

# Modulated Branching Processes and Origins of Power Laws

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**Abstract**—Power law distributions have been repeatedly observed in a wide variety of socioeconomic, biological and technological areas, including: distributions of wealth, species-area relationships, populations of cities, values of companies, sizes of living organisms and, more recently, distributions of documents and visitors on the Web, etc. In the vast majority of these observations, e.g., city populations and sizes of living organisms, the objects of interest evolve due to the replication of their many independent components, e.g., births-deaths of individuals and replications of cells. Furthermore, the rates of replication of the many components are often controlled by exogenous parameters causing periods of expansion and contraction, e.g., baby booms and busts, economic booms and recessions, etc. In addition, the sizes of these objects often either have reflective lower boundaries, e.g., cities do not fall below a certain size, low income individuals are subsidized by the government, companies are protected by bankruptcy laws, etc; or have porous/absorbing lower boundaries, e.g., cities may degenerate, bankruptcy protection may fail and a company can be liquidated.

Hence, it is natural to propose reflected modulated branching processes as generic models for many of the preceding observations of power laws. Indeed, our main results show that these apparently new mathematical objects result in power law distributions under quite general “polynomial Gärtner-Ellis” conditions. The generality of our results could explain the ubiquitous nature of power law distributions. Furthermore, an informal interpretation of our main results suggests that alternating periods of expansion and reduction, e.g., economic booms and recessions, are primarily responsible for the appearance of power law distributions.

Our results also establish a general asymptotic equivalence between the reflected branching processes and the corresponding reflected multiplicative processes. Furthermore, in the course of our analysis, we discover a duality between the reflected multiplicative processes and queueing theory. Essentially, this duality demonstrates that the power law distributions play an equivalent role for reflected multiplicative processes as the exponential/geometric distributions do in queueing analysis.

**Index Terms**—Modulated branching processes, reflected multiplicative processes, proportional growth models, power law distributions, heavy tails, subexponential distributions, queueing processes, reflected additive random walks, Cramér large deviations, polynomial Gärtner-Ellis conditions.

## I. INTRODUCTION

Power law distributions are found in a wide range of domains, ranging from socioeconomic to biological and technological areas. Specifically, these types of distributions describe the city populations, species-area relationships, sizes of living organisms, value of companies, distribution of

wealth, and more recently, the distribution of documents on the Web, the number of visitors per Web site, etc. Hence, one would expect that there exist universal mathematical laws that explain this ubiquitous nature of power law distributions. To this end, we propose an apparently new class of models, termed reflected modulated branching processes, which, under quite general polynomial Gärtner-Ellis conditions, result in power law distributions.

Empirical observations of power laws have a long history, starting from the discovery by Pareto [1] in 1897 that a plot of the logarithm of the number of incomes above a level against the logarithm of that level yields points close to a straight line, which is essentially equivalent to saying that the income distribution follows a power law. Hence, power law distributions are often called Pareto distributions; for more recent study on income distributions see [2]–[6]. In a different context, early work by Arrhenius [7] in 1921 conjectured a power law relationship between the number of species and the census area, which was followed by Preston’s prediction in [8] that the slope on the log/log species-area plot has a canonical value equal to 0.262; for additional information and measurements on species-area relationships see [9]–[11]. Interestingly, there also exists a power law relationship between the rank of the cities and the population of the corresponding cities. This was proposed by Auerbach [12] in 1913 and later studied by Zipf [13], after whom power law is also known as Zipf’s law. Ever since, much attention on both empirical examinations and explanations on city size distributions have been drawn [13]–[18]. Similar observations have been made for firm sizes [19], and even the gene family and protein statistics [20]–[23]. It is maybe even more surprising that many features of the Internet are governed by power laws, including the distribution of pages per Web site [24], the page request distribution [25], [26], the file size distribution [27], [28], Ethernet LAN traffic [29], World Wide Web traffic [30], the number of visitors per Web site [31], [32], the distribution of scenes in MPEG video streams [33] and the distribution of the indegrees and outdegrees in the Web graph as well as the physical network connectivity graph [34]–[37]. In socio-economic areas, in addition to income distributions, the fluctuations in stock prices have also been observed to be characterized by power laws [38], [39].

Hence, these repeated empirical observations of power laws, over a period of more than a hundred years, strongly suggest that there exist general mathematical laws that gov-

ern these phenomena. In this regard, after carefully examining the situations that result in power laws, we discover that most of them are characterized by the following three features. First, in the vast majority of these observations, e.g., city populations and sizes of living organisms, the objects of interest evolve due to the replication of their many independent components, e.g., birth-deaths of individuals and replications of cells. Secondly, the rate of replication of the many components is often controlled by exogenous parameters causing periods of baby booms and busts, economic growths and recessions, etc. Thirdly, the sizes of these objects often have lower boundaries, e.g., cities do not fall below a certain size, low income individuals are subsidized by the government, companies are protected by bankruptcy laws, etc.

In order to capture the preceding features, it is natural to propose *reflected modulated branching processes* as generic models for many of the observations of power laws. Indeed, one of our main results, presented in Theorem 2, shows that these apparently new mathematical objects result in power law distributions under quite general polynomial Gärtner-Ellis conditions. The generality of our results could explain the ubiquitous nature of power law distributions. Furthermore, an informal interpretation of our main results, stated in Theorems 2 and 3 of Section III, suggests that alternating periods of expansions and contractions, e.g., economic booms and recessions, are primarily responsible for the appearance of power law distributions. From a mathematical perspective, we develop a novel mathematical technique for analyzing reflected modulated branching processes since these objects appear new and the traditional methods for investigating branching processes [40] do not directly apply.

Formal description of our reflected modulated branching process (RMBP) model is given in Section II. In the singular case when the number of individuals born in each state of the modulating process is constant, our model reduces to a reflected multiplicative process. In Subsection II-A we establish a rigorous connection (duality) between the reflected multiplicative processes (RMPs) and queueing theory. We would like to point out that this duality, although a minor point of our paper, makes a vast literature on queueing theory directly applicable to the analysis of RMPs. As a direct consequence of this connection, in Subsection II-A we translate several well known queueing results to the context of RMPs. Informally, these results show that the role which exponential distributions play in queueing theory, and in additive reflected random walks in general, is represented by power law distributions in the framework of RMPs/RMBPs. Furthermore, this relationship appears to reduce the debate on the relative importance of power law versus exponential distributions/models to the analogous question of the prevalence of proportional growth versus additive phenomena. In addition, this duality immediately implies and generalizes many of the prior results in the area of RMPs and power laws. Some of these prior results include the work of Levy and Solomon that appears to be the first to show how power laws can be obtained by adding a reflection condition to

a multiplicative process [39], [41], [42]; this was further analyzed by Sornette and Rama [43].

In addition, while the reduction of RMBPs to RMPs is apparent in the special case when constant number of individuals are born in each state of the modulating process, our main result, Theorem 2, reveals a deeper general asymptotic equivalence between the power law exponent of a RMBP and the corresponding RMP.

In some domains, e.g., the growth of living organisms, the objects always grow (basically never shrink) up until a certain random time. Huberman and Adamin [24] also propose this model as an explanation of the growth dynamics of the World Wide Web by arguing that the observation time is an exponential random variable. This notion has been revisited in [5] and generalized to a larger family of random processes observed at an exponential random time [44]. In this regard, in Subsection V-A.2, we study randomly stopped modulated branching processes and show, under more general conditions than the preceding studies, that the resulting variables follow power laws.

In many areas, objects of interest may not have a strictly reflecting barrier, but rather a porous one, e.g., cities may degenerate, bankruptcy protection may sometimes fail and a company can be liquidated. Hence, in Subsection V-B, we study MBP with an absorbing barrier and show that it leads to power law distributions as well. The result, under somewhat more restrictive conditions, is basically a direct corollary of Theorem 2 on RMBPs. We argue that these types of models can be natural candidates for describing the bursts of requests at popular Internet Web sites, often referred to as hotspots. In the prior study of city sizes, Bland and Solomon show [45], using heuristic arguments, that a multiplicative process with an absorbing barrier can result in power laws.

Based on our new model, we discuss two related phenomena: truncated power laws and double Pareto distributions. We argue that one can obtain a truncated power law distribution by adding an upper barrier to RMBP, similarly as the truncated geometric distributions appear in queueing theory, e.g., finite buffer  $M/M/1$  queue. Furthermore, by the duality of RMBP and queueing theory, we give two new natural explanations of the origins of double Pareto distributions that have been widely observed in practice. In the queueing context, it has been shown that the tail of the queue length distribution exhibits different decay rates in the heavy-traffic and large deviation regime, respectively [46]; similar behavior of the queue length distribution was attributed to the multiple time scale arrivals in [47]. We claim that the preceding two mechanisms, when translated to the proportional growth context, provide natural explanations of the double Pareto distributions.

Finally, we would like to mention that there might be other mechanisms that result in power law distributions, e.g., the “randomly typing model” used to explain the power law distribution of frequencies of words in natural languages (see [48]) and the “highly optimized tolerance” studied in [49]; for a recent survey on various mechanisms that result in power laws see [48].

## II. REFLECTED MODULATED BRANCHING PROCESSES

In this section we formally describe our model. Let  $\{J_n\}_{n>-\infty}$  be a stationary and ergodic modulating process that takes values in positive integers. Define a family of independent, non-negative, integer-valued random variables  $\{B_n^i(j)\}$ ,  $-\infty < i, j, n < \infty$ , which are independent of the modulation process  $\{J_n\}$ . In addition, for fixed  $j$ , variables  $\{B_n^i(j)\}$  are identically distributed with  $\mu(j) \triangleq \mathbb{E}[B_1^1(j)] < \infty$ .

*Definition 1:* A modulated branching process  $\{Z_n\}_{n=0}^\infty$  is recursively defined by

$$Z_{n+1} = \sum_{i=1}^{Z_n} B_n^i(J_n), \quad (1)$$

where the initial value  $Z_0$  is a positive integer. For convenience, when  $Z_0 = l$ , we denote the process by  $\{Z_n^l\}$ .

*Definition 2:* For any  $l \in \mathbb{N}$  and an integer valued  $\Lambda_0$ , a Reflected Modulated Branching Process (RMBP)  $\{\Lambda_n\}_{n=0}^\infty$  is recursively defined as

$$\Lambda_{n+1} = \max\left(\sum_{i=1}^{\Lambda_n} B_n^i(J_n), l\right). \quad (2)$$

*Remark 1:* These types of modulated branching processes, with or without a reflecting barrier, appear to be new and, thus, the traditional methods for the analysis of branching processes [40] do not seem to directly apply.

*Remark 2:* A more general framework would be to define

$$Z_{n+1} = \int_0^{Z_n} B_n^t(J_n(t)) d\nu(t), \quad (3)$$

for any real measure  $\nu$ , and, similarly,

$$\Lambda_{n+1} = \max\left(\int_0^{\Lambda_n} B_n^t(J_n(t)) d\nu(t), l\right), \quad (4)$$

where  $l > 0$  and  $B_n^t(J_n(t))$  is  $\nu$ -measurable. We refrain from this generalization since it introduces additional technical difficulties without any new insight. Now, we present the basic limiting results on the convergence to stationarity of  $Z_n$  and  $\Lambda_n$ .

*Lemma 1:* If  $\mathbb{E} \log \mu(J_0) < 0$ , then a.s., we have

$$\lim_{n \rightarrow \infty} Z_n = 0.$$

*Proof:* For all  $n \geq 1$ , let  $W_n = Z_n / \Pi_{n-1}$ , where  $\Pi_n = \prod_{i=0}^n \mu(J_i)$ . It is easy to check that  $W_n$  is a positive martingale with respect to the filtration  $\mathcal{F}_n = \sigma(J_i, Z_i, 0 \leq i \leq n-1)$ . Hence, by the martingale convergence theorem (see Theorem 35.5. of [50]), as  $n \rightarrow \infty$ ,

$$W_n \rightarrow W \text{ a.s.},$$

where  $W$  is a.s. positive finite. Next, since  $\{J_n\}$  is stationary and ergodic, so is  $\{\mu(J_n)\}$ , and, therefore,

$$\frac{\log \Pi_n}{n} = \frac{1}{n} \sum_{i=1}^n \log \mu(J_i) \rightarrow \mathbb{E} \log \mu(J_0) < 0 \text{ as } n \rightarrow \infty.$$

Thus,  $\Pi_n \rightarrow 0$  as  $n \rightarrow \infty$ , which yields the statement of the lemma. ■

Next, let  $Z_{-n}$  be the number of individuals at time 0 in an unrestricted branching process that starts at time  $-n$  with  $l$  individuals; when needed for clarity, we will use the notation  $Z_{-n}^l$  to explicitly indicate the initial state  $l$ .

*Lemma 2:* Assume  $\mathbb{E} \log \mu(J_0) < 0$ , then, for any a.s. finite initial condition  $\Lambda_0$ ,  $\Lambda_n$  converges in distribution to

$$\Lambda \stackrel{d}{=} \max_{n \geq 0} Z_{-n}.$$

*Proof:* First, assume that  $\Lambda_0 = l$  and let  $Z_n^k$  be the number of individuals at time  $n$  in an unrestricted branching process that starts at time  $k$  with  $l$  individuals. Then, by stationarity of  $\{J_n\}$ , we have  $Z_n^k \stackrel{d}{=} Z_{k-n}$ . Clearly,

$$\Lambda_1 = \max\left(\sum_{i=1}^l B_1^i(J_1), l\right) \stackrel{d}{=} \max\{Z_{-1}, Z_0\},$$

and, by induction and stationarity, it is easy to show

$$\Lambda_n \stackrel{d}{=} \max(Z_{-n}, Z_{-(n-1)}, \dots, Z_{-1}, Z_0),$$

which, by monotonicity, yields

$$\mathbb{P}[\Lambda_n > x] \rightarrow \mathbb{P}[\Lambda > x] \text{ as } n \rightarrow \infty.$$

Now, if  $\Lambda_n^{\Lambda_0}$  is a process with initial condition  $\Lambda_0 \geq l$ , then, it is easy to see that

$$\Lambda_n^{\Lambda_0} \geq \Lambda_n \geq l, \text{ for all } n,$$

implying

$$\mathbb{P}[\Lambda_n^{\Lambda_0} > x] \geq \mathbb{P}[\Lambda_n > x]. \quad (5)$$

If we define the stopping time  $\tau$  to be the first time when  $\Lambda_n^{\Lambda_0}$  hits the boundary  $l$ , then, the preceding monotonicity implies that  $\Lambda_n = \Lambda_n^{\Lambda_0}$  for all  $n \geq \tau$ . Using this observation, we obtain

$$\begin{aligned} \mathbb{P}[\Lambda_n^{\Lambda_0} > x] &= \mathbb{P}[\Lambda_n^{\Lambda_0} > x, \tau > n] + \mathbb{P}[\Lambda_n^{\Lambda_0} > x, \tau \leq n] \\ &\leq \mathbb{P}[\Lambda_n^{\Lambda_0} > x, \tau > n] + \mathbb{P}[\Lambda_n > x, \tau \leq n] \\ &\leq \mathbb{P}[\tau > n] + \mathbb{P}[\Lambda_n > x]. \end{aligned} \quad (6)$$

Next, by Lemma 1,  $\tau$  is a.s. finite and, thus, by (5) and (6), we conclude

$$\lim_{n \rightarrow \infty} \mathbb{P}[\Lambda_n > x] = \lim_{n \rightarrow \infty} \mathbb{P}[\Lambda_n^{\Lambda_0} > x] = \mathbb{P}[\Lambda > x].$$

■

### A. Reflected Multiplicative Processes and Queueing Duality

Note that in the special case  $B_n^i(J_n) \equiv J_n$ , reflected modulated branching processes reduce to reflected multiplicative processes with  $J_n$  being integer valued. In general, by using the definition in (3),  $J_n$  can be relaxed to take any positive real values. Hence, in this subsection we assume that  $\{J_n\}_{n \geq 0}$  is a positive, real valued process.

*Definition 3:* For  $l > 0$  and  $M_0 < \infty$  define Reflected Multiplicative Process (RMP) as

$$M_{n+1} = \max(M_n \cdot J_{n+1}, l), \quad n \geq 0. \quad (7)$$

RMP has been previously proposed and studied in literature [15], [27], [41]–[43] as the explanation of the origin of power laws. In this section we show a direct connection

(duality) between RMP and queuing theory, by which most of the previously obtained results on RMP follow directly from the well-known queuing results.

Without loss of generality we can assume  $l = 1$ , since we can always divide (7) by  $l$  and define  $M_n^1 = M_n/l$ . Now, let  $X_n = \log J_n$  and  $Q_n = \log M_n$  with the standard conventions  $\log 0 = -\infty$  and  $e^{-\infty} = 0$ . Then, for  $l = 1$ , equation (7) is equivalent to

$$Q_{n+1} = \max(Q_n + X_{n+1}, 0), \quad (8)$$

which is the workload (waiting-time) recursion in a single server (FIFO) queue.

*Lemma 3:* If  $\mathbb{E} \log J_n < 0$ , then  $M_n$  converges in distribution to an a.s. finite random variable  $M$ , and

$$M \stackrel{d}{=} \sup_{n \geq 0} \Pi_n, \quad (9)$$

where  $\Pi_0 = 1, \Pi_n = \prod_{i=-n}^{-1} J_i, n \geq 1$ .

*Proof:* By the classical result of Loynes [51],  $Q_n$ , defined by (8), converges to an a.s. finite stationary limit  $Q$  if  $\mathbb{E} X_n = \mathbb{E} \log J_n < 0$  and, furthermore,

$$Q \stackrel{d}{=} \sup_{n \geq 0} S_n,$$

where  $S_0 = 0$  and  $S_n = \sum_{i=-n}^{-1} X_i$ . This implies the convergence of  $M_n$  and

$$M \stackrel{d}{=} e^{\sup_{n \geq 0} S_n} = \sup_{n \geq 0} e^{S_n} = \sup_{n \geq 0} \Pi_n. \quad \blacksquare$$

The following theorem is a direct corollary of Theorem 1 in [52].

*Theorem 1:* Let  $\{J_n\}_{n \geq 1}$  be stationary and ergodic. If there exists a function  $\Psi$  and positive constants  $\alpha^*$  and  $\varepsilon^*$  such that

- 1)  $n^{-1} \log \mathbb{E}[(\Pi_n)^\alpha] \rightarrow \Psi(\alpha)$  as  $n \rightarrow \infty$  for  $|\alpha - \alpha^*| < \varepsilon^*$ ,
- 2)  $\Psi$  is finite in a neighborhood of  $\alpha^*$  and differentiable at  $\alpha^*$  with  $\Psi(\alpha^*) = 0, \Psi'(\alpha^*) > 0$ , and
- 3)  $\mathbb{E}[(\Pi_n)^{\alpha^*}] < \infty$  for  $n \geq 1$ ,

then

$$\lim_{x \rightarrow \infty} \frac{\log \mathbb{P}[M > x]}{\log x} = -\alpha^*. \quad (10)$$

*Remark 3:* We refer to assumptions 1) and 2) as the *polynomial Grtner-Ellis conditions*. Also, it is worth noting that the multiplicative process  $\Pi_n$  without the reflective boundary would essentially follow the Lognormal distribution, as it was recently observed in [53] (this is similar to the fact that the unrestricted additive random walk is approximated well by Normal distribution). However, we would like to emphasize that the lower boundary  $l$  is not just a mathematical artifact, but a very natural condition since no physical object can approach zero arbitrarily close without either repelling (reflecting) from it or vanishing (absorbing).

Here, we illustrate the preceding theorem by the following examples. Assume that  $\{A_i\}, \{C_i\}$  are two mutually independent sequences, and let  $J_n = e^{A_n - C_n}$ . Recall that  $M_n$  is

defined in (7), then  $Q_n = \log M_n$  satisfies

$$Q_{n+1} = (Q_n + A_n - C_n)^+. \quad (11)$$

The first two examples assume that  $\{A_i\}, \{C_i\}$  are two i.i.d. sequences, the third example takes  $\{J_n\}$  to be a Markov chain, and in the last example,  $\{J_n\}$  is modulated by a Markov chain  $X(n)$ .

*Example 1:* If  $\{A_i\}, \{C_i\}$  follow exponential distributions,  $\mathbb{P}[C_i > x] = e^{-\mu x}, \mathbb{P}[A_i > x] = e^{-\lambda x}$  and  $\lambda < \mu$ , then  $Q_n$  represents the waiting time in a  $M/M/1$  queue. By Theorem 9.1 of [54], the stationary waiting time in a  $M/M/1$  queue is distributed as

$$\mathbb{P}[Q > x] = \frac{\lambda}{\mu} e^{-(\mu - \lambda)x}, \quad x \geq 0,$$

which equivalently yields a power law distribution for  $M$ ,

$$\mathbb{P}[M > x] = \mathbb{P}[Q > \log x] = \frac{\lambda}{\mu x^{\mu - \lambda}}, \quad x \geq 1$$

with power exponent  $\alpha = \mu - \lambda$ .

*Example 2:* If  $\{A_i\}, \{C_i\}$  are two i.i.d Bernoulli processes with  $\mathbb{P}[A_n = 1] = 1 - \mathbb{P}[A_n = 0] = p, \mathbb{P}[C_n = 1] = 1 - \mathbb{P}[C_n = 0] = q, p < q$ . Then, the elementary queuing/Markov chain theory shows that the stationary distribution of  $Q_n$ , as defined in (11), is geometric  $\mathbb{P}[Q \geq j] = (1 - \rho)\rho^j, j \geq 0$ , where  $\rho = p(1 - q)/q(1 - p) < 1$ . Therefore,

$$\begin{aligned} \mathbb{P}[M \geq x] &= \mathbb{P}[Q \geq \log x] \\ &= \rho^{\lfloor \log x \rfloor}, \quad x \geq 1. \end{aligned}$$

Since  $\log x - 1 < \lfloor \log x \rfloor \leq \log x$ , it is easy to conclude that

$$\frac{1}{x^{\log(1/\rho)}} \leq \mathbb{P}[M \geq x] < \frac{1}{\rho x^{\log(1/\rho)}}.$$

*Example 3:* If  $\{J_n\}$  is a Markov chain taking values in a finite set  $\Sigma$  and possessing an irreducible transition matrix  $Q = (q(i, j))_{i, j \in \Sigma}$ , then the function  $\Psi$  defined in Theorem 1 can be explicitly computed. Define matrix  $Q_\alpha$  with elements

$$q_\alpha(i, j) = q(i, j)j^\alpha, \quad i, j \in \Sigma.$$

By Theorem 3.1.2 of [55], we have as  $n \rightarrow \infty$ ,

$$n^{-1} \log \mathbb{E}[(\Pi_n)^\alpha] \rightarrow \log \text{dev}(Q_\alpha),$$

where  $\text{dev}(Q_\alpha)$  is the Perron-Frobenius eigenvalue of matrix  $Q_\alpha$ . To illustrate this result, we take  $\Sigma = \{u, d\}$  where  $u \cdot d = 1, u > 1$ , and  $q(d, u) = q, q(d, d) = 1 - q, q(u, d) = p, q(u, u) = 1 - p$  where  $p > q$ . It is easy to compute

$$Q_\alpha = \begin{pmatrix} (1 - p)u^\alpha & pd^\alpha \\ qu^\alpha & (1 - q)d^\alpha \end{pmatrix},$$

and, by letting  $\log \text{dev}(Q_\alpha) = 0$ , we obtain

$$\alpha^* = \frac{\log(1 - q) - \log(1 - p)}{\log u}.$$

*Example 4 (double Pareto):* If  $\{J_n \equiv J(X(n))\}$  is modulated by a Markov chain  $X(n)$ , we argue that  $\mathbb{P}[M > x]$  can have different asymptotic decay rates over multiple time scales. This phenomenon was investigated in [47] in the

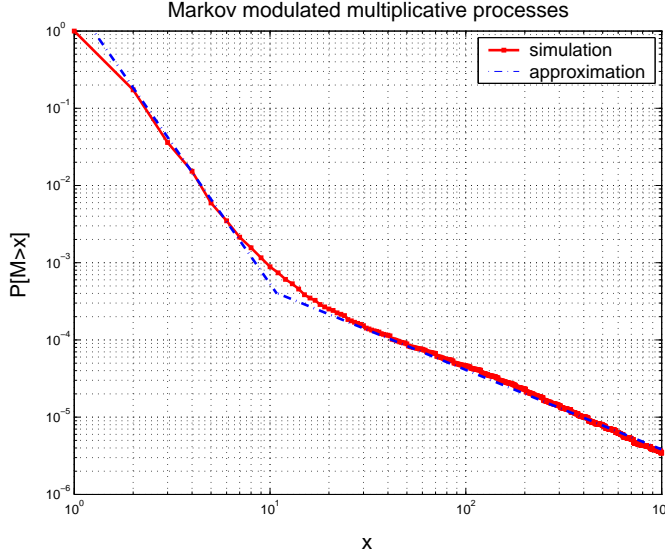


Fig. 1. Illustration for Example 4 of the double Pareto distribution.

queuing context and formulated as Theorem 3 therein. To visualize this phenomena, we study the following example. Consider a Markov process  $X(n)$  of two states (say  $\{1, 2\}$ ) with transition probabilities  $p_{12} = 1/5000$ ,  $p_{21} = 1/10$ , and  $\mathbb{P}[J(1) = 1.2] = 1 - \mathbb{P}[J(2) = 0.6] = 0.5$ ,  $\mathbb{P}[J(2) = 1.7] = 1 - \mathbb{P}[J(2) = 0.25] = 0.6$ . The corresponding simulation result for  $5 * 10^7$  trials is presented in Figure 1. We observe from this figure a double Pareto distribution for  $M$ , which provides a new explanation to the origins of double Pareto distributions as compared to the one in [56].

### III. MAIN RESULTS

This section presents our main results in Theorems 2 and 3. In this regard, we define  $\bar{B}_n^i \triangleq \sup_k B_n^i(k)$ , and, to avoid technical difficulties, assume  $\underline{\mu} \triangleq \inf_j \mu(j) > 0$ . With a small abuse of notation, as compared to the preceding Subsection II-A, we redefine here  $\Pi_n = \prod_{i=-n}^{-1} \mu(J_i)$ ,  $n \geq 1$ ,  $\Pi_0 = l$  and  $M = \sup_{n \geq 0} \Pi_n$ .

**Theorem 2:** Assume that the process  $\{\Pi_n\}$  satisfies the polynomial Gärtner-Ellis conditions (conditions 1) and 2) of Theorem 1), and  $\mathbb{E}(\Pi_n)^{\alpha^* + \varepsilon} < \infty$ ,  $\mathbb{E}[e^{\theta \bar{B}_n^i}] < \infty$  for some  $\varepsilon, \theta > 0$  and all  $n \geq 1$ , then,

$$\lim_{x \rightarrow \infty} \frac{\log \mathbb{P}[\Lambda > x]}{\log x} = \lim_{x \rightarrow \infty} \frac{\log \mathbb{P}[M > x]}{\log x} = -\alpha^*. \quad (12)$$

**Theorem 3:** If  $\sup_j \mu(j) < 1$  and  $\mathbb{E}[e^{\theta \bar{B}_n^i}] < \infty$  for  $\theta > 0$ , then,  $\mathbb{P}[\Lambda > x] = o(e^{-\xi x})$  for some  $\xi > 0$ , implying

$$\lim_{x \rightarrow \infty} \frac{\log \mathbb{P}[\Lambda > x]}{\log x} = -\infty. \quad (13)$$

**Remark 4:** Informally speaking, these two theorems show that the alternating periods of contractions and expansions, e.g., economic booms and recessions, are primarily responsible for the appearance of power law distributions; in other words, if there are no periods of expansions, i.e., the condition  $\sup_j \mu(j) < 1$  of Theorem 3 is satisfied, then  $\Lambda$  has a lighter (exponential) tail than power laws.

Furthermore, the first equality in (12) of Theorem 2 reveals a general asymptotic equivalence between the reflected modulated branching process and the corresponding reflected multiplicative process.

In the following subsections, we present the proof of Theorem 2 and, due to space limitations, the proof of Theorem 3 is deferred to the extended version of this paper [57].

#### A. Proof of Theorem 2

In this paper we use the following standard notation. For any two real functions  $a(t)$  and  $b(t)$ , we use  $a(t) = o(b(t))$  to denote that  $\lim_{t \rightarrow \infty} a(t)/b(t) = 0$ .

1) *Upper Bound:* The proof of the upper bound uses the following technical lemmas that will be proven in Section VI. Since the proof is based on the change (increase) of boundary  $l$ , we denote this dependence explicitly as  $\Lambda^l \equiv \Lambda$ . According to Lemma 2, the initial value of  $\{\Lambda_n\}$  has no impact on  $\Lambda$  and, therefore, in this subsection we simply assume that  $\Lambda_0^l = l$ .

**Lemma 4:** For any  $\beta > 0$ , the branching process  $Z_n^l$  defined by (1) satisfies, as  $x \rightarrow \infty$ ,

$$\sum_{n > x}^{\infty} \mathbb{P}[Z_n^l > x] = o\left(\frac{1}{x^\beta}\right). \quad (14)$$

**Lemma 5:** If  $\Lambda_n^l$  is the reflected branching process, as defined in (2), and  $\varepsilon > 0$ , then

$$\mathbb{P}[\Lambda_n^l > x] \leq \mathbb{P}\left[\max_{1 \leq j \leq n} \Pi_j (1 + \varepsilon)^j > x/l\right] + n\mathbb{P}[\mathcal{B}_1^{l, \varepsilon}],$$

where  $\mathcal{B}_n^{l, \varepsilon} \triangleq \bigcup_{j \geq l} \{ \sum_{i=1}^j B_n^i(J_n) > j\mu(J_n)(1 + \varepsilon) \}$  and  $\Pi_j = \prod_{i=-1}^{-j} \mu(J_i)$ .

**Lemma 6:** If we set  $l_x = \lfloor x^\varepsilon \rfloor$ ,  $0 < \varepsilon < 1$  in the definition of  $\mathcal{B}_1^{l_x, \varepsilon}$  in Lemma 5, then, for any  $\beta > 1$ , we obtain

$$\mathbb{P}[\mathcal{B}_1^{l_x, \varepsilon}] = o\left(\frac{1}{x^\beta}\right) \text{ as } x \rightarrow \infty.$$

**Lemma 7:** If  $l_1 \geq l_2$ , then for all  $n \geq 0$ ,

$$\mathbb{P}[\Lambda_n^{l_1} > x] \geq \mathbb{P}[\Lambda_n^{l_2} > x].$$

Now, we are ready to complete the proof of the upper bound. Choosing  $l_x = \lfloor x^\varepsilon \rfloor \geq l$ , and using Lemma 7, we derive

$$\begin{aligned} \mathbb{P}[\Lambda^l > x] &= \mathbb{P}\left[\sup_{j \geq 1} Z_{-j}^l > x\right] \\ &\leq \mathbb{P}\left[\sup_{1 \leq j \leq x} Z_{-j}^{l_x} > x\right] + \mathbb{P}\left[\sup_{j > x} Z_{-j}^l > x\right] \\ &\leq \mathbb{P}[\Lambda_{\lfloor x \rfloor}^{l_x} > x] + \sum_{j > x} \mathbb{P}[Z_j^l > x] \\ &\leq \mathbb{P}\left[\sup_{j \geq 1} \Pi_j (1 + \varepsilon)^j > x^{1-\varepsilon}\right] + x\mathbb{P}[\mathcal{B}_1^{l_x, \varepsilon}] \\ &\quad + \sum_{j > x} \mathbb{P}[Z_j^l > x] \\ &\triangleq I_1(x) + I_2(x) + I_3(x), \end{aligned} \quad (15)$$

where the last inequality follows from Lemma 5.

Next, define a new process  $\{\mu^\epsilon(J_n) = \mu(J_n)(1 + \epsilon)\}_{n \geq 1}$  and  $\Pi_n^\epsilon = \prod_{i=-n}^{-1} \mu^\epsilon(J_i)$ . Then, for  $\epsilon$  small enough, we have

- 1)  $n^{-1} \log \mathbb{E}(\Pi_n^\epsilon)^\alpha \rightarrow \Psi^\epsilon(\alpha) = \Psi(\alpha) + \alpha \log(1 + \epsilon)$  as  $n \rightarrow \infty$  for  $|\alpha - \alpha^*| < \epsilon^*$ ,
- 2)  $\Psi^\epsilon$  is finite in a neighborhood of  $\alpha_\epsilon^*$  where  $\alpha_\epsilon^* < \alpha^*$  and differentiable at  $\alpha_\epsilon^*$  with  $\Psi(\alpha_\epsilon^*) + \alpha_\epsilon^* \log(1 + \epsilon) = 0$  and  $\Psi'(\alpha_\epsilon^*) > 0$ , and
- 3)  $\mathbb{E}(\Pi_n^\epsilon)^{\alpha_\epsilon^*} < \infty$  for  $n \geq 1$ .

Therefore, by Theorem 1,

$$\lim_{x \rightarrow \infty} \frac{\log \mathbb{P}[\sup_{i \geq 1} \Pi_i(1 + \epsilon)^i > x^{1-\epsilon}]}{\log x} = -\alpha_\epsilon^*. \quad (16)$$

Using (16), Lemma 4 and Lemma 6, we obtain

$$\begin{aligned} \frac{\log \mathbb{P}[\Lambda^l > x]}{\log x} &\leq \frac{\log(I_1(x))}{\log x} + \frac{\log\left(1 + \frac{I_2(x) + I_3(x)}{I_1(x)}\right)}{\log x} \\ &\rightarrow \alpha_\epsilon^* \quad \text{as } x \rightarrow \infty. \end{aligned}$$

Since  $\Psi^\epsilon(\alpha)$  is continuous in a neighborhood of  $\alpha^*$  in both  $\alpha$  and  $\epsilon$ , we derive

$$\lim_{\epsilon \rightarrow 0} \alpha_\epsilon^* = \alpha^*,$$

implying,

$$\limsup_{x \rightarrow \infty} \frac{\log \mathbb{P}[\Lambda > x]}{\log x} \leq -\alpha^*. \quad (17)$$

2) *Lower Bound:* Similarly as in the proof of the upper bound, we use the following lemmas that will be proven in Section VI.

*Lemma 8:* If  $\{\Lambda_n^{y_1}\}$  and  $\{\Lambda_n^{y_2}\}$  are two conditionally independent reflected branching processes given  $\{J_n\}_{n \geq 0}$ , then,

$$\Lambda_n^{y_1 + y_2} \leq \Lambda_n^{y_1} + \Lambda_n^{y_2}.$$

*Lemma 9:* For any  $0 < \epsilon < 1$ , there exist  $\beta, h > 0$  such that, when  $x \rightarrow \infty$ ,

$$\mathbb{P}\left[\sup_{i > h \log x} \Pi_i(1 - \epsilon)^i > x\right] = o\left(\frac{1}{x^{\alpha^* + \beta}}\right).$$

*Lemma 10:* For  $1 > \delta, \epsilon > 0$ , let  $y_x = \lfloor x^\delta \rfloor$ , and  $\mathcal{B}_n^{y_x, \epsilon} \triangleq \bigcup_{j \geq y_x} \{\sum_{i=1}^j B_n^i(J_n) < j\mu(J_n)(1 - \epsilon)\}$ , then, there exists  $\beta > 1$ , such that

$$\mathbb{P}[\mathcal{B}_1^{y_x, \epsilon}] = o\left(\frac{1}{x^{\alpha^* + \beta}}\right).$$

Now, we proceed to complete the proof of the lower bound. First, observe that for any integer  $y \geq 1$ ,

$$\begin{aligned} \mathbb{P}[\Lambda_n^l > x] &\geq \mathbb{P}[\Lambda_n^1 > x] \\ &= \frac{y \mathbb{P}[\Lambda_n^1 > x]}{y} \\ &\geq \frac{\mathbb{P}[\sum_{j=1}^y \Lambda_{n,j}^1 > yx]}{y} \\ &\geq \frac{\mathbb{P}[\Lambda_n^y > yx]}{y}, \end{aligned} \quad (18)$$

where the last inequality follows from Lemma 8 and the fact that  $\{\Lambda_{n,j}^1\}$  are conditionally i.i.d. copies of  $\Lambda_n^1$  given  $\{J_n\}$ . Let  $\Pi_n^i = \mu(J_i)\mu(J_{i+1}) \cdots \mu(J_n)$ ,  $\mathcal{B}_n^{y, \epsilon} =$

$\bigcup_{j \geq y} \{\sum_{i=1}^j B_n^i(J_n) < j\mu(J_n)(1 - \epsilon)\}$  where  $0 < \epsilon < 1$ . Then, we derive

$$\begin{aligned} \mathbb{P}[\Lambda_n^y > yx] &\geq \mathbb{P}\left[\sup_{0 \leq i \leq n-1} \Pi_n^i(1 - \epsilon)^{n-i} > x\right] - \mathbb{P}[\mathcal{B}_1^{y, \epsilon}] \\ &\quad - \cdots - \mathbb{P}[\mathcal{B}_n^{y, \epsilon}] \\ &= \mathbb{P}\left[\sup_{1 \leq i \leq n} \Pi_i(1 - \epsilon)^i > x\right] - n\mathbb{P}[\mathcal{B}_1^{y, \epsilon}] \\ &\geq \mathbb{P}\left[\sup_{i \geq 1} \Pi_i(1 - \epsilon)^i > x\right] \\ &\quad - \mathbb{P}\left[\sup_{i > n} \Pi_i(1 - \epsilon)^i > x\right] - n\mathbb{P}[\mathcal{B}_1^{y, \epsilon}] \\ &\triangleq I_1 - I_2 - I_3. \end{aligned} \quad (19)$$

Next, similarly as in the proof of the upper bound, define a new process  $\{\mu_\epsilon(J_n) = \mu(J_n)(1 - \epsilon)\}_{n \geq 1}$  and let  $\Pi_n^\epsilon = \prod_{i=-n}^{-1} \mu_\epsilon(J_i)$ . For  $\epsilon$  small enough, we have

- 1)  $n^{-1} \log \mathbb{E}(\Pi_n^\epsilon)^\alpha \rightarrow \Psi^\epsilon(\alpha) = \Psi(\alpha) + \alpha \log(1 - \epsilon)$  as  $n \rightarrow \infty$  for  $|\alpha - \alpha^*| < \epsilon^*$ ,
- 2)  $\Psi^\epsilon$  is finite in a neighborhood of  $\alpha_\epsilon^*$  where  $\alpha_\epsilon^* > \alpha^*$  and differentiable at  $\alpha_\epsilon^*$  with  $\Psi(\alpha_\epsilon^*) + \alpha_\epsilon^* \log(1 - \epsilon) = 0$  and  $\Psi'(\alpha_\epsilon^*) > 0$ , and
- 3)  $\mathbb{E}(\Pi_n^\epsilon)^{\alpha_\epsilon^*} < \infty$  for  $n \geq 1$ .

Therefore, by Theorem 1, we obtain

$$\lim_{x \rightarrow \infty} \frac{\log \mathbb{P}[\sup_{i \geq 1} \Pi_i(1 - \epsilon)^i > x]}{\log x} = -\alpha_\epsilon^*, \quad (20)$$

and

$$\lim_{\epsilon \rightarrow 0} \alpha_\epsilon^* = \alpha^*. \quad (21)$$

Then, for  $0 < \delta < 1$ , by choosing  $y_x = \lfloor x^\delta \rfloor$  and  $n_x = \lfloor x \rfloor$  in (18), we derive

$$\begin{aligned} \log \mathbb{P}[\Lambda > x] &\geq \log \mathbb{P}[\Lambda_{n_x}^l > x] \\ &\geq \log \frac{\mathbb{P}[\Lambda_{n_x}^{y_x} > y_x x]}{y_x} \\ &= \log \mathbb{P}[\Lambda_{n_x}^{y_x} > y_x x] - \delta \log x \\ &\geq \log(I_1 - I_2 - I_3) - \delta \log x, \end{aligned} \quad (22)$$

which, by Lemmas 9, 10, and passing  $\delta, \epsilon \rightarrow 0$ , yields

$$\liminf_{x \rightarrow \infty} \frac{\log \mathbb{P}[\Lambda > x]}{\log x} \geq -\alpha^*.$$

The last inequality, in conjunction with (17), completes the proof of Theorem 2.  $\blacksquare$

#### IV. EXACT ASYMPTOTICS

This section presents the exact asymptotic approximations of the RMPs and RMBPs in the following two subsections, respectively.

### A. On the Exact Asymptotics of Reflected Multiplicative Processes

The following theorems are direct translations from the corresponding queueing theory results. Theorem 4 is the large deviation result, Theorem 5 is the heavy traffic approximation, and they are basically corollaries of Theorem 5.2 in Chapter XIII and Theorem 7.1 in Chapter X of [54].

For a sequence of i.i.d. random variables  $\{J_n\}_{n \geq 1}$ , define  $G_+$  to be the ladder height distribution of the random walk  $\{S_n = \sum_{i=1}^n \log J_i\}_{n \geq 1}$  with  $\|G_+\| = \mathbb{P}[S_n \leq 0 \text{ for all } n \geq 1]$ .

**Theorem 4:** If the sequence  $\{\log J_n\}_{n \geq 1}$  is i.i.d. and nonlattice,  $\mathbb{E}[\log J_1] < 0$ ,  $\mathbb{E}[J_1^{\alpha^*}] = 1$ ,  $\mathbb{E}[J_1^\alpha] < \infty$  for  $\alpha^* - \epsilon < \alpha < \alpha^* + \epsilon$ , and  $\|G_+\| < 1$ , then

$$\lim_{x \rightarrow \infty} \mathbb{P}[M > x] \cdot x^{\alpha^*} = \frac{1 - \|G_+\|}{\alpha^* \int_0^\infty x e^{\alpha^* x} G_+(dx)}.$$

*Proof:* The result is a direct consequence of Theorem 5.2 in Chapter XIII of [54]. ■

**Remark 5:** If  $S_n$  is lattice valued, see Remark 5.4 of [54].

**Theorem 5:** If  $\{J_n^{(k)}\}_{n \geq 1}$ , indexed by  $k$ , are i.i.d. for each fixed  $k$  with  $m_k \triangleq \mathbb{E}[\log J_1^{(k)}]$ ,  $\sigma_k^2 \triangleq \text{Var}[\log J_1^{(k)}]$ , and the random walks  $\{S_n^{(k)} = \sum_{i=1}^n \log J_i^{(k)}\}_{n \geq 1}$  satisfy  $m_k < 0$ ,  $\lim_{k \rightarrow \infty} m_k = 0$ ,  $\lim_{k \rightarrow \infty} \sigma_k^2 > 0$ , and  $(\log J_1^{(k)})^2$  is uniformly integrable, then,

$$\lim_{k \rightarrow \infty} \mathbb{P}\left[M_{(k)}^{-m_k/\sigma_k^2} > y\right] = 1/y^2.$$

*Proof:* From Theorem 7.1 in [54], we have

$$\lim_{k \rightarrow \infty} \mathbb{P}\left[-\frac{m_k}{\sigma_k^2} \log M^{(k)} > z\right] = e^{-2z},$$

and, by letting  $z = \log y$ , we obtain Theorem 5. ■

**Remark 6:** The preceding two theorems essentially provide a new general explanation of the measured double Pareto phenomena, e.g., see [56], [58].

### B. On the Exact Asymptotics of Reflected Branching Processes

Deriving the exact asymptotics for RBMPs is a difficult problem. However, in the scaling region when the boundary  $l$  grows as well, albeit slowly, one can derive an explicit asymptotic characterization. In this subsection, assume that  $\{J_n\}_{n \geq 1}$  is i.i.d. and nonlattice, and let  $G_+$  be the ladder height distribution of the nonlattice random walk  $\{S_n = \sum_{i=1}^n \log \mu(J_i)\}_{n \geq 1}$  with  $\|G_+\| = \mathbb{P}[S_n \leq 0 \text{ for all } n \geq 1] < 1$ .

**Theorem 6:** If  $\inf_j \mu(j) \triangleq \underline{\mu} > 0$ ,  $\mathbb{E}[\log \mu(J_1)] < 0$ ,  $\mathbb{E}[\mu(J_1)^{\alpha^*}] = 1$ ,  $\mathbb{E}[\mu(J_1)^\alpha] < \infty$  for  $\alpha^* - \epsilon < \alpha < \alpha^* + \epsilon$  and  $\mathbb{E}\left[e^{\theta \sup_k \{|B_1^{(k)} - \mu(k)|\}}\right] < \infty$ ,  $\theta > 0$ , then for any  $\gamma > 0$ ,

$$\lim_{l_x \geq \frac{(\log x)^{3+\gamma}}{x} \rightarrow \infty} \mathbb{P}[\Lambda^{l_x}/l_x > x] x^{\alpha^*} = \frac{1 - \|G_+\|}{\alpha^* \int_0^\infty x e^{\alpha^* x} G_+(dx)}.$$

Again, the proof of this theorem is deferred to the extended version of the paper [57]. Instead, we illustrate it with the following simulation example.

**Example 5:** Assume that  $\{J_n\}_{n \geq 1}$  is a Bernoulli process with  $\mathbb{P}[J_n = 1] = 0.4 = 1 - \mathbb{P}[J_n = 0]$ , variables  $\{B_n^i(1)\}_{i \geq 1}$  follows Poisson distribution with mean 1.5 and  $\{B_n^i(0)\}_{i \geq 1}$  with mean 0.6. The simulation results, for  $l = 1, 5, 13, 21$ , are drawn in Figure 5. From the figure we can clearly see that  $\mathbb{P}[\Lambda^{l_x}/l_x > x]$  approaches the limiting value very quickly, i.e., for  $l = 13$  and  $l = 21$ , the plots of  $\mathbb{P}[\Lambda^l/l > x]$  are basically indistinguishable.

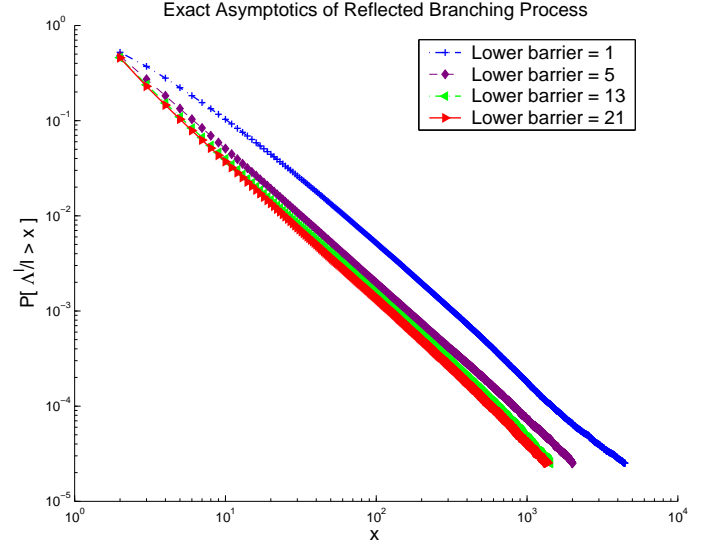


Fig. 2. Simulation of  $\mathbb{P}[\Lambda^l/l > x]$  versus  $x$  parameterized by  $l$ .

## V. DISCUSSION OF RELATED MODELS

Here, we briefly address the two related models: randomly stopped processes and modulated branching processes with absorbing barriers.

### A. Randomly Stopped Processes

In this subsection we study randomly stopped multiplicative and branching processes, respectively.

**1) Multiplicative Processes:** Following the approach of Chapter VIII of [54], we study the ladder heights of a multiplicative process. For any RMP with independent multipliers,  $M$  can be represented in terms of the ladder heights. To this end, define  $\Pi_n \triangleq \prod_{i=0}^n J_i$  and the ladder height process  $\{H_i\}_{i \geq 1}$  of  $\{S_n = \sum_{i=1}^n \log J_i\}_{n \geq 1}$  with  $\|G_+\| = \mathbb{P}[S_n \leq 0 \text{ for all } n \geq 1] < 1$ , and  $H_i^e \triangleq e^{H_i}$ .

**Theorem 7:** Suppose that  $\{J_n\}_{n \geq 1}$  is an i.i.d. sequence with  $\mathbb{E}[\log J_1] < 0$ , then,

$$M \stackrel{d}{=} \prod_{i=1}^N H_i^e, \quad (23)$$

where  $N$  is independent of  $\{H_i^e\}_{i \geq 1}$  and follows a geometric distribution  $\mathbb{P}[N > n] = \|G_+\|^n$ .

*Proof:* Based on the well-known Pollaczek-Khinchin representation (see Chapter VIII of [54])

$$Q \stackrel{d}{=} \sum_{i=1}^N H_i,$$

where  $\mathbb{P}[N > n] = \|G_+\|^n$ , it immediately follows that

$$\mathbb{P}[M > x] = \mathbb{P}\left[e^{\sum_{i=1}^N H_i} > x\right] = \mathbb{P}\left[\prod_{i=1}^N H_i^e > x\right].$$

Conversely, we can prove that for a stationary multiplicative process, if the observation time has exponential tail, the stopped process has a power law tail under rather general conditions.

*Theorem 8:* Let  $N$  be an integer random variable independent of  $\Pi_n$ ,  $\lambda > 0$  and

$$\lim_{x \rightarrow \infty} \frac{\log \mathbb{P}[N > x]}{x} = -\lambda.$$

If  $\Psi$  is finite in a neighborhood of  $\alpha^*$  and differentiable at  $\alpha^*$  with  $\Psi(\alpha^*) = \lambda$ ,  $\Psi'(\alpha^*) > 0$ , and  $\mathbb{E}[(\Pi_n)^{\alpha^*}] < \infty$  for  $n \geq 1$ , then,

$$\lim_{x \rightarrow \infty} \frac{\log \mathbb{P}[\Pi_N > x]}{\log x} = -\alpha^*. \quad (24)$$

The complete proof of Theorem 8 will be presented in [57]. Here, we only prove a special case, stated in Theorem 9 below, which establishes a connection between the RMPs and geometrically stopped multiplicative processes through a  $M/GI/1$  queue. Assume that  $\{J_n\}_{n \geq 1}$  is an i.i.d. process,  $\Pi_n$  is the corresponding multiplicative process,  $N$  is a geometric random variable that is independent of  $\Pi_n$  with  $\mathbb{P}[N > n] = \rho^n$ , and  $\bar{G}(t), t \geq 0$ , is a positive decreasing function.

*Theorem 9:* If, for some  $\alpha^*, \epsilon > 0$ ,  $\int_0^\infty e^{\alpha^* x} \bar{G}(x) dx = \rho^{-1}$ ,  $K \triangleq \int_0^\infty x e^{\alpha^* x} \bar{G}(x) dx < \infty$ ,  $\int_0^\infty \bar{G}(\log s / \epsilon) ds < \infty$  and  $\mathbb{P}[\log J_1 \leq t] = \int_0^t \bar{G}(s) ds / \int_0^\infty \bar{G}(s) ds, t \geq 0$ , then, we can always construct a RMP such that  $M \stackrel{d}{=} \Pi_N$ , and,

$$\lim_{x \rightarrow \infty} \mathbb{P}[\Pi_N > x] \cdot x^{\alpha^*} = \frac{1 - \rho}{\alpha^* \rho K}.$$

*Proof:* We will give a constructive proof based on the connection (duality) that we establish between the  $M/GI/1$  queue and the geometrically stopped multiplicative process.

Consider a  $M/GI/1$  queue with service distribution  $\mathbb{P}[S > t] = \bar{G}(t)/\bar{G}(0), t \geq 0$  and Poisson arrivals of rate  $\lambda = \rho \mathbb{E}S$ . Then, by Pollaczek-Khintchine formula, the variable  $Q$  is equal in distribution to  $\sum_{i=1}^N H_i$ , where  $\mathbb{P}[N > n] = \rho^n$  and

$$\mathbb{P}[H_i > x] = \frac{\int_0^x \mathbb{P}[S > s] ds}{\mathbb{E}S} = \frac{\int_0^x \bar{G}(s) ds}{\int_0^\infty \bar{G}(s) ds} = \mathbb{P}[\log J_i > x].$$

Therefore,

$$\mathbb{P}[Q > \log x] = \mathbb{P}\left[\sum_{i=1}^N H_i > \log x\right] = \mathbb{P}[\Pi_N > x],$$

and, by Cramér-Lundberg theorem (e.g., see Theorem 5.2 in Chapter XIII of [54]), we obtain

$$\lim_{x \rightarrow \infty} \mathbb{P}[Q > \log x] x^{\alpha^*} = \frac{1 - \rho}{\alpha^* \rho K},$$

which completes the proof.  $\blacksquare$

2) *Branching Processes:* In the following theorem, we extend the preceding results to the context of randomly stopped branching processes. Similarly as before, we postpone the proof of the theorem to the full version of the paper [57].

*Theorem 10:* Suppose  $B_n^k(J_n) \geq 1$  for all  $n, k$ . Under the same conditions as in Theorem 8, we have,

$$\lim_{x \rightarrow \infty} \frac{\log \mathbb{P}[Z_N > x]}{\log x} = -\alpha^*.$$

## B. Branching Processes with Absorbing Barriers

For many dynamic processes (e.g., city sizes), quite often when the size of the object of interest falls below a threshold (e.g., urban decay), the whole object disappears. Therefore, we study a branching process with an absorbing barrier. This can also model the arrivals to popular Web sites (hotspots), since information (news) is distributed according to a branching process, e.g.,  $A$  tells  $B, C$  and further  $B$  may tell  $D$ , etc. Empirical examination shows that Web requests follow power law distributions, e.g., see [31], [32].

For a threshold  $l > 0$ , define stopping time  $P \triangleq \inf\{n > 0 : Z_n^l \leq l\}$  to be the life cycle, within which the branching process is modulated by a sequence of i.i.d. random variables  $\{J_n\}$ , and after  $P$  the process is absorbed/disappears. We denote this process by  $Z_P$ . Let the arrival process  $\{A_t\}_{t > -\infty}$  be a sequence of i.i.d. Poisson random variables with parameter  $\mathbb{E}[A_t] = q$ . At time  $t$ ,  $A_t$  objects are created, each evolving according to an i.i.d. copy of the modulated branching process  $Z_P$ . This system converges to a stationary process with  $N(t)$  objects alive at time  $t$ . Assume that the system has reached its stationarity, then, by Little's Law,  $\mathbb{E}[N(t)] = q \mathbb{E}[P]$ . Object  $j$  observed at time  $t = 0$  is generated at time  $-P_j^r, 1 \leq j \leq N(0)$ , with a size  $Z_{-P_j^r}^l$ .

*Lemma 11:* The total size of all objects  $Z_s$  observed at time  $t = 0$  in stationarity can be represented as

$$Z_s \stackrel{d}{=} \sum_{j=1}^{N_0} Z_{-P_j^r}^l,$$

where

$$\mathbb{P}[P_j^r > x] = \frac{\int_x^\infty \mathbb{P}[P > u] du}{\mathbb{E}[P]}.$$

Next, we show that  $Z_s$  follows a power law. The proof of the following theorem, which is essentially a corollary of Theorem 2, will be given in [57].

*Theorem 11:* Suppose that the conditions described in this subsection hold,  $\{J_n\}$  satisfies the assumptions of Theorem 4 (except for the nonlattice one) and  $\mathbb{E}[e^{\theta B_n^i}] < \infty$  for  $\theta > 0$ , then,

$$\lim_{x \rightarrow \infty} \frac{\log \mathbb{P}[Z_s > x]}{\log x} = -\alpha^*.$$

## VI. PROOFS

This section contains the proofs of the technical lemmas 4, 5, 6, 8 and 10.  $\blacksquare$



### A. proof of Lemma 4

Similarly as in the proof of Lemma 1,  $W_n = Z_n^l/\Pi_n$  is a martingale. Then for any  $\epsilon > 0$ ,

$$\begin{aligned}\mathbb{P}[Z_n^l > x] &= \mathbb{P}[W_n \cdot \Pi_n > x] \\ &= \mathbb{P}[(W_n e^{-\epsilon n}) \cdot (\Pi_n e^{\epsilon n}) > x] \\ &\leq \mathbb{P}[W_n e^{-\epsilon n} > 1] + \mathbb{P}[\Pi_n e^{\epsilon n} > x] \\ &\leq \mathbb{E}[W_n e^{-\epsilon n}] + \mathbb{P}[\Pi_n e^{\epsilon n} > x].\end{aligned}\quad (25)$$

Observe that

$$\sum_{n>x} \mathbb{E}[W_n e^{-\epsilon n}] = \sum_{n>x} e^{-\epsilon n} \leq \frac{e^{-\epsilon x}}{1 - e^{-\epsilon}} = o\left(\frac{1}{x^\beta}\right). \quad (26)$$

By the first condition of Theorem 2, we can select  $\delta, \epsilon > 0$  small enough and  $n_0$  large enough such that  $\Psi(\alpha^* - \delta) + 2\epsilon(\alpha^* - \delta) = -\zeta < 0$  and  $n^{-1} \log \mathbb{E}[\Pi_n^{(\alpha^* - \delta)}] < \Psi(\alpha^* - \delta) + \epsilon(\alpha^* - \delta)$ , then, for  $x > n_0$ ,

$$\begin{aligned}\sum_{n>x} \mathbb{P}[\Pi_n e^{\epsilon n} > x] &\leq \sum_{n>x} \mathbb{E}[\Pi_n^{(\alpha^* - \delta)}] e^{\epsilon(\alpha^* - \delta)n} / x^{(\alpha^* - \delta)} \\ &\leq \sum_{n>x} e^{-\zeta n} / x^{(\alpha^* - \delta)} \leq \frac{e^{-\zeta x}}{(1 - e^{-\zeta})x^{\alpha^* - \delta}} \\ &= o\left(\frac{1}{x^\beta}\right) \text{ as } x \rightarrow \infty.\end{aligned}\quad (27)$$

Therefore, replacing (26) and (27) into (25) completes the proof.  $\blacksquare$

### B. Proof of Lemma 5

Observe that

$$\begin{aligned}\mathbb{P}[\Lambda_n^l > x] &= \mathbb{P}[\Lambda_n^l > x, (\mathcal{B}_n^{l,\epsilon})^C] + \mathbb{P}[\Lambda_n^l > x, \mathcal{B}_n^{l,\epsilon}] \\ &\leq \mathbb{P}\left[\left\{\sum_{i=1}^{\Lambda_{n-1}^l} B_n^i(J_{n-1}) > x\right\}\right. \\ &\quad \left.\cup \left\{\sum_{i=1}^l B_n^i(J_{n-1}) > x\right\}, (\mathcal{B}_n^{l,\epsilon})^C\right] + \mathbb{P}[\mathcal{B}_n^{l,\epsilon}] \\ &\leq \mathbb{P}\left[\left\{\Lambda_{n-1}^l \mu(J_{n-1})(1 + \epsilon) > x\right\}\right. \\ &\quad \left.\cup \left\{\mu(J_{n-1})(1 + \epsilon) > x/l\right\}\right] + \mathbb{P}[\mathcal{B}_1^{l,\epsilon}] \\ &\leq \mathbb{P}\left[\left\{\Lambda_{n-1}^l \mu(J_{n-1})(1 + \epsilon) > x\right\}\right. \\ &\quad \left.\cup \left\{\mu(J_{n-1})(1 + \epsilon) > x/l\right\}, (\mathcal{B}_{n-1}^{l,\epsilon})^C\right] \\ &\quad + \mathbb{P}[\mathcal{B}_{n-1}^{l,\epsilon}] + \mathbb{P}[\mathcal{B}_1^{l,\epsilon}] \\ &\leq \mathbb{P}\left[\left\{\Lambda_{n-2}^l \mu(J_{n-1})\mu(J_{n-2})(1 + \epsilon)^2 > x\right\}\right. \\ &\quad \left.\cup \left\{\max(\mu(J_{n-1})\mu(J_{n-2})(1 + \epsilon)^2,\right.\right. \\ &\quad \left.\left.\mu(J_{n-1})(1 + \epsilon)) > x/l\right\}\right] + 2\mathbb{P}[\mathcal{B}_1^{l,\epsilon}].\end{aligned}$$

Then, by continuing the induction, one can easily obtain

$$\begin{aligned}\mathbb{P}[\Lambda_n^l > x] &\leq \mathbb{P}\left[\max_{1 \leq j \leq n} (1 + \epsilon)^j \prod_{i=1}^j \mu(J_{n-i}) > x/l\right] \\ &\quad + n\mathbb{P}[\mathcal{B}_1^{l,\epsilon}],\end{aligned}$$

which, by stationarity of  $\{\mu(J_n)\}$ , yields

$$\mathbb{P}[\Lambda_n^l > x] \leq \mathbb{P}\left[\max_{1 \leq j \leq n} \Pi_j (1 + \epsilon)^j > x/l\right] + n\mathbb{P}[\mathcal{B}_1^{l,\epsilon}].$$

### C. Proof of Lemma 6

Due to space limitations, the detailed proof of this lemma can be found in the extended version of this paper [57].

### D. Proof of Lemma 8

First, it is easy to show that, for any real numbers  $x_1, x_2, y_1, y_2$ ,

$$\max(x_1 + x_2, y_1 + y_2) \leq \max(x_1, y_1) + \max(x_2, y_2).$$

Now, by the preceding inequality,

$$\begin{aligned}\Lambda_1^{y_1+y_2} &= \max\left(\sum_{i=1}^{y_1+y_2} B_1^i(J_1), y_1 + y_2\right) \\ &\leq \max\left(\sum_{i=1}^{y_1} B_1^i(J_1) + \sum_{i=y_1+1}^{y_2} B_1^i(J_1), y_1 + y_2\right) \\ &\stackrel{d}{\leq} \Lambda_1^{y_1} + \Lambda_1^{y_2}.\end{aligned}$$

The proof is completed by induction

$$\begin{aligned}\Lambda_{n+1}^{y_1+y_2} &= \max\left(\sum_{i=1}^{\Lambda_n^{y_1+y_2}} B_1^i(J_n), y_1 + y_2\right) \\ &\stackrel{d}{\leq} \max\left(\sum_{i=1}^{\Lambda_n^{y_1} + \Lambda_n^{y_2}} B_1^i(J_n), y_1 + y_2\right) \\ &\stackrel{d}{\leq} \Lambda_n^{y_1} + \Lambda_n^{y_2}.\end{aligned}$$

### E. Proof of Lemma 10

Similarly as for Lemma 6, the proof is presented in [57].

## REFERENCES

- [1] V. Pareto, *Cours d'Economie Politique*, vol. 2, Pichou, Paris, 1897.
- [2] D. Champernowne, "A model of income distribution," *Economic Journal*, vol. 63, pp. 318–351, 1953.
- [3] B. Mandelbrot, "The Pareto-Levy law and the distribution of income," *International Economic Review*, vol. 1, pp. 79–106, 1960.
- [4] C. Dagum, "A systematic approach to the generation of income distribution models," *Journal of Income Distribution*, vol. 61, pp. 105–126, 1996.
- [5] W. J. Reed, "The Pareto, Zipf and other power laws," *Economics Letters*, vol. 74, pp. 15–19, 2001.
- [6] —, "The Pareto law of incomes—an explanation and an extension," *Physica A*, 319:469–486, 2003.
- [7] O. Arrhenius, *J. Ecol.*, vol. 9, no. 95–99, 1921.
- [8] F. Preston, "The canonical distribution of commonness and rarity," *Ecology*, no. 43, pp. 185–215, 410–430, 1962.

- [9] E. F. Connor and E. D. McCoy, "The statistics and biology of the species-area relationship," *The American Naturalist*, vol. 113, no. 6, pp. 791–833, Jun., 1979.
- [10] J. B. Plotkin, M. D. Potts, D. W. Yu, S. Bunyavejchewin, R. Condit, R. Foster, S. Hubbell, J. LaFrankie, N. Manokaran, H.-S. Lee, R. Sukumar, M. A. Nowak, and P. S. Ashton, "Predicting species diversity in tropical forests," *PNAS*, vol. 97, no. 20, pp. 10850–10854, September 26, 2000.
- [11] J. E. Keeley, "Relating species abundance distributions to species-area curves in two mediterranean-type shrublands," *Diversity and Distributions*, no. 9, pp. 253–259, 2003.
- [12] F. Auerbach, "Das gesetz der bevölkerungskonzentration," *Petermanns Geographische Mitteilungen*, vol. 59, pp. 74–76, 1913.
- [13] G. Zipf, *Human Behavior and the Principle of Least Effort*. Cambridge, MA: Addison-Wesley, 1949.
- [14] Y. M. Ioannides and H. G. Overman, "Zipf's law for cities: an empirical examination," *Regional science and urban economics*, vol. 33, no. 2, pp. 127–137, 2003.
- [15] X. Gabaix, "Zipf's law for cities: an explanation," *The Quarterly Journal of Economics*, vol. 114, no. 3, pp. 739–767, 1999.
- [16] K. Rosen and M. Resnick, "The size distribution of cities: An examination of the Pareto law and primacy," *Journal of Urban Economics*, VIII, pp. 165–186, 1980.
- [17] J. Parr, "A note on the size distribution of cities over time," *Journal of Urban Economics*, VIII, pp. 199–212, 1985.
- [18] A. P. Allen, B. Li, and E. L. Charnov, "Population fluctuations, power laws and mixtures of lognormal distributions," *Ecology Letters*, no. 4, pp. 1–3, 2001.
- [19] L. A. N. Amaral, S. V. Buldyrev, S. Havlin, H. Leschhorn, P. Maass, M. A. Salinger, H. E. Stanley, and M. H. Stanley, "Scaling behavior in economics: Empirical results for company growth," *Journal de Physique I*, vol. 7, pp. 621–633, 1997.
- [20] M. A. Huynen and E. van Nimwegen, "The frequency distribution of gene family sizes in complete genomes," *Mol. Biol. Evol.*, vol. 15, no. 5, pp. 583–9, May 1998.
- [21] A. Rzhetsky and S. M. Gomez, "Birth of scale-free molecular networks and the number of distinct DNA and protein domains per genome," *Bioinformatics*, vol. 17, pp. 988–996, 2001.
- [22] N. M. Luscombe, J. Qian, Z. Zhang, T. Johnson, and M. Gerstein, "The dominance of the population by a selected few: power-law behaviour applies to a wide variety of genomic properties," *Genome Biology*, vol. 3, no. 8, 2002.
- [23] J. Brujic, R. I. H. Z., K. A. Walther, and J. M. Fernandez, "Single-molecule force spectroscopy reveals signatures of glassy dynamics in the energy landscape of ubiquitin," *Nature Physics*, vol. 2, no. 4, pp. 282–286, 2006.
- [24] L. Adamic and B. Huberman, "Evolutionary dynamics of the World Wide Web," *Nature*, vol. 401, p. 131, 1999.
- [25] C. Cunha, A. Bestavros, and M. Crovella, "Characteristics of World Wide Web client-based traces," *Technical Report TR-95010*, Boston University, pp. 126–134, 1995.
- [26] L. Breslau, P. Cao, L. Fan, G. Phillips, and S. Shenker, "Web caching and Zipf-like distributions: Evidence and implications," in *INFOCOM'99*, vol. 1, New York, NY, USA, March 1999, pp. 126–134.
- [27] A. B. Downey, "The structural cause of file size distributions," in *Ninth IEEE International Symposium on Modeling, Analysis, and Simulation of Computer and Telecommunications Systems (MASCOTS'01)*, 2001, p. 0361.
- [28] P. R. Jelenković and P. Momčilović, "Capacity regions for network multiplexers with heavy-tailed fluid on-off sources," in *INFOCOM'01*, Anchorage, AK, April 2001.
- [29] W. E. Leland, W. Willinger, M. S. Taqqu, and D. V. Wilson, "On the self-similar nature of Ethernet traffic," *SIGCOMM Comput. Commun. Rev.*, vol. 25, no. 1, pp. 202–213, 1995.
- [30] M. Crovella and A. Bestavros, "Self-similarity in World Wide Web traffic: evidence and possible causes," *IEEE/ACM Transactions on Networking*, vol. 5:6, pp. 835–846, 1997.
- [31] B. A. Huberman and L. A. Adamic, "The nature of markets in the World Wide Web," *Quarterly Journal of Economic Commerce I*, pp. 5–12, 2000.
- [32] L. A. Adamic and B. A. Huberman, "Zipf's law and the Internet," *Glottometrics*, vol. 3, pp. 143–150, 2002.
- [33] P. R. Jelenković, A. A. Lazar, and N. Semret, "The effect of multiple time scales and subexponentiality in MPEG video streams on queuing behavior," *IEEE Journal of Selected Areas in Communications*, vol. 15, no. 6, pp. 1052–1071, 1997.
- [34] M. Faloutsos, P. Faloutsos, and C. Faloutsos, "On power-law relationships of the Internet topology," in *Proceedings of the ACM SIGCOMM 1999 Conference*, 1999, pp. 251–262.
- [35] A. L. Barabási, R. Albert, and H. Jeong, "Mean-field theory for scale-free random networks," *Physica A* 272, pp. 173–189, 1999.
- [36] R. Kumar, P. Raghavan, S. Rajagopalan, D. Sivakumar, A. Tomkins, and E. Upfal, "Stochastic models for the web graph," in *Proceedings of the 41st Annual Symposium on Foundations of Computer Science*, pp. 57–65, 2000.
- [37] A. Medina, I. Matta, and J. Byers, "On the origin of power laws in Internet topologies," *Computer Communication Review*, vol. 30, no. 2, pp. 18–28, 2000.
- [38] X. Gabaix, P. Gopikrishnan, V. Plerou, and H. E. Stanley, "A theory of power-law distributions in financial market fluctuations," *Nature*, vol. 423, pp. 267–270, 15 May 2003.
- [39] M. Levy and S. Solomon, "Dynamical explanation for the emergence of power law in a stock market model," *International Journal of Modern Physics C*, vol. 7, pp. 65–72, 1996.
- [40] K. Athreya and A. N. Vidyashankar, "Branching processes," *Springer-Verlag*, 1972.
- [41] M. Levy and S. Solomon, "Spontaneous scaling emergence in generic stochastic systems," *International Journal of Modern Physics C*, vol. 7, no. 5, p. 745, 1996.
- [42] —, "Power laws are logarithmic boltzmann laws," *International Journal of Modern Physics C*, vol. 7, pp. 595–601, 1996.
- [43] D. Sornette and R. Cont, "Convergent multiplicative processes repelled from zero: power laws and truncated power laws," *Journal de Physique I*, vol. 7:3, pp. 431–444, March 1997.
- [44] W. J. Reed and B. D. Hughes, "From gene families and genera to incomes and internet file sizes: Why power laws are so common in nature," *Phys. Rev. E*, vol. 66, no. 6, p. 067103, Dec 2002.
- [45] A. Blank and S. Solomon, "Power laws in cities population, financial markets and Internet sites (scaling in systems with a variable number of components)," *Physica A*, vol. 287, pp. 279–288, 2000.
- [46] M. Olvera-Cravioto, J. Blanchet, and P. W. Glynn, "On the transition from heavy traffic to heavy tails for the M/G/1 queue: the regularly varying case," 2006, private communication.
- [47] P. R. Jelenković and A. Lazar, "On the dependence of the queue tail distribution on multiple time scales of ATM multiplexers," in *Proceedings of the Conference on Information Sciences and Systems*, Baltimore, MD, March 1995, pp. 435–440.
- [48] M. Mitzenmacher, "A brief history of generative models for power law and lognormal distributions," *Internet Mathematics*, vol. 1, no. 2, pp. 226–251, 2004.
- [49] J. M. Carlson and J. Doyle, "Highly optimized tolerance: A mechanism for power laws in designed systems," *Physics Review E*, vol. 60, no. 2, 1999.
- [50] P. Billingsley, *Probability and Measure*, 3rd ed. John Wiley & Sons, Inc., 1995.
- [51] R. M. Loynes, "The stability of a queue with non-independent inter-arrival and service times," *Proc. Camb. Phil. Soc.*, vol. 58, pp. 497–520, 1962.
- [52] P. W. Glynn and W. Whitt, "Logarithmic asymptotics for steady-state tail probabilities in a single-server queue," *Studies in Appl. Prob.*, vol. 31, pp. 131–156, 1994.
- [53] W.-B. Gong, Y. Liu, V. Misra, and D. Towsley, "Self-similarity and long range dependence on the Internet: a second look at the evidence, origins and implications," *Computer Networks*, vol. 48, pp. 377–399, June 2005.
- [54] S. Asmussen, *Applied Probability and Queues*. Wiley, New York, 1987.
- [55] A. Dembo and O. Zeitouni, *Large Deviations Techniques and Applications*. Springer, 1998.
- [56] W. J. Reed and M. Jorgensen, "The double Pareto - lognormal distribution - a new parametric model for size distributions," *Communications in Statistics: Theory and Methods*, vol. 33, no. 8, pp. 1733 – 1753, 2004.
- [57] P. R. Jelenković and J. Tan, "Modulated branching processes, origins of power laws and queueing duality," Dept. of Electrical Engineering, Columbia University, Tech. Rep., October 2006, in preparation, URL: [http://www.comet.columbia.edu/~predrag/pub\\_list.html](http://www.comet.columbia.edu/~predrag/pub_list.html).
- [58] M. Mitzenmacher, "Dynamic models for file sizes and double Pareto distributions," *Internet Mathematics*, vol. 1, no. 3, pp. 305 – 333, 2004.