Solutions Homework #1
Introduction to Communication Systems – E3701

Problem 1:
a)

$$\mathcal{F}(s(t-\tau)) = \int_{-\infty}^{\infty} s(t-\tau)e^{-j2\pi ft}dt$$

Defining $u=t-\tau$:

$$\mathcal{F}(s(t-\tau)) = \int_{-\infty}^{\infty} s(u)e^{-j2\pi f(u+\tau)}du = e^{-j2\pi f\tau}\int_{-\infty}^{\infty} s(u)e^{-j2\pi fu}du = e^{-j2\pi f\tau}S(f)$$

b)

$$\mathcal{F}(s(t)\cos(2\pi f_o t)) = \mathcal{F}\left(s(t)\left(\frac{e^{j2\pi f_o t}+e^{-j2\pi f_o t}}{2}\right)\right) = \int_{-\infty}^{\infty} s(t)\left(\frac{e^{j2\pi f_o t}+e^{-j2\pi f_o t}}{2}\right)e^{-j2\pi ft}dt =$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} s(t)\left(e^{-j2\pi(f+f_o)t}+e^{-j2\pi(f-f_o)t}\right)dt = \frac{1}{2}\left(S(f+f_0)+S(f-f_0)\right)$$

c)

Since $S(f) = \int_{-\infty}^{\infty} s(t)e^{-j2\pi ft}dt$ by definition, we obtain $S(0)$ by evaluating the previous expression in f=0, which yields:

$$S(0) = \int_{-\infty}^{\infty} s(t)dt$$

d)

Since $s(t) = \int_{-\infty}^{\infty} S(f)e^{j2\pi ft}df$ by definition, we obtain $s(0)$ by evaluating the previous expression in t=0, which yields:

$$s(0) = \int_{-\infty}^{\infty} S(f)df$$

e)

S(f)= $\int_{-\infty}^{\infty} s(t)e^{-j2\pi ft}\,dt$ by definition thus taking the complex conjugate on both sides we obtain:

$S^*(f) = \int_{-\infty}^{\infty} s^*(t)e^{j2\pi ft}\,dt = \int_{-\infty}^{\infty} s(t)e^{j2\pi ft}\,dt$, since when *s(t)* is real, $s(t) = s^*(t)$.

*** *Note:* we here use the notation $s^*(t)$ to express the complex conjugate of *s(t)* ***

Now if we define u=-t in the previous equality, and given that when *s(t)* is even, then *s(t)=s(-t)*; we obtain:

$S^*(f) = \int_{-\infty}^{\infty} s(-u)e^{-j2\pi fu}\,du = \int_{-\infty}^{\infty} s(u)e^{-j2\pi fu}\,du$ =*S(f)*

So $S(f) = S^*(f)$, which means that *S(f)* is real.

A function s(t) is odd if *s(t)=-s(-t)*.
Reasoning in the same way as done with the even functions, we can conclude that when s(t) is real and odd, then
$S(f) = -S^*(f)$, which means that *S(f)* is imaginary.

f)

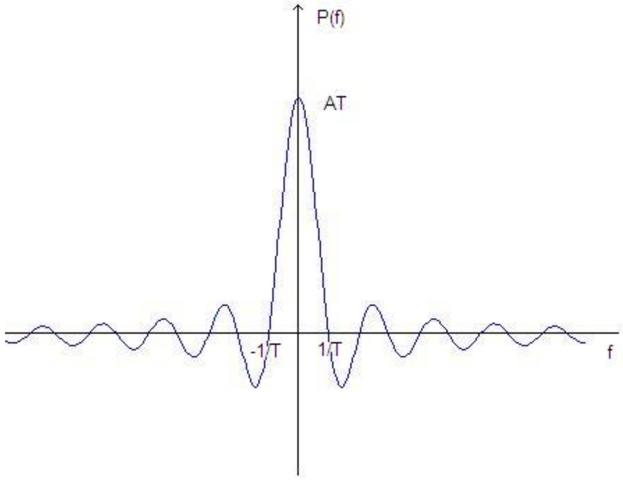
S(f)= $\int_{-\infty}^{\infty} s(t)e^{-j2\pi ft}\,dt$

since s(t) is a real function and the complex conjugate of $e^{-j2\pi ft}$ can be obtained by substituting –f for f, we can conclude that the Fourier transform of a real function for the negative frequencies is the complex conjugate of itself for the positive frequencies.

Problem 2:

$$P(f)=\int_{-\infty}^{\infty} p(t)e^{-j2\pi ft}\,dt=\int_{-T/2}^{T/2} Ae^{-j2\pi ft}\,dt=\mathrm{A}\left.\frac{e^{-j2\pi ft}}{-j2\pi f}\right|_{-T/2}^{T/2}=\frac{A}{2\pi f}\;\frac{e^{j2\pi f_o T/2}-e^{-j2\pi f_o T/2}}{j}=$$

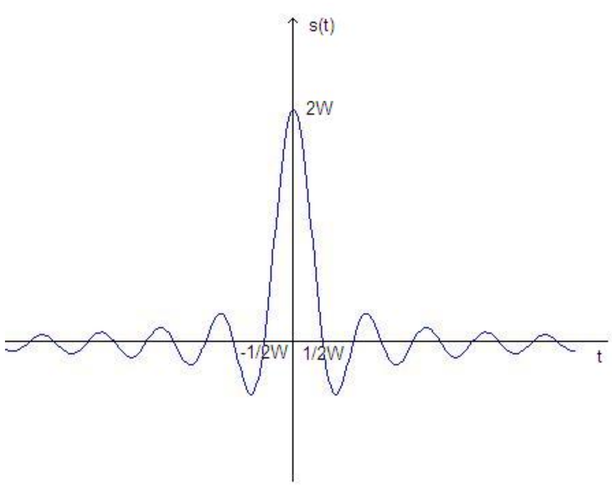
$$=\frac{A}{2\pi f}\,2\sin\left(\frac{2\pi fT}{2}\right)=\frac{A}{2\pi f}\;\frac{\sin\left(\frac{2\pi fT}{2}\right)}{\frac{2\pi fT}{2}}=\mathrm{AT}\ \mathrm{sinc}\left(\frac{2\pi fT}{2}\right)$$



Problem 3:

$$s(t)=\int_{-\infty}^{\infty} S(f)e^{j2\pi ft}\,df=\int_{-W}^{W} e^{j2\pi ft}\,df=\left.\frac{e^{j2\pi ft}}{j2\pi t}\right|_{-W}^{W}=\frac{e^{j2\pi Wt}-e^{-j2\pi Wt}}{j2\pi t}=$$

$$=\frac{1}{\pi.t}\ \sin(2\pi Wt)=2W\ \frac{\sin(2\pi Wt)}{2\pi Wt}=2W\ \mathrm{sinc}(2\pi Wt)$$

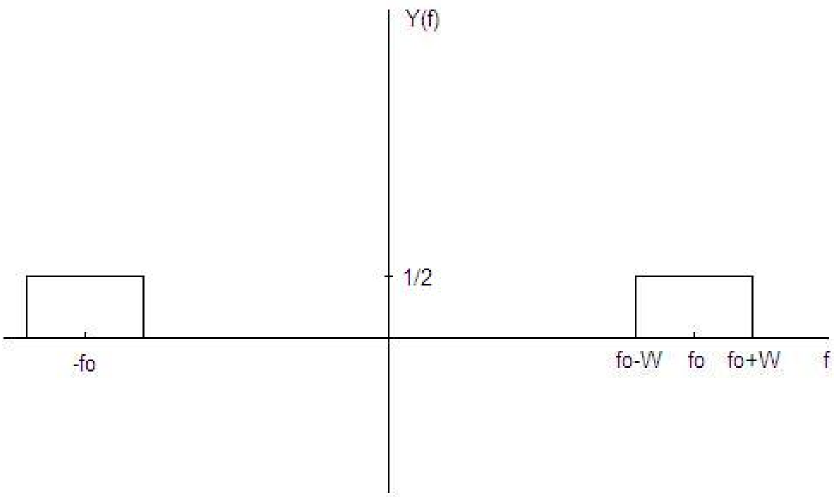


Problem 4:

$y(t=)s(t)cos(2\pi f_o t)$

According to problem 1.b:
$\mathscr{F}(s(t)cos(2\pi f_o t))= \frac{1}{2}(S(f+f_0)+S(f-f_0))$

Which means that if $s(t)= 2W\, sinc(2\pi Wt)$, then $Y(f)$ is half of $S(f)$ shifted and centered at f_o plus $S(f)$ shifted and centered at $-f_o$, as can be seen in the following figure:



The new bandwidth is twice the bandwidth of the original, s(t).

Problem 5:

$$Y(f)=H(f)X(f)= H(f) \int_{-\infty}^{\infty} x(\tau)e^{-j2\pi f\tau}d\tau = \int_{-\infty}^{\infty} x(\tau)H(f)e^{-j2\pi f\tau}d\tau$$

But according to Problem1.a: $= H(f)e^{-j2\pi f\tau} = \mathscr{F}(h(t-\tau))= \int_{-\infty}^{\infty} h(t-\tau)e^{-j2\pi ft}dt$, so:

$$Y(f)=H(f)X(f)= \int_{-\infty}^{\infty} x(\tau)\int_{-\infty}^{\infty} h(t-\tau)e^{-j2\pi ft}dt.d\tau \quad = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau\right]e^{-j2\pi ft}d\tau =$$

$$\mathscr{F}\left(\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \right)$$

But by definition, $Y(f) = \mathscr{F}(y(t))$, which means that

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = h(t)*x(t)$$