Power Laws in ALOHA Systems

E6083: lecture 8

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March 6, 2007

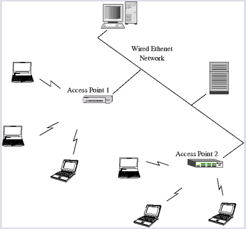
Outline

Renewed Interest in ALOHA

Multiple Access: nodes share a common medium

- A receiver can hear several transmitters
- A transmitter can broadcast to several receivers

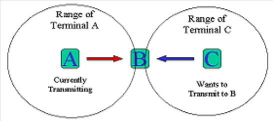
How to share the medium unaware of the others?



Properties of ALOHA

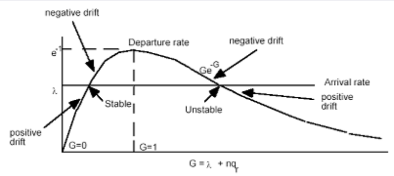
- low complexity
- distributed, without coordination
- scalable

Hidden terminal



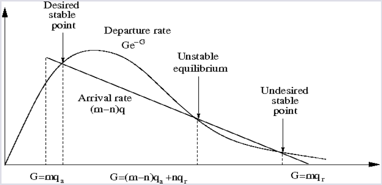
Instability of Slotted ALOHA with Infinite Number of Users

If backlog increases beyond unstable point, then the departure rate will drop to 0.



Positive throughput with finite number of users

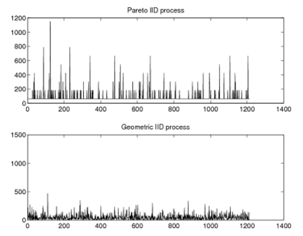
m: total number of users
n: number of backlog
q_a: the arrival probability
q_r: retransmission probability
with $q_r > q_a$



Can power law arise in ALOHA?

Drawback of power law

- Compare the sample path of the power law with the geometric distribution of the same mean and variance...
- Power law delay may impede the system periodically, even cause 0 throughput.



- Delay in ALOHA with fixed packet length and number of users is light-tailed.
- In reality, both the packet size and the number of users can be variable.

Outline

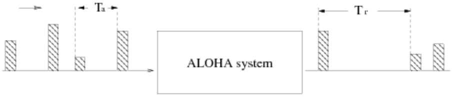
1. ALOHA type of Protocols

2. Power Laws in ALOHA
   - ALOHA with Variable Size Packets
   - Power Laws in Slotted ALOHA with Random Number of Users

ALOHA with Variable Size Packets

View AlOHA system as a converter

T_a: arrival interval
T_r: transmission time



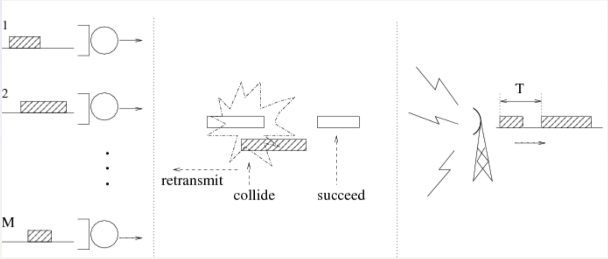
Polling Questions?

- Variable packet sizes can amplify the delay, but how much?
- If packet sizes are concentrated (light tailed), is the transmission delay also light-tailed?
- If the number of users is finite and fixed, is the throughput always positive?

Model Description

- Finite number of users M, $U(t)$ is the number of backoffs at time t.
- Each user can hold at most one packet in its queue.
- New packet is generated after an independent (from all other variables) exponential time with mean $1/\lambda$.
- Each packet has an independent length that is equal in distribution to a random variable L.
- After a collision, each participating user waits (backoffs) for an independent exponential period of time with mean $1/\nu$ and then attempts to retransmit its packet.

Visualized Scheme



We study:

- N - number of transmission attempts between 2 successful transmissions.
- T - time between 2 successful transmissions.

Power Laws in the Finite Population ALOHA with Variable Size Packets

Theorem

If

$$\lim_{x\to\infty} \frac{\log \mathbb{P}[L > x]}{x} = -\mu, \quad \mu > 0,$$

then, we have

$$\lim_{n\to\infty} \frac{\log \mathbb{P}[N > n]}{\log n} = -\frac{M\mu}{(M-1)\nu}$$

and

$$\lim_{t\to\infty} \frac{\log \mathbb{P}[T > t]}{\log t} = -\frac{M\mu}{(M-1)\nu}.$$

Simulations

Example

4 experiments: $M = 2, 4, 10, 20$;
packet sizes $\sim$ i.i.d. exp(1);
arrival intervals and backoffs $\sim$ exp(2/3);
simulation samples $= 10^5$.
As M gets large ($M = 10, 20$), the slopes of the distributions on the log / log plot are essentially the same.

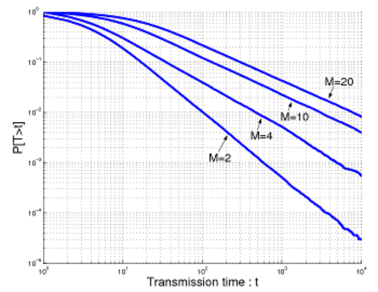
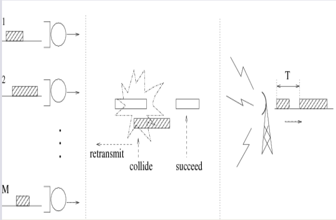


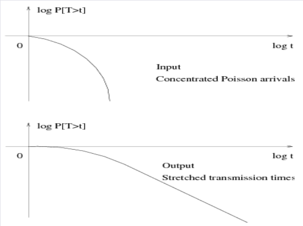
Figure: Interval distribution between successfully transmitted packets.

Implications

- the distribution tails of N and T are essentially power laws when the packet distribution $\approx e^{-\mu x}$.
- The finite population ALOHA may exhibit high variations and possible zero throughput.
  - $0 < M\mu/(M-1)\nu < 1 \Rightarrow$ zero throughput;
  - $1 < M\mu/(M-1)\nu < 2 \Rightarrow \mathbb{V}ar[T] = \infty$.
- For large M, $M\mu/(M-1) \approx \mu/\nu$ and thus, the system has zero throughput if $\nu \gtrapprox \mu$.
- 0 throughput may occur even when $\mathbb{E}L \ll 1/\nu$.

Nonlinear amplifier





Proof of power laws in ALOHA with variable packet sizes

1. assuming that a collision happens at time $t = 0$, i.e., $U(0) = M$.
2. For $x_\epsilon > 0$,

$$\begin{aligned}\mathbb{P}[N > n] &= \mathbb{E}\left[\left(1 - \frac{1}{M}\left(\sum_{i=1}^{M} e^{-L_i(M-1)\nu}\right)\right)^n\right] \\ &= \mathbb{E}\left[\left(1 - \frac{1}{M}\left(\sum_{i=1}^{M} e^{-L_i(M-1)\nu}\right)\right)^n \mathbf{1}\left(\bigcap_{i=1}^{M}\{L_i > x_\epsilon\}\right)\right] \\ &\quad + \mathbb{E}\left[\left(1 - \frac{1}{M}\left(\sum_{i=1}^{M} e^{-L_i(M-1)\nu}\right)\right)^n \mathbf{1}\left(\bigcup_{i=1}^{M}\{L_i \le x_\epsilon\}\right)\right].\end{aligned}$$

Proof of the first Theorem

3. Upper bound. By using $1 - x \le e^{-x}$ and the independence of L_i,

$$\mathbb{P}[N > n] \le \left(\mathbb{E}\left[e^{-\frac{n}{M}e^{-L(M-1)\nu}}\mathbf{1}(L > x_\epsilon)\right]\right)^M + \left(1 - \frac{1}{M}e^{-x_\epsilon(M-1)\nu}\right)^n$$

$$\le \left(\mathbb{E}\left[e^{-\frac{n}{M}e^{-L\mathbf{1}(L>x_\epsilon)(M-1)\nu}}\right]\right)^M + \eta^n.$$

For any $0 < \epsilon < \mu$, there exits x_ϵ such that $\mathbb{P}[L > x] \le e^{-(\mu-\epsilon)x}$ for all $x \ge x_\epsilon$, which, by defining random variable L_ϵ with $\mathbb{P}[L_\epsilon > x] = e^{-(\mu-\epsilon)x}, x \ge 0$, implies $L\mathbf{1}(L > x_\epsilon) \overset{d}{\le} L_\epsilon$, therefore,

$$\mathbb{P}[N > n] \le \left(\mathbb{E}\left[e^{-\frac{n}{M}U^{(M-1)\nu/(\mu-\epsilon)}}\right]\right)^M + \eta^n.$$

By using the identity $\mathbb{E}[e^{-\theta U^{1/\alpha}}] = \Gamma(\alpha+1)/\theta^\alpha$, one can easily obtain

$$\overline{\lim_{n\to\infty}} \frac{\log \mathbb{P}[N > n]}{\log n} \le -\frac{M(\mu-\epsilon)}{(M-1)\nu}.$$

4. Lower bound. Define $L_o \triangleq \min\{L_1, L_2, \cdots, L_M\}$, and observe that

$$\mathbb{P}[N > n] = \mathbb{E}\left[\left(1 - \frac{1}{M}\left(\sum_{i=1}^{M} e^{-L_i(M-1)\nu}\right)\right)^n\right]$$

$$\geq \mathbb{E}\left[\left(1 - e^{-L_o(M-1)\nu}\right)^n\right].$$

For any $\epsilon > 0$, there exists x_ϵ such that $\mathbb{P}[L_o > x] \geq e^{-(M\mu+\epsilon)x}$ for all $x \geq x_\epsilon$. Define random variable L_o^ϵ with $\mathbb{P}[L_o^\epsilon > x] = e^{-(M\mu+\epsilon)x}, x \geq 0$, then, $L_o \overset{d}{\geq} L_o^\epsilon \mathbf{1}(L_o^\epsilon > x_\epsilon)$, therefore

$$\mathbb{P}[N > n] \geq \mathbb{E}\left[\left(1 - e^{-L_o^\epsilon(M-1)\nu}\right)^n \mathbf{1}(L_o^\epsilon > x_\epsilon)\right],$$

which, by using similar techniques in the proof of the upper bound, implies

$$\varliminf_{n\to\infty} \frac{\log \mathbb{P}[N > n]}{\log n} \geq -\frac{M\mu + \epsilon}{(M-1)\nu}.$$

5. Passing $\epsilon \to 0$ in the lower and upper bound, we finish the proof for the case $U(0) = M$.
6. Define $N_s \triangleq \min\{n \geq 0 : U(T_n) = M\}$, $N_l \triangleq \min\{N, N_s\}$ and $N_e \triangleq N - N_l$. It can be shown that $\mathbb{P}[N_l > n] \leq \mathbb{P}[N_s > n] = o(e^{-\theta n})$, and we obtain

$$\begin{aligned}\mathbb{P}[N_e > n] &= \mathbb{P}[N - N_s > n, N > N_s] \\ &= \mathbb{P}[N > N_s]\mathbb{P}[N - N_s > n \mid N > N_s] \\ &= \mathbb{P}[N > N_s]\mathbb{P}[N > n \mid N_s = 0],\end{aligned}$$

which yields

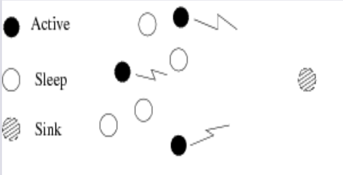
$$\lim_{n\to\infty} \frac{\log \mathbb{P}[N > n]}{\log n} = \lim_{n\to\infty} \frac{\log \mathbb{P}[N_l + N_e > n]}{\log n} = -\frac{M\mu}{(M-1)\nu}.$$

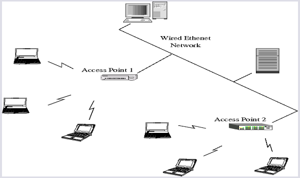
Outline

Power Laws in Slotted ALOHA with Random Number of Users

Power laws can be eliminated by reducing the variability of the packet sizes, however, when the number of active users M is random, we may also have power laws. Here, backoff $\sim$ Geo$(e^{-\nu}), \nu > 0$.

Examples





Power Laws in Slotted ALOHA with Random Number of Users

Theorem

If there exists $\alpha > 0$, *such that,* $\lim_{x\to\infty} \frac{\log \mathbb{P}[M>x]}{x} = -\alpha$, *then, we have*

$$\lim_{n\to\infty} \frac{\log \mathbb{P}[N > n]}{\log n} = \lim_{t\to\infty} \frac{\log \mathbb{P}[T > t]}{\log t} = -\frac{\alpha}{\nu}.$$

Theorem (Exact asymptotics)

If $\lambda = \nu$ *and* $\bar{F}(x) \triangleq \mathbb{P}[M > x]$ *satisfies* $H\left(-\log \bar{F}(x)\right) \bar{F}(x)^{1/\beta} \sim xe^{-\nu x}$ *with* $H(x)$ *being continuous and regularly varying, then, as* $t \to \infty$,

$$\mathbb{P}[T > t] \sim \frac{\Gamma(\beta+1)(e^{\nu}-1)^{\beta}}{t^{\beta} H(\beta \log t)^{\beta}}.$$

Power Laws in Slotted ALOHA with Random Number of Users

Example (Setting)

- If active users M is bounded $\Rightarrow \mathbb{P}[T > x]$ is exponentially bounded.
- However, this exponential behavior may happen for very small probabilities, while the delays of interest can fall inside the region of the distribution (main body) that behaves as the power law.
- Assume that initially $M \geq 1$ users have unit size packets ready to send and M follows geometric distribution with mean 3.
- The backoff times of colliding users are independent and geometrically distributed with mean 2.
- M has finite support $[1, K]$ where K ranges from 6 to 14 and we set the number of users to be equal to $M_K = \min(M, K)$.

Power Laws in Slotted ALOHA with Random Number of Users

Example (Simulation Results)

- The support of the main body of $\mathbb{P}[T > t]$ grows exponentially fast.
- If $K = 14$, the probabilities of interest for $\mathbb{P}[T > t]$ are bigger than $1/500$, then the result of this experiment is basically the same as for $K = \infty$.

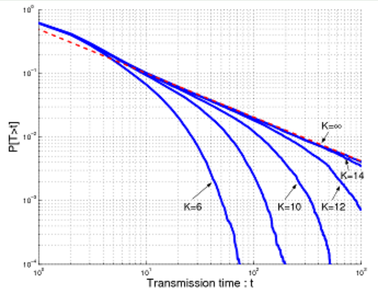


Figure: Stretched support of the power law main body when the number of users is $\min(M, K)$.

Proof of power laws in ALOHA with variable number of users

1. First consider a situation where all the users are backlogged, i.e., have a packet to send.

$$\mathbb{P}[N > n] = \mathbb{E}\left[\left(1 - \frac{Me^{-(M-1)\nu}(1-e^{-\nu})}{1-e^{-M\nu}}\right)^n\right].$$

   On the other hand, we have

$$\mathbb{P}[T > t] = \mathbb{E}\left[\left(1 - Me^{-(M-1)\nu}(1-e^{-\nu})\right)^t\right], \quad t \in \mathbb{N}.$$

2. one can show that the same asymptotic results hold if the initial number of backlogged users is less than M.