$$

Problem 2

Part a

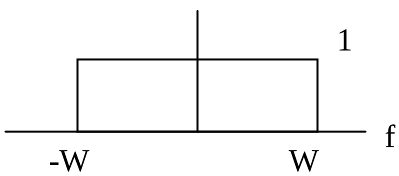
$$\Im\{s(t)-s(t-2T)\} = S(f) - S(f)e^{-j2\pi f(2T)} = S(f)\left[1-e^{-j2\pi f\frac{1}{W}}\right]$$

$$= S(f)e^{-j\pi f\frac{1}{W}}\left[e^{j\pi f\frac{1}{W}} - e^{-j\pi f\frac{1}{W}}\right]$$

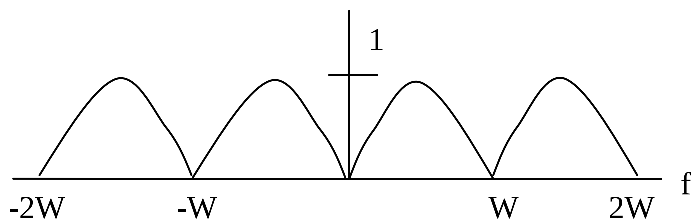
$$= S(f)e^{-j\pi f\frac{1}{W}}(2j)\sin\left(\pi f\frac{1}{W}\right)$$

$$= 2S(f)\sin\left(\pi f\frac{1}{W}\right)e^{-j\pi\left(\frac{f}{W}-\frac{1}{2}\right)}$$

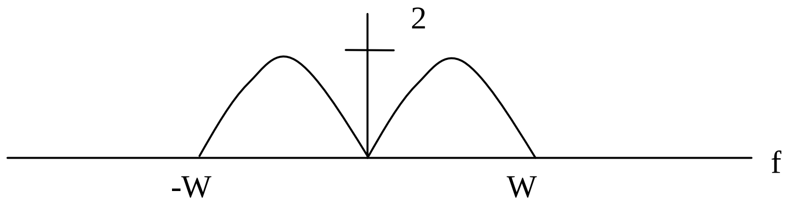
Magnitude spectrum $= 2S(f)\sin\left(\pi f\frac{1}{W}\right)$



S(f)



$\mathbf{\sin\left(\pi f\frac{1}{W}\right)}$



$\mathbf{2S(f)\sin\left(\pi f\frac{1}{W}\right)}$

Part b

$$S(f) = rect\left(\frac{f}{2W}\right) \quad \Leftrightarrow \quad s(t) = 2Wsinc(2Wt)$$

$$s(t-2T) = s\left(t - \frac{1}{W}\right) = 2Wsinc\left(2W\left(t - \frac{1}{W}\right)\right)$$

$$s(t-2T) = 2Wsinc(2Wt-2)$$

$$p_{mod-duo}(t) = s(t) - s(t-2T) = 2W[sinc(2Wt) - sinc(2Wt-2)]$$

$$= 2W\left[\frac{\sin(2W\pi t)}{2W\pi t} - \frac{\sin(2W\pi t - 2\pi)}{2W\pi t - 2\pi}\right]$$

$$= 2W\left[\frac{\sin(2W\pi t)}{2W\pi t} - \frac{\sin(2W\pi t)}{2W\pi t - 2\pi}\right]$$

$$= 2W\sin(2W\pi t)\left[\frac{1}{2W\pi t} - \frac{1}{2W\pi t - 2\pi}\right]$$

$$= 2W\sin(2W\pi t)\left[\frac{-1}{2W\pi t(Wt-1)}\right]$$

$$= \sin(2W\pi t)\left[\frac{-1}{\pi W t^2 - \pi t}\right]$$

Part c

$$E_x \equiv \int_{-\infty}^{\infty} x(t)^2\, dt = \int_{-\infty}^{\infty} |X(f)|^2\, df$$

$$E_{S(f)} = \int_{-W}^{W} (1)^2\, df = 2W$$

$$E_{P(f)} = \int_{-W}^{W} \left|2\sin\left(\pi f \frac{1}{W}\right)\right|^2 df = 4W$$

$$E_{P(f)} \big/ E_{S(f)} = \frac{4W}{2W} = 2$$

The modified duobinary signal has two times more energy than the perfect Nyquist signal.