Solutions Homework #5
Introduction to Communication Systems – E3701

Problem 2.44

Part a

We first write out our standard equation for an FM signal:

$$x_{FM}(t) = A\cos\left(2\pi f_0 t + 2\pi h \int_{-\infty}^{t} m(\tau)d\tau\right)$$

We next use a standard trigonometric identity to rewrite the equation:

$$x_{FM}(t) = A\cos\left(2\pi h \int_{-\infty}^{t} m(\tau)d\tau\right)\cos(2\pi f_0 t) - A\sin\left(2\pi h \int_{-\infty}^{t} m(\tau)d\tau\right)\sin(2\pi f_0 t)$$

Now, we are told that the signal is narrowband so we know that h<<1. This enables us to use the trigonometric approximations

$$\cos(small) \approx 1 \qquad \sin(small) \approx small$$

Wherever we see an 'h' multiplying something inside of a sine or cosine we can use these approximations. Applying these approximations results in

$$x_{FM}(t) = A\cos(2\pi f_0 t) - A2\pi h \int_{-\infty}^{t} m(\tau)d\tau \sin(2\pi f_0 t)$$

Recall some common Hilbert transform properties:

$$H\{\cos(2\pi f_0 t)\} = \sin(2\pi f_0 t) \qquad H\{s(t)\cos(2\pi f_0 t)\} = s(t)\sin(2\pi f_0 t)$$
$$H\{\sin(2\pi f_0 t)\} = -\cos(2\pi f_0 t) \qquad H\{s(t)\sin(2\pi f_0 t)\} = -s(t)\cos(2\pi f_0 t)$$

If we evaluate the integral in our FM signal it becomes simply a function of it, ie s(t). We can use this knowledge to apply the Hilbert transform properties to our FM signal to arrive at our final solution:

$$\boxed{H(x_{FM}(t)) = \hat{x}_{FM}(t) = A\sin(2\pi f_0 t) + A2\pi h\left[\int_{-\infty}^{t} m(\tau)d\tau\right]\cos(2\pi f_0 t)}$$

Part b

If we integrate the given modulating signal, m(t), we arrive at the following FM signal:

$$x_{FM}(t) = A\cos\left(2\pi f_0 t + \beta\sin\left(2\pi f_m t\right)\right)$$

We can use another trigonometric identity here.

$$x_{FM}(t) = A\cos\left(\beta\sin\left(2\pi f_m t\right)\right)\cos\left(2\pi f_0 t\right) - A\sin\left(\beta\sin\left(2\pi f_m t\right)\right)\sin\left(2\pi f_0 t\right)$$

Now, since $\beta\sin\left(2\pi f_m t\right)$ is a small, slowly time varying signal (relative to f_0) we can treat it as our s(t) signal. Using this notation, the above Hilbert transform properties, and some trigonometric identities we arrive at

$$\begin{aligned}
H\left\{x_{FM}(t)\right\} = \hat{x}_{FM}(t) &= A\cos\left(\beta\sin\left(2\pi f_m t\right)\right)\sin\left(2\pi f_0 t\right) + A\sin\left(\beta\sin\left(2\pi f_m t\right)\right)\cos\left(2\pi f_0 t\right) \\
&= A\frac{1}{2}\left[\sin\left(\beta\sin\left(2\pi f_m t\right) - 2\pi f_0 t\right) + \sin\left(\beta\sin\left(2\pi f_m t\right) + 2\pi f_0 t\right)\right] \\
&\quad + A\frac{1}{2}\left[\sin\left(2\pi f_0 t - \beta\sin\left(2\pi f_m t\right)\right) + \sin\left(2\pi f_0 t + \beta\sin\left(2\pi f_m t\right)\right)\right] \\
&= A\sin\left(2\pi f_0 t + \beta\sin\left(2\pi f_m t\right)\right) \\
&\quad + A\frac{1}{2}\sin\left(\beta\sin\left(2\pi f_m t\right) - 2\pi f_0 t\right) + \sin\left(2\pi f_0 t - \beta\sin\left(2\pi f_m t\right)\right) \\
&= A\sin\left(2\pi f_0 t + \beta\sin\left(2\pi f_m t\right)\right) \\
&\quad + A\frac{1}{2}\sin\left(\beta\sin\left(2\pi f_m t\right) - 2\pi f_0 t\right) - \sin\left(\beta\sin\left(2\pi f_m t\right) - 2\pi f_0 t\right)
\end{aligned}$$

We thus arrive at the final solution:

$$\boxed{x_{FM}(t) = A\sin\left(2\pi f_0 t + \beta\sin\left(2\pi f_m t\right)\right)}$$

Problem 2

Part a

This is identical to the integral in problem 2.44, part b:

$$\boxed{x_{FM}(t) = A\cos\left(2\pi f_0 t + \beta \sin(2\pi W t)\right), \quad \beta = \frac{hA_m}{W}}$$

Part b

$$\boxed{\Delta f = hA_m}$$

Part c

This was covered in class. The first BPF eliminates any noise in the spectrum. The output of the hard limiter looks like a variable frequency square wave. W can write the output of the hard limiter as follows:

$$x_{HL}(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\theta(t)}, \quad c_n = k\frac{\sin\left(n\pi/2\right)}{n\pi/2} \quad \theta(t) = 2\pi f_0 t + \beta \sin(2\pi W t) \quad and \quad k = 1$$

Recall that complex exponentials can be expanded using Euler's formula:

$$e^{jx} = \cos x + j\sin x \qquad e^{-jx} = \cos x - j\sin x$$

Thus, if we evaluate our Fourier sum from minus infinity to plus infinity, all of the jsin terms cancel while all of the cosine terms add. We are left with

$$\begin{aligned} x_{HL}(t) &= \sum_{n=-\infty}^{\infty} c_n e^{jn\theta(t)} = \sum_{n=1}^{\infty} 2k\frac{\sin\left(n\pi/2\right)}{n\pi/2}\cos(n\theta(t)) \\ &= \frac{4k}{\pi}\cos\left(2\pi f_0 t + \beta\sin(2\pi W t)\right) - \frac{4k}{3\pi}\cos\left(6\pi f_0 t + \beta\sin(2\pi W t)\right) + \cdots \end{aligned}$$

All of the high frequency terms are eliminated when the hard-limited signal passes through the second BPF. Also note that the maximum amplitude of the hard limiter is k. Thus, due to the nonlinear nature of the hard limiter none of our terms can have an amplitude larger than k. We are left with

$$e(t) = k\cos(2\pi f_0 t + \beta\sin(2\pi Wt))$$

Passing through the differentiator results in

$$\begin{aligned}\frac{1}{2\pi}\frac{d}{dt}e(t) &= \frac{1}{2\pi}k(-1)\sin(2\pi f_0 t + \beta\sin(2\pi Wt))(2\pi f_0 + \beta 2\pi W\cos(2\pi Wt)) \\ &= -k\sin(2\pi f_0 t + \beta\sin(2\pi Wt))(f_0 + \beta W\cos(2\pi Wt))\end{aligned}$$

This looks similar to our AM signal, with the HF part inside the sine term and the right hand factor representing the modulating signal. Thus, after we pass the above signal through the envelope detector we arrive at

$$v_{env}(t) = kf_0 + k\beta W\cos(2\pi Wt), \quad \beta W = \Delta f$$

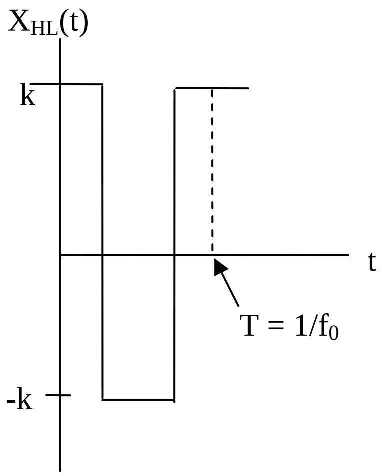
The final step is to pass this signal through the DC blocking capacitor. We thus arrive at the final output signal:

$$\boxed{v_{out}(t) = k\Delta f\cos(2\pi Wt)}$$

Part d

$$x_{AM}(t) = A[1 + ms(t)]\cos(2\pi f_0 t), \quad s(t) = B\cos(2\pi Wt)$$

We travel through the hard limiter. This time, due to the nature of AM, our signal is truly periodic and looks as follows:



We can see that now our signal coming out of the hard limiter, x_{HL}, is truly periodic and looks like a square wave with amplitude k. The second BPF following the hard limiter eliminates all frequencies above f_0. If you recall, a square wave can be written as a sum of sinusoids of increasing frequencies. Since f_0 is the fundamental of this square wave it is allowed to pass while all of the remaining HF terms are eliminated. Thus our square wave is smoothed and the remaining signal is

$$e(t) = k\cos(2\pi f_0 t)$$

We pass this signal through the differentiator.

$$\frac{1}{2\pi}\frac{d}{dt}e(t) = \frac{k}{2\pi}(-2\pi f_0)\sin(2\pi f_0 t) = -kf_0\sin(2\pi f_0 t)$$

This signal is in turn passed through the envelope detector producing the following signal.

$$v_{env}(t) = kf_0$$

The DC blocking capacitor eliminates this signal to produce the final result.

$$\boxed{v_{out}(t) = 0}$$

Part e

AM Case, SSB

$$e(t) = k\cos(2\pi f_0 t)$$

The first BPF has no effect since it is centered at our carrier frequency, f_0. Multiplying by the cosine squares our cosine term. We can use a trigonometric property.

$$k\cos^2(2\pi f_0 t) = \frac{k}{2}\left[1 + \cos(2\pi(2f_0)t)\right]$$

The final LPF eliminates the HF cosine term. We are left with a constant.

$$\boxed{v_{out}(t) = \frac{k}{2}}$$

AM Case, DSB

The only difference here is the BW of the first BPF. Since our signal is simply a spike in the frequency domain the different BW has no effect on the signal. Thus we obtain the same answer as we did in the SSB case.

$$\boxed{v_{out}(t) = \frac{k}{2}}$$

FM Case, SSB

$$e(t) = k\cos(2\pi f_0 t + \beta\sin(2\pi Wt))$$

We know from our class notes that this signal can be written as

$$e(t) = k\,\mathrm{Re}\left\{e^{j2\pi f_0 t}e^{j\beta\sin(2\pi Wt)}\right\}$$

And from here we know that $e^{j\beta\sin(2\pi Wt)}$ can be written as a Fourier Series. Thus, we can think of our incoming signal as

$$e(t) = k\sum_{n=-\infty}^{\infty} J_n(\beta)\cos(2\pi(f_0 + nW)t)$$

Again, once we pass this signal through the first BPF it eliminates the HF terms where n ≥ 2, leaving us with

$$e_{BPF}(t) = kJ_1(\beta)\cos(2\pi(f_0 + W)t)$$

Note that since our BPF only extends to the right of our center frequency we only pass the term f_0 + W and we eliminate the term f_0 – W. We multiply be the carrier cosine and use a trigonometric property.

$$\begin{aligned} e_{BPF}(t)\cos(2\pi f_0 t) &= kJ_1(\beta)\cos(2\pi(f_0 + W)t)\cos(2\pi f_0 t) \\ &= \frac{k}{2}J_1(\beta)\cos(2\pi(2f_0 + W)t) + \frac{k}{2}J_1(\beta)\cos(2\pi Wt) \end{aligned}$$

Our final LPF eliminates the HF term and we are left with the solution.

$$\boxed{v_{out}(t) = \frac{k}{2}J_1(\beta)\cos(2\pi Wt)}$$

FM Case, DSB

$$e(t) = k\cos(2\pi f_0 t + \beta\sin(2\pi Wt))$$

For this final case we can again express our input signal using the Fourier Series expression.

$$e(t) = k\sum_{n=-\infty}^{\infty} J_n(\beta)\cos(2\pi(f_0 + nW)t)$$

We pass this signal through the BPF. However, this time notice that the filter extends *left* by W as well as right. So now we not only pass the low frequency term f_0 + W but we *also* pass the low frequency term f_0 – W. We arrive at

$$e_{BPF}(t) = kJ_1(\beta)\cos(2\pi(f_0 + W)t) - kJ_1(\beta)\cos(2\pi(f_0 - W)t)$$

Note that the Bessel function is an odd function, ie $J_{-1}(\beta) = -J_1(\beta)$. This accounts for the minus sign in the middle of our $e_{BPF}(t)$ term.
We multiply by our carrier cosine and apply the same trigonometric property.

$$\begin{aligned} e_{BPF}(t)\cos(2\pi f_0 t) &= kJ_1(\beta)\cos(2\pi(f_0 + W)t)\cos(2\pi f_0 t) - kJ_1(\beta)\cos(2\pi(f_0 - W)t)\cos(2\pi f_0 t) \\ &= \frac{k}{2}J_1(\beta)[\cos(2\pi(2f_0 + W)t) + \cos(2\pi Wt)] - \frac{k}{2}J_1(\beta)[\cos(2\pi(2f_0 - W)t) + \cos(-2\pi Wt)] \end{aligned}$$

The LPF eliminates the HF terms as usual and we are left with the following. Note that cosine is also an odd function.

$$v_{out}(t) = \frac{k}{2}J_1(\beta)\cos(2\pi Wt) - \frac{k}{2}J_1(\beta)\cos(-2\pi Wt) = \frac{k}{2}J_1(\beta)\cos(2\pi Wt) - \frac{k}{2}J_1(\beta)\cos(2\pi Wt) = 0$$

$$\boxed{v_{out}(t) = 0}$$